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Bayesian Estimation of Traffic Intensity in M/D/1 Queue Relative to Balanced Loss Function

D. Nadayil[*]

Department of Statistics, Bangalore University, Bangalore, India

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Abstract: In the Imbedded Markov Chain analysis of M/G/1 queue, $X_1, X_2, \ldots X_n$, ... is a sequence of i.i.d random variables. In particular, for the M/D/1 queue, the distribution of common random variable turns out to be well known Poisson distribution with mean ρ, the traffic intensity. In this article, the Bayes estimator of traffic intensity in steady state relative to the Balanced loss function (BLF) has been derived based on observations on X, the number of customer's arrival during the customer's service period. Admissibility and inadmissibility of the linear estimator under unconstrained optimization are also obtained.

Keywords: Admissibility, Bayes estimator, Balanced Loss function, Inadmissibility, M/D/1 queue.

1 Introduction

Queuing systems have attracted researchers from various disciplines like mathematics, probability, statistics, operations research and industrial engineering. The problems tackled, evidently, include mathematical, probabilistic, statistical and operational problems. An important aspect is the application of queuing theory. The problems of statistical inference in various queuing systems have attracted statisticians and queuing theorists as well. Bayesian inference has also been drawn to infer parameters and parametric functions in queuing systems. Muddapur [1] appears to be an early contributor to this area of research. A good review of literature on statistical and Bayesian inference in queuing systems is by Bhat et.al [2]. The recent literature has been enriched by contributions from Armero and Bayarri [3], Chowdhury and Borthakur [4], Srinivas et.al [5], Srinivas and Udupa [6]. Classical estimation in the M/D/1 queue along with the consistent asymptotic normality of various measures were discussed by Srinivas and Kale [7]. For estimating the weighted coefficients of the Balanced Loss function, F.Al.Duais & Alhagyan [8] utilised a nonlinear programming method and two balanced loss functions for estimating the parameters are discussed. In queueing literature, the M/D/1 queuing system is extensively used. In actual life, there are many instances where service times are constant. A washing machine cycle, for example, requires a fixed amount of time to complete one service. Another example is the automatic bus wash system, in which the amount of time each bus spends being washed remains consistent.

The objective of this article is to find the Bayes estimator in a steady state relative to the balanced loss function which is carried out in the second section. Zellner [9] introduced the Balanced Loss Function (BLF), which is a two-part loss function and it emulates two criteria, with the first term representing goodness of fit and the other representing accuracy of estimation and is given by

$$L_B(\hat{\rho},\rho) = \frac{w}{n}\sum_{i=1}^{n}(X_i-\hat{\rho})^2 + (1-w)(\hat{\rho}-\rho)^2 \, where 0 < w < 1 \tag{1}$$

[*]Corresponding author e-mail: deenunadayil@gmail.com

The concept of BLF and estimation relative to this loss function are also referred to by Chung et.al. [10] and Sanjari & Asgharzadeh [11]. The admissibility and inadmissibility of the linear estimator of traffic intensity are also obtained in the third section of this article.

2 Bayes estimator of ρ, in stable state relative to Balanced Loss Function

In M/D/1 queue, let $X_1, X_2, \ldots X_n$ be a random sample of n observations from well-known Poisson distribution with pmf ,

$$P(X = x) = \frac{e^{-\rho}\rho^{x}}{x!}, x = 0,1,2,\ldots \tag{2}$$

where X_i represents number of client arrivals during the service time of i^{th} client.
The M/D/1 queue is said to be in stable state if ρ is confined in ⴱ= [ρ: 0<ρ<1]. Our objective here is to find Bayes estimator of ρ in stable state. This estimator is obtained by minimizing the posterior expected BLF given by

$$E^{\pi(\rho/x)}L(\hat{\rho},\rho) = \min E^{\pi(\rho/x)}L(\hat{\rho},\rho)$$

$$= \min E\left[\frac{w}{n}\sum_{i=1}^{n}(X_i - \hat{\rho})^2 + (1-w)(\hat{\rho}-\rho)^2\right] \text{ s.t } 0<\hat{\rho}<1 \tag{3}$$

