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Estimate the Parameters of the Gamma / Gompertz Distribution based on Different Sampling Schemes of Ordered Sets

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Abstract: In this paper, based on different types of ordered set sampling methods, the maximum likelihood (ML) method is used to estimate the unknown parameters of the Gamma / Gompertz distribution. The results of the computer simulation show the efficiency of neoteric ranked set sampling (NRSS), ranked set sampling (RSS), median ranked set sampling (MRSS), and percentile ranked set sampling (PRSS) based on simple random sampling (SRS).

Keywords: Ranked set sampling; Neoteric ranked set sampling; Median ranked set sampling; Gamma/Gompertz distribution; Maximum likelihood method.

1 Introduction

In real life where the variables of interest in the experimental unit are easier to categorize rather than quantify, it turns out that for the estimation of the population mean, an old concept of McIntyre [1] that RSS is beneficial and far superior to SRS. Takahasi and Wakimoto [2] developed the basic theory of the RSS process under the assumption of perfect ordering. The McIntyre technique was first used by Halls and Dell [3]. They found through experiments that RSS is more effective than SRS. They also described the currently used name-sorted set sampling. According to Dell and Clutter [4], the estimator of the mean under an imperfect RSS is still an unbiased estimator of the population mean. The relative efficiency (RP) of the two methods is equal to 1 only if the ranking is not better than random. Dell and Clutter emphasized that RP depends on the population characteristics and the size of the classification error. When used in this way, RP is also known as relative efficiency (RE).
Muttlak [5] used MRSS method to estimate the population mean and he showed that it is more efficient than the usual RSS method. Muttlak [6] introduced PRSS method with different value of p (where $0 \leq p \leq 1$) for estimating the population mean. Hassan [7] obtained the ML estimation and Bayesian estimation of the shape and scale parameters of the exponential distribution based on SRS and RSS. Nadjafi [8] estimated Gamma/Gompertz distribution parameters based on the ML estimation method using SRS method. Sameh and Qtait [9] used the MRSS-based ML method to estimate the shape and scale parameters of the exponential exponential distribution. Zamanzade and Al-Omari [10] proposed the NRSS procedure and demonstrated that the efficiency of estimators based on NRSS is higher than that used by RSS and SRS schemes.
The Gamma/Gompertz (G/G) distribution has been used in the fields of biology and human mortality. Missov[11] studied the life expectancy resulting from G/G force of mortality. The probability density function (PDF) and the cumulative distribution function (CDF) of the G/G distribution are, respectively, given by [12]

$$f(x;b,s,\beta) = \frac{bse^{bx}\beta^{s}}{(\beta-1+e^{bx})^{s+1}};\ x \geq 0; b,s,\beta > 0, \quad (1)$$

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and

$$F(x;b,s,\beta) = 1 - \frac{\beta^s}{(\beta - 1 + e^{bx})^s}; \ \ x \geq 0; b, s, \beta > 0. \tag{2}$$

where b is the scale parameter, s and β are the shape parameters.
In this article, the performance of the NRSS method is compared to other set sampling methods using the ML method on G/G distribution to estimate the unknown parameters (b, s, β) based on SRS scheme. The rest of this paper is organized as follows: In Section 2, the classification set sampling method is introduced. The ML estimation method will be discussed in Section 3. In Section 4, the results of computer simulations are presented to compare the efficiency of SRS-based estimators with their RSS counterparts MRSS, PRSS, and NRSS. The results show that for the G/G distribution considered in this study, NRSS has less mean square error and higher efficiency than all other methods. The conclusion is given in Section 5.

2 Some Ranked Set Sampling Techniques

In this section, the different sampling techniques used to select units in the sample are RSS, MRSS, PRSS and NRSS, and the definition of the likelihood function (LF) of each of these techniques will be considered. All the samples in this section are independent random samples.

2.1 Ranked Set Sampling Method

The original idea of RSS was put forward by McIntyre's [1]. RSS is a method that can improve technical efficiency such as estimation without making a large number of substantive observations. In addition, it aims to reduce the number of measurement observations required to achieve the required accuracy when making inferences. The RSS scheme can be described as follows:[13]
Step 1. In order to extract a random sample of size n, we determined n^2 units from the target population.
Step 2. These units are randomly assigned to n sets each of size n. The n units of each sample are ranked visually or in an inexpensive way in relation to the variable of interest.
Step 3. From the first set of n units, measure the smallest taxa. Starting from the second set of n units, measure the second smallest unit. This process continues until the largest ranked is measured from n^{th} set of n units.
Step 4. Repeat steps 1 through 3. This whole process is called a m cycle. Using these cycles to obtain a RSS sample of size nm.
Let $\{X_{(i;n)j}, i = 1, 2, ..., n; j = 1, 2, ..., m\}$ be an independent ranked set sample where n is the set size and m is the number of cycles. Then PDF of $X_{(i;i)j}$ is given by [14]

$$f_n(x_{(i;i)j}) = \frac{n!}{(i-1)!(n-i)!} f(x_{(i;i)j};\theta)\ [F(x_{(i;i)j};\theta)]^{i-1}\ [1 - F(x_{(i;i)j};\theta)]^{n-i}, \tag{3}$$

The LF corresponding to RSS technique is as follows:

$$L_{RSS}(x_{(i;i)j};\theta) = \prod_{j=1}^{m}\prod_{i=1}^{n}\left[\frac{n!}{(i-1)!(n-i)!} f(x_{(i;i)j};\theta)\ [F(x_{(i;i)j};\theta)]^{i-1}\ [1 - F(x_{(i;i)j};\theta)]^{n-i}\right]. \tag{4}$$

2.2 Median Set Sampling Method

Muttlak [5] suggested median ranked set sampling (MRSS) method to estimate the population mean. The MRSS scheme is explained as follows: [15]
Step 1. To extract a random sample of size n, the target population identifies n^2 units.
Step 2. These units are randomly assigned to n sets of size n. The n units of each sample are classified visually or by any economic method according to the variable of interest.
Step 3. For an even-numbered n sample, determine n^2units from the target population. Each sample classifies itself according to the ranked set sampling design. Choose the smallest rank of $\frac{n}{2}^{th}$ from the first $\frac{n}{2}$ sets and choose the $\frac{n+2}{2}^{th}$ smallest rank from the other $\frac{n}{2}$ sets. Similarly, for an odd sample size n, choose the smallest rank of $\frac{n+1}{2}^{th}$ from all sets.

*Step*4. Repeat steps 1 through 3 to obtain m cycles to draw a MRSS sample of size nm.
When n is an even, the LF corresponding to MRSS is shown as [14]

$$L_{MRSS_{even}}(x_{(i;i)j};\theta)=\prod_{j=1}^{m}\prod_{i=1}^{\frac{n}{2}}\left[\frac{n!}{[(\frac{n}{2}-1)!]^2(\frac{n}{2})}f(x_{(i;\frac{n}{2})j};\theta)\,[F(x_{(i;\frac{n}{2})j};\theta)]^{\frac{n-2}{2}}\,[1-F(x_{(i;\frac{n}{2})j};\theta)]^{\frac{n}{2}}\right]$$
$$\times\prod_{j=1}^{m}\prod_{i=\frac{n}{2}+1}^{n}\left[\frac{n!}{[(\frac{n}{2}-1)!]^2(\frac{n}{2})}f(x_{(i;\frac{n+2}{2})j};\theta)\,[F(x_{(i;\frac{n+2}{2})j};\theta)]^{\frac{n}{2}}\,[1-F(x_{(i;\frac{n+2}{2})j};\theta)]^{\frac{n-2}{2}}\right],\tag{5}$$

Otherwise, when n is an odd, it is given by

$$L_{MRSS_{odd}}(x_{(i;i)j};\theta)=\prod_{j=1}^{m}\prod_{i=1}^{n}\left[\frac{n!}{[(\frac{n-1}{2})!]^2}f(x_{(i;\frac{n+1}{2})j};\theta)\,[F(x_{(i;\frac{n+1}{2})j};\theta)]^{\frac{n-1}{2}}\,[1-F(x_{(i;\frac{n+1}{2})j};\theta)]^{\frac{n-1}{2}}\right].\tag{6}$$