The constraint 0< $\hat{\rho}$ <1,which is considered in steady state in M/D/1 queue is modified by 0< $\hat{\rho}$ <1^-
where 1^- is a value below one but close to 1. [7]

$$Max - f(\hat{\rho}) = -E\left[\frac{w}{n}\sum_{i=1}^{n}(X_i - \hat{\rho})^2 + (1-w)(\hat{\rho}-\rho)^2\right] \tag{4}$$

$$\text{s.t } \hat{\rho} \geq 0$$
$$\hat{\rho} \leq 1^-$$

This optimization problem is nonlinear program in a single variable $\hat{\rho}$,and –f($\hat{\rho}$) being strictly concave function and we are required to solve this nonlinear program. Kuhn-Tucker conditions by Hilear and Liberman [12] provide the optimal solution to the above nonlinear problem which is given below.

1. $\frac{df}{d\hat{\rho}} - a\frac{dg}{d\hat{\rho}} \leq 0$

   $2w\bar{x} - 2w\hat{\rho}^* - (1-w)(2\hat{\rho}^* - 2\bar{\rho}) - a \leq 0$

2. $\hat{\rho}^*\left[-2w\hat{\rho}^* + 2w\bar{x}_n - 2(1-w)(\hat{\rho}^* - \bar{\rho}) - a\right] = 0$

3. $(\hat{\rho}^* - 1^-) \leq 0$

4. $a * (\hat{\rho}^* - 1^-) = 0$

5. $\hat{\rho}^* \geq 0$

6. $a \geq 0$

where a is a real number.

Conditions 2 and 4 requires that product of two quantities be zero. Therefore, each of these conditions is really saying that at least one of the quantities must be zero.

From condition 2,

$$\hat{\rho}^* = 0 \quad \text{or} \quad w\bar{x}_n + (1-w)\bar{\rho} = \frac{a}{2}$$

Clearly $\hat{\rho}^* = 0$ is a trivial solution.

Thus $\hat{\rho}^* = w\bar{x}_n + (1-w)\bar{\rho} - \frac{a}{2}$ where $\bar{\rho}$ is the posterior mean.

As a result, optimal solution is

$$\hat{\rho}^* = \begin{cases} 1^- & ; a>0 \\ w\bar{x}_n + (1-w)\bar{\rho} & ; a=0 \end{cases} \tag{5}$$

3 The Bayes estimator under unconstrained optimization its admissibility and inadmissibility of traffic intensity

We presume that some previous understanding about the queuing parameter can be acquired from expert knowledge or opinion in the Bayesian approach. In Bayesian statistics, we combine likelihood with the prior information in order to calculate the posterior distribution. Here, in this problem, we consider the following prior, which is the conjugate prior. A conjugate prior distribution of any distribution is one in which the posterior and prior densities belong to the same distribution family. Such priors provide the advantage of being analytically complaisant, flexible, and easily explicable. We investigate the situation when the prior density is Gamma prior which is given as

$$g_1(\rho) = \frac{1}{\Gamma\beta} \propto^{\beta} e^{-\propto\rho} \rho^{\beta-1}, 0 < \rho < \infty; \propto, \beta > 0 \tag{6}$$

The Bayes theorem is used to combine the likelihood function of probability mass function (2) with the conjugate prior density (6) to generate the posterior density which is Gamma distribution with parameters $(\sum xi + \beta)$ and $(n+\alpha)$.

3.1 Admissibility

Following Chung et.al. [10], the admissibility of the linear estimator is established. Relative to the balanced loss function, the Bayes estimator of ρ is,

$$\hat{\rho} = w\bar{x} + (1-w)\frac{\sum x_i + \beta}{n+\alpha}$$

$$= \frac{n+\alpha w}{n+\alpha}\bar{x} + (1-w)\frac{\beta}{n+\alpha} \tag{7}$$

which is in the form of linear estimator , $c\bar{x}+d$ where c= $\frac{n+\alpha w}{n+\alpha}$ and d= $(1-w)\frac{\beta}{n+\alpha}$.