2.3 Percentile Set Sampling Method

Muttlak [6] used the method of percentile ranked set sampling (PRSS), using different values of p (where $0\leq p\leq 1$) to estimate the population mean. The PRSS scheme is explained as follows:[16]
*Step*1. Identifies n^2 units in the target population from which random samples of size n are drawn.
*Step*2. Randomly allocate these units to n sets each of size n. The n units in each sample are ranked visually or by any inexpensive method with respect to the variable of interest.
*Step*3. For the even sample n, select the $(p(n+1))^{th}$ smallest classification unit of the first $\frac{n}{2}$ sets and select the $(q(n+1))^{th}$ smallest sorting units of other $\frac{n}{2}$ sets. Similarly, for an odd sample size n, select the $(p(n+1))^{th}$ smallest ranked unit from first $\frac{n-1}{2}$ sets. Select the $(q(n+1))^{th}$ smallest ranked unit from second $\frac{n-1}{2}$ sets and the middle rank $\frac{n+1}{2}$ from the remaining set.
*Step*4. Repeat steps 1 through 3 to obtain m cycles to draw a PRSS sample of size nm.
The LF corresponding to PRSS when n is an even is given by

$$L_{PRSS_{even}}(x_{(i;i)j};\theta)=\prod_{j=1}^{m}\prod_{i=1}^{\frac{n}{2}}\left[\frac{n!}{(p(n+1)-1)!(n-p(n+1))!}f(x_{(i;p(n+1))j};\theta)\,[F(x_{(i;p(n+1))j};\theta)]^{p(n+1)-1}\,[1-F(x_{(i;p(n+1))j};\theta)]^{n-p(n+1)}\right]$$
$$\times\prod_{j=1}^{m}\prod_{i=\frac{n}{2}+1}^{n}\left[\frac{n!}{(q(n+1)-1)!(n-q(n+1))!}f(x_{(i;q(n+1))j};\theta)\,[F(x_{(i;q(n+1))j};\theta)]^{q(n+1)-1}[1-F(x_{(i;q(n+1))j};\theta)]^{n-q(n+1)}\right],\tag{7}$$

Otherwise, when n is odd, it is shown as

$$L_{PRSS_{odd}}(x_{(i;i)j};\theta)=\prod_{j=1}^{m}\prod_{i=1}^{\frac{n}{2}}\left[\frac{n!}{(p(n+1)-1)!(n-p(n+1))!}f(x_{(i;p(n+1))j};\theta)\,[F(x_{(i;p(n+1))j};\theta)]^{p(n+1)-1}\right]$$
$$\times[1-F(x_{i;(p(n+1))j};\theta)]^{n-p(n+1)}\times\prod_{j=1}^{m}\prod_{i=\frac{n}{2}+1}^{n-1}\left[\frac{n!}{(q(n+1)-1)!(n-q(n+1))!}f(x_{(i;q(n+1))j};\theta)\,[F(x_{(i;q(n+1))j};\theta)]^{q(n+1)-1}\right]$$
$$\times[1-F(x_{(i;q(n+1))j};\theta)]^{n-q(n+1)}\times\left[\frac{n!}{[(\frac{n-1}{2})!]^2}f(x_{(i;\frac{n+1}{2})j};\theta)\,[F(x_{(i;\frac{n+1}{2})j};\theta)]^{\frac{n-1}{2}}\,[1-F(x_{(i;\frac{n+1}{2})j};\theta)]^{\frac{n-1}{2}}\right].\tag{8}$$

2.4 Neoteric Set Sampling Method

Zamanzade and Al-Omari [10] developed an RSS modification called NRSS. The difference between NRSS and the original RSS scheme is that it consists of a group of n^2 units instead of n groups of size n. This strategy has proven to be effective and can generate a more effective estimator for the population mean. The NRSS procedure can be described as follows: [13]
*Step*1. Choose a simple random sample of size n^2 units rom the target finite population.
*Step*2. Sort the n^2selected units in increments of the variables of interest based on visual inspection or any other free method.

Step 3. For an odd sample size n, select $(\frac{n+1}{2}+(i-1)n)^{th}$ rank units for $(i=1,2,...n)$. For even sample size n, there are two cases for i. If i is odd, then select the$(\frac{n+2}{2}+(i-1)n)^{th}$ ranked unit. Otherwise, if i is even, then select the$(\frac{n}{2}+(i-1)n)^{th}$ ranked unit for $(i=1,2,...n)$.
Step 4. Repeat steps 1 through 3 to obtain m cycles to build a NRSS of size nm.
The LF corresponding to NRSS is given by [14]

$$L_{NRSS}(x_{(i)j};\theta)=\prod_{j=1}^{m}\left[\prod_{i=1}^{n+1}\frac{n^2!}{(k(i)-k(i-1)-1)!}\prod_{i=1}^{n}f(x_{k(i)j};\theta)\prod_{i=1}^{n+1}[F(x_{k(i)j};\theta)-F(x_{k(i-1)j};\theta)]^{k(i)-k(i-1)-1}\right], \quad (9)$$

where

$$k(i)=\begin{cases}\frac{n+1}{2}+(i-1)n & n\ odd\\ \frac{n+2}{2}+(i-1)n & n\ even, i\ odd\\ \frac{n}{2}+(i-1)n & n\ even, i\ even\end{cases} \qquad \text{where } (i=1,2,...n)\ , k(0)=0 \text{ and } x_{k(0)}\cong-\infty.$$

3 Estimation Based on maximum likelihood method

In this section, the unknown parameters of the G/G distribution are estimated using ML based on SRS, RSS, MRSS, PRSS, and NRSS.