$c\bar{x}+d$ is Bayes estimator with c being greater than w and less than 1 for w<1.Also d is greater than zero for any choice of w, n,α, β.Since the balanced loss function is strictly convex, hence $c\bar{x}+d$ will be unique Bayes estimator provided w<c<1 and d>0 with finite Bayes risk and hence $\{ c\bar{x}+d; w<c<1, d>0 \}$is admissible.

3.2 Inadmissibility

If any of the following conditions are satisfied, then $c\bar{x}+d$ is inadmissible with respect to the BLF,

i. c>1, d≥ 0
ii. c=1 and d not equal to zero
iii. c<w, d<0
iv. c=w, d<0

Proof:

Consider the frequentist risk of $c\bar{x}+d$ given by

$$R(\rho, c\bar{X}+d) = ((c-1)\rho + d)^2 + \frac{\rho}{n}\left[(c-w)^2 + w(n-w)\right] \quad (8)$$

i.Assume that c>1,

i.e. $(c-w)^2 > (1-w)^2$ and hence from (8),

Now consider the risk,

$$R(\rho, c\bar{x}+d) \geq \frac{\rho}{n}\left[(c-w)^2 + (nw-w^2)\right]$$
$$> \frac{\rho}{n}\left[(1-w)^2 + (nw-w^2)\right]$$
$$= R(\rho, \bar{X})$$

By definition, $\bar{X}$ seems to be R-better decision rule than $c\bar{X}+d$ when c>1.

Hence $c\bar{X}+d$ is inadmissible estimator when c >1 and d≥ 0.

ii. Assume that c=1 and d not equal to zero

$$R(\rho, \bar{X}+d) = d^2 + \frac{\rho}{n}\left[(1-2w+w^2) + (nw-w^2)\right]$$
$$> \frac{\rho}{n}\left[(1-2w+w^2) + (nw-w^2)\right]$$
$$= R(\rho, \bar{X})$$

$\bar{X}$ seems to be R-better relative to $\bar{X}+d$ when d not equal to zero. Therefore, $\bar{X}$ dominates $\bar{X}+d$ when d not equal to zero.

iii. c<w, d<0

Assume that c<w, then

$(c-1)^2 > (w-1)^2$, then

$$R(\rho, c\bar{X}+d) = \frac{\rho}{n}\left[(c^2-2cw+w^2) + (nw-w^2)\right] + \left[(c-1)\rho + d\right]^2$$
$$= \frac{\rho}{n}\left[(c^2-2cw+w^2) + (nw-w^2)\right] + (c-1)^2\left[\rho + \frac{d}{c-1}\right]^2$$
$$> \frac{\rho}{n}\left[nw-w^2\right] + (w^2-2w+1)\left[\rho + \frac{d}{c-1}\right]^2$$
$$= \frac{\rho}{n}\left[nw-w^2\right] + \left[(w-1)\rho + \frac{d(w-1)}{(c-1)}\right]^2$$
$$= R(\rho, w\bar{X} + d\frac{(w-1)}{c-1})$$

i.e. $w\bar{X} + d\dfrac{(w-1)}{a-1}$ is R better relative to $c\bar{X}+d$ when d<0.

i.e. $w\bar{X}+d\dfrac{(w-1)}{a-1}$ dominates $c\bar{X}+d$ when d<0.

iv. c=w, d<0

Assume that c=w, then

$$R(w\bar{x}+d,\rho)-R(w\bar{x},\rho)=\left[d^2+(w^2-2w+1)\rho^2+2(w-1)d\rho\right]-\left[(w-1)^2\rho^2\right]$$
$$=\left[2(w-1)d\rho\right]+d^2>0$$

Thus $w\bar{X}$ is R better relative to $w\bar{X}+d$ when d<0.

That is $w\bar{X}$ dominates $w\bar{X}+d$ when d<0.

Hence $c\bar{X}+d$ is inadmissible when c=w, d<0.

4 Conclusion

The Bayes estimator of traffic intensity with respect to BLF is derived based on the observations ($X_1, X_2, \ldots X_n$) where X_j represents the number of customers who arrive during the service time of j^{th} period. It is also shown that the linear estimator of traffic intensity is admissible and inadmissible.

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