3.1 Estimation Based on SRS

Nadjafi [8] estimated the parameters of the G/G distribution using the SRS-based ML method. $\{X_1,X_2,..,X_n\}$ are used as independent random samples of G/G distribution using PDF, which is given by Eq.(1). The LF and the Log-likelihood function $(logL)$ of $(b,s$ and $\beta)$ are given by

$$L_{SRS}(x_i;b,s,\beta)=\prod_{i=1}^{n}\frac{bse^{bx_i}\beta^s}{(\beta-1+e^{bx_i})^{s+1}}, \quad (10)$$

and

$$logL_{SRS}=nlog(b)+nlog(s)+nslog(\beta)+b\sum_{i=1}^{n}x_i-(s+1)\sum_{i=1}^{n}log(\beta-1+e^{bx_i}). \quad (11)$$

First partial derivatives of the $logL$ for b, s and β are as follows:[11]

$$\frac{\partial logL_{SRS}}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n}x_i-(s+1)\sum_{i=1}^{n}\frac{x_ie^{bx_i}}{\beta-1+e^{bx_i}}, \quad (12)$$

$$\frac{\partial logL_{SRS}}{\partial s}=\frac{n}{s}+nlog(\beta)-\sum_{i=1}^{n}log(\beta-1+e^{bx_i}), \quad (13)$$

and

$$\frac{\partial logL_{SRS}}{\partial \beta}=\frac{ns}{\beta}-(s+1)\sum_{i=1}^{n}\frac{1}{\beta-1+e^{bx_i}}. \quad (14)$$

Setting the derivatives to zero, the three non-linear equations can be solved by numerical methods.

3.2 Estimation Based on RSS

In this subsection, we substitute Eq.(1) and Eq.(2) in Eq.(4) to derive MLs for the G/G distribution according to the RSS scheme. $\{X_{(1;1)1},X_{(1;1)2},..,X_{(1;1)n_1};X_{(2;2)1},X_{(2;2)2},...,X_{(2;2)n_2};...;X_{(n;n)1},X_{(n;n)2},...,X_{(n;n)n_n}\}$ are independent random variables. It is said that $X_{(i;i)j}$ represents the i^{th} order statistic from the i^{th}sample of size n where $(i=1,2,...,n)$. Repeat this cycle m times to earn nm units. Let $X_{(i;i)j}$ shows j^{th} cycles of size m where $(j=1,2,...,m)$. The LF and $logL$ functions of the RSS scheme are given by

$$L_{RSS}(x_{(i;i)j};b,s,\beta)=\prod_{j=1}^{m}\prod_{i=1}^{n}\left[\frac{n!}{(i-1)!(n-i)!}\frac{bse^{bx_{(i;i)j}}\beta^{s}}{(\beta-1+e^{bx_{(i;i)j}})^{s+1}}\left[1-\frac{\beta^{s}}{(\beta-1+e^{bx_{(i;i)j}})^{s}}\right]^{(i-1)}\left[\frac{\beta}{(\beta-1+e^{bx_{(i;i)j}})}\right]^{s(n-i)}\right], \tag{15}$$

and

$$logL_{RSS}=log\left[\prod_{j=1}^{m}\prod_{i=1}^{n}\frac{n!}{(i-1)!(n-i)!}\right]+mn\,log(b)+mn\,log(s)+mns\,log(\beta)+b\sum_{j=1}^{m}\sum_{i=1}^{n}x_{(i;i)j}$$
$$-(s+1)\sum_{j=1}^{m}\sum_{i=1}^{n}log(\beta-1+e^{bx_{(i;i)j}})+\sum_{j=1}^{m}\sum_{i=1}^{n}s(n-i)log\left(\frac{\beta}{\beta-1+e^{bx_{(i;i)j}}}\right)+\sum_{j=1}^{m}\sum_{i=1}^{n}(i-1)log\left[1-\frac{\beta^{s}}{(\beta-1+e^{bx_{(i;i)j}})^{s}}\right]. \tag{16}$$

First partial derivatives of the $logL_{RSS}$ with respect to b, s and β, respectively, are

$$\frac{\partial logL_{RSS}}{\partial b}=\frac{mn}{b}+\sum_{j=1}^{m}\sum_{i=1}^{n}x_{(i;i)j}-(s+1)\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{x_{(i;i)j}e^{bx_{(i;i)j}}}{\beta-1+e^{bx_{(i;i)j}}}-\sum_{j=1}^{m}\sum_{i=1}^{n}s(n-i)\frac{x_{(i;i)j}e^{bx_{(i;i)j}}}{\beta-1+e^{bx_{(i;i)j}}}$$
$$+\sum_{j=1}^{m}\sum_{i=1}^{n}(i-1)\frac{s\beta^{s}x_{(i;i)j}e^{bx_{(i;i)j}}}{(\beta-1+e^{bx_{(i;i)j}})^{s+1}-\beta^{s}\left(\beta-1+e^{bx_{(i;i)j}}\right)}, \tag{17}$$

$$\frac{\partial logL_{RSS}}{\partial s}=\frac{mn}{s}+mn\,log(\beta)-\sum_{j=1}^{m}\sum_{i=1}^{n}log(\beta-1+e^{bx_{(i;i)j}})+\sum_{j=1}^{m}\sum_{i=1}^{n}(n-i)log\left(\frac{\beta}{\beta-1+e^{bx_{(i;i)j}}}\right)$$
$$-\sum_{j=1}^{m}\sum_{i=1}^{n}(i-1)\frac{\beta^{s}}{(\beta-1+e^{bx_{(i;i)j}})^{s}-\beta^{s}}ln\left(\frac{\beta}{\beta-1+e^{bx_{(i;i)j}}}\right), \tag{18}$$

and

$$\frac{\partial logL_{RSS}}{\partial \beta}=\frac{mns}{\beta}-\sum_{j=1}^{m}\sum_{i=1}^{n}(i-1)\frac{s\beta^{s-1}\left(-1+e^{bx_{(i;i)j}}\right)}{(\beta-1+e^{bx_{(i;i)j}})^{s+1}-\beta^{s}\left(\beta-1+e^{bx_{(i;i)j}}\right)}+\sum_{j=1}^{m}\sum_{i=1}^{n}s(n-i)\frac{\left(-1+e^{bx_{(i;i)j}}\right)}{\beta\left(\beta-1+e^{bx_{(i;i)j}}\right)}$$
$$-(s+1)\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{1}{\left(\beta-1+e^{bx_{(i;i)j}}\right)}. \tag{19}$$

Equating the derivatives to zero, the three non-linear equations can be solved by numerical methods.

3.3 Estimation Based on MRSS

Substituting Eq.(1) and Eq.(2) to get the ML of the G/G distribution according to the MRSS scheme in Eq.(5) when n in the equation is an even and in Eq.(6) when n is an odd. The LF and $logL$ functions of MRSS scheme for n even are followed by

$$L_{MRSS_{even}}(x_{(i;i)j};b,s,\beta)=\prod_{j=1}^{m}\prod_{i=1}^{\frac{n}{2}}\left[\frac{n!}{[(\frac{n}{2}-1)!]^{2}(\frac{n}{2})}\frac{bse^{bx_{(i;\frac{n}{2})j}}\beta^{s}}{(\beta-1+e^{bx_{(i;\frac{n}{2})j}})^{s+1}}\left[1-\frac{\beta^{s}}{(\beta-1+e^{bx_{(i;\frac{n}{2})j}})^{s}}\right]^{\frac{n-2}{2}}\left[\frac{\beta}{(\beta-1+e^{bx_{(i;\frac{n}{2})j}})}\right]^{\frac{sn}{2}}\right]$$
$$\times\prod_{j=1}^{m}\prod_{i=\frac{n}{2}+1}^{n}\left[\frac{n!}{[(\frac{n}{2}-1)!]^{2}(\frac{n}{2})}\frac{bse^{bx_{(i;\frac{n+2}{2})j}}\beta^{s}}{(\beta-1+e^{bx_{(i;\frac{n+2}{2})j}})^{s+1}}\left[1-\frac{\beta^{s}}{(\beta-1+e^{bx_{(i;\frac{n+2}{2})j}})^{s}}\right]^{\frac{n}{2}}\left[\frac{\beta}{(\beta-1+e^{bx_{(i;\frac{n+2}{2})j}})}\right]^{\frac{s(n-2)}{2}}\right], \tag{20}$$

and

$$logL_{MRSS_{even}} = log\left[\prod_{j=1}^{m}\prod_{i=1}^{\frac{n}{2}}\frac{n!}{[(\frac{n}{2}-1)!]^2(\frac{n}{2})}\right] + mn\,log(b) + mn\,log(s) + mns\,log(\beta) + b\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}x_{(i;\frac{n}{2})j} + b\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}x_{(i;\frac{n+2}{2})j}$$
$$+\frac{sn}{2}\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}log\left(\frac{\beta}{\beta-1+e^{bx_{(i;\frac{n}{2})j}}}\right) + (\frac{n-2}{2})\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}log\left[1-\frac{\beta^s}{(\beta-1+e^{bx_{(i;\frac{n}{2})j}})^s}\right] + log\left[\prod_{j=1}^{m}\prod_{i=\frac{n}{2}+1}^{n}\frac{n!}{[(\frac{n}{2}-1)!]^2(\frac{n}{2})}\right]$$
$$-(s+1)\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}log(\beta-1+e^{bx_{(i;\frac{n+2}{2})j}}) + \frac{s(n-2)}{2}\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}log\left(\frac{\beta}{\beta-1+e^{bx_{(i;\frac{n+2}{2})j}}}\right)$$
$$-(s+1)\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}log(\beta-1+e^{bx_{(i;\frac{n}{2})j}}) + \frac{n}{2}\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}log\left[1-\frac{\beta^s}{(\beta-1+e^{bx_{(i;\frac{n+2}{2})j}})^s}\right]. \quad (21)$$

First partial derivatives of the $logL_{MRSS_{even}}$ with respect to b, s and β, respectively are as follows:

$$\frac{\partial logL_{MRSS_{even}}}{\partial b} = \frac{mn}{b} + \sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}x_{(i;\frac{n}{2})j} - (s+1)\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}\frac{x_{(i;\frac{n}{2})j}e^{bx_{(i;\frac{n}{2})j}}}{\beta-1+e^{bx_{(i;\frac{n}{2})j}}} - \frac{sn}{2}\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}\frac{x_{(i;\frac{n}{2})j}e^{bx_{(i;\frac{n}{2})j}}}{\beta-1+e^{bx_{(i;\frac{n}{2})j}}} + \sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}x_{(i;\frac{n+2}{2})j}$$
$$+(\frac{n-2}{2})\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}\frac{s\beta^s x_{(i;\frac{n}{2})j}e^{bx_{(i;\frac{n}{2})j}}}{(\beta-1+e^{bx_{(i;\frac{n}{2})j}})^{s+1}-\beta^s\left(\beta-1+e^{bx_{(i;\frac{n}{2})j}}\right)} - (s+1)\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}\frac{x_{(i;\frac{n+2}{2})j}e^{bx_{(i;\frac{n+2}{2})j}}}{\beta-1+e^{bx_{(i;\frac{n+2}{2})j}}}$$
$$+\frac{n}{2}\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}\frac{s\beta^s x_{(i;\frac{n+2}{2})j}e^{bx_{(i;\frac{n+2}{2})j}}}{(\beta-1+e^{bx_{(i;\frac{n+2}{2})j}})^{s+1}-\beta^s\left(\beta-1+e^{bx_{(i;\frac{n+2}{2})j}}\right)} - \frac{s(n-2)}{2}\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}\frac{x_{(i;\frac{n+2}{2})j}e^{bx_{(i;\frac{n+2}{2})j}}}{\beta-1+e^{bx_{(i;\frac{n+2}{2})j}}}, \quad (22)$$

$$\frac{\partial logL_{MRSS_{even}}}{\partial s} = \frac{mn}{s} + mn\,log(\beta) - \sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}log(\beta-1+e^{bx_{(i;\frac{n}{2})j}}) + \frac{n}{2}\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}log\left(\frac{\beta}{\beta-1+e^{bx_{(i;\frac{n}{2})j}}}\right)$$
$$-(\frac{n-2}{2})\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}\frac{\beta^s}{(\beta-1+e^{bx_{(i;\frac{n}{2})j}})^s-\beta^s}ln\left(\frac{\beta}{\beta-1+e^{bx_{(i;\frac{n}{2})j}}}\right) - \sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}log(\beta-1+e^{bx_{(i;\frac{n+2}{2})j}})$$
$$+\frac{(n-2)}{2}\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}log\left(\frac{\beta}{\beta-1+e^{bx_{(i;\frac{n+2}{2})j}}}\right) - (\frac{n}{2})\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}\frac{\beta^s}{(\beta-1+e^{bx_{(i;\frac{n+2}{2})j}})^s-\beta^s}ln\left(\frac{\beta}{\beta-1+e^{bx_{(i;\frac{n+2}{2})j}}}\right), \quad (23)$$

and

$$\frac{\partial logL_{MRSS_{even}}}{\partial \beta} = \frac{mns}{\beta} - \frac{(n-2)}{2}\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}\frac{s\beta^{s-1}\left(-1+e^{bx_{(i;\frac{n}{2})j}}\right)}{(\beta-1+e^{bx_{(i;\frac{n}{2})j}})^{s+1}-\beta^s\left(\beta-1+e^{bx_{(i;\frac{n}{2})j}}\right)} + \frac{sn}{2}\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}\frac{\left(-1+e^{bx_{(i;\frac{n}{2})j}}\right)}{\beta\left(\beta-1+e^{bx_{(i;\frac{n}{2})j}}\right)}$$
$$-(s+1)\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}\frac{1}{\left(\beta-1+e^{bx_{(i;\frac{n}{2})j}}\right)} - \frac{n}{2}\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}\frac{s\beta^{s-1}\left(-1+e^{bx_{(i;\frac{n+2}{2})j}}\right)}{(\beta-1+e^{bx_{(i;\frac{n+2}{2})j}})^{s+1}-\beta^s\left(\beta-1+e^{bx_{(i;\frac{n+2}{2})j}}\right)}$$
$$+\frac{s(n-2)}{2}\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}\frac{\left(-1+e^{bx_{(i;\frac{n+2}{2})j}}\right)}{\beta\left(\beta-1+e^{bx_{(i;\frac{n+2}{2})j}}\right)} - (s+1)\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}\frac{1}{\left(\beta-1+e^{bx_{(i;\frac{n+2}{2})j}}\right)}. \quad (24)$$

Setting the derivatives to zero, the three non-linear equations can be solved by numerical methods. The LF and $logL$ functions of MRSS technique for n odd are given by

$$L_{MRSS_{odd}}(x_{(i;i)j};b,s,\beta) = \prod_{j=1}^{m}\prod_{i=1}^{n}\left[\frac{n!}{[(\frac{n-1}{2})!]^2}\frac{bse^{bx_{(i;\frac{n+1}{2})j}}\beta^s}{(\beta-1+e^{bx_{(i;\frac{n+1}{2})j}})^{s+1}}\left[1-\frac{\beta^s}{(\beta-1+e^{bx_{(i;\frac{n+1}{2})j}})^s}\right]^{\frac{n-1}{2}}\left[\frac{\beta}{(\beta-1+e^{bx_{(i;\frac{n+1}{2})j}})}\right]^{\frac{s(n-1)}{2}}\right], \quad (25)$$

and

$$logL_{MRSS_{odd}} = log\left[\prod_{j=1}^{m}\prod_{i=1}^{n}\frac{n!}{[(\frac{n-1}{2})!]^2}\right] + mnlog(b) + mnlog(s) + mnslog(\beta) + b\sum_{j=1}^{m}\sum_{i=1}^{n}x_{(i;\frac{n+1}{2})j}$$
$$-(s+1)\sum_{j=1}^{m}\sum_{i=1}^{n}log(\beta-1+e^{bx_{(i;\frac{n+1}{2})j}}) + \frac{s(n-1)}{2}\sum_{j=1}^{m}\sum_{i=1}^{n}log\left(\frac{\beta}{\beta-1+e^{bx_{(i;\frac{n+1}{2})j}}}\right)$$
$$+(\frac{n-1}{2})\sum_{j=1}^{m}\sum_{i=1}^{n}log\left[1-\frac{\beta^s}{(\beta-1+e^{bx_{(i;\frac{n+1}{2})j}})^s}\right]. \tag{26}$$

First partial derivatives of the $logL_{MRSS_{odd}}$ with respect to b, s and β, respectively, are

$$\frac{\partial logL_{MRSS_{odd}}}{\partial b} = \frac{mn}{b} + \sum_{j=1}^{m}\sum_{i=1}^{n}x_{(i;\frac{n+1}{2})j} - (s+1)\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{x_{(i;\frac{n+1}{2})j}e^{bx_{(i;\frac{n+1}{2})j}}}{\beta-1+e^{bx_{(i;\frac{n+1}{2})j}}} - \frac{s(n-1)}{2}\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{x_{(i;\frac{n+1}{2})j}e^{bx_{(i;\frac{n+1}{2})j}}}{\beta-1+e^{bx_{(i;\frac{n+1}{2})j}}}$$
$$+(\frac{n-1}{2})\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{s\beta^s x_{(i;\frac{n+1}{2})j}e^{bx_{(i;\frac{n+1}{2})j}}}{(\beta-1+e^{bx_{(i;\frac{n+1}{2})j}})^{s+1} - \beta^s\left(\beta-1+e^{bx_{(i;\frac{n+1}{2})j}}\right)}, \tag{27}$$

$$\frac{\partial logL_{MRSS_{odd}}}{\partial s} = \frac{mn}{s} + mnlog(\beta) - \sum_{j=1}^{m}\sum_{i=1}^{n}log(\beta-1+e^{bx_{(i;\frac{n+1}{2})j}}) + (\frac{n-1}{2})\sum_{j=1}^{m}\sum_{i=1}^{n}log\left(\frac{\beta}{\beta-1+e^{bx_{(i;\frac{n+1}{2})j}}}\right)$$
$$-(\frac{n-1}{2})\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{\beta^s}{(\beta-1+e^{bx_{(i;\frac{n+1}{2})j}})^s-\beta^s}\,ln\left(\frac{\beta}{\beta-1+e^{bx_{(i;\frac{n+1}{2})j}}}\right), \tag{28}$$

and

$$\frac{\partial logL_{MRSS_{odd}}}{\partial \beta} = \frac{mns}{\beta} - \frac{(n-1)}{2}\sum_{j=1}^{m}\sum_{i=1}^{n}(i-1)\frac{s\beta^{s-1}\left(-1+e^{bx_{(i;\frac{n+1}{2})j}}\right)}{(\beta-1+e^{bx_{(i;\frac{n+1}{2})j}})^{s+1}-\beta^s\left(\beta-1+e^{bx_{(i;\frac{n+1}{2})j}}\right)}$$
$$+\frac{s(n-1)}{2}\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{\left(-1+e^{bx_{(i;\frac{n+1}{2})j}}\right)}{\beta\left(\beta-1+e^{bx_{(i;\frac{n+1}{2})j}}\right)} - (s+1)\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{1}{\left(\beta-1+e^{bx_{(i;\frac{n+1}{2})j}}\right)}. \tag{29}$$

Equating the derivatives to zero, the three non-linear equations can be solved by numerical methods.

3.4 Estimation Based on PRSS

By substituting into the Eq.(7) when n is an even and into the Eq.(8) when n is an odd. The LF and $logL$ functions of PRSS scheme for n even are followed by

$$L_{PRSS_{even}}(x_{(i;i)j};b,s,\beta) = \prod_{j=1}^{m}\prod_{i=1}^{\frac{n}{2}}[\frac{n!}{(p(n+1)-1)!(n-p(n+1))!}\frac{bse^{bx_{(i;p(n+1))j}}\beta^s}{(\beta-1+e^{bx_{(i;p(n+1))j}})^{s+1}}\left[1-\frac{\beta^s}{(\beta-1+e^{bx_{(i;p(n+1))j}})^s}\right]^{p(n+1)-1}$$
$$\times\left[\frac{\beta}{(\beta-1+e^{bx_{(i;p(n+1))j}})}\right]^{s(n-p(n+1))}] \times \prod_{j=1}^{m}\prod_{i=\frac{n}{2}+1}^{n}[\frac{n!}{(q(n+1)-1)!(n-q(n+1))!}\frac{bse^{bx_{(i;q(n+1))j}}\beta^s}{(\beta-1+e^{bx_{(i;q(n+1))j}})^{s+1}}$$
$$\times\left[1-\frac{\beta^s}{(\beta-1+e^{bx_{(i;q(n+1))j}})^s}\right]^{q(n+1)-1}\left[\frac{\beta}{(\beta-1+e^{bx_{(i;q(n+1))j}})}\right]^{s(n-q(n+1))}], \tag{30}$$

and

$$logL_{PRSS_{even}} = log\left[\prod_{j=1}^{m}\prod_{i=1}^{\frac{n}{2}}\frac{n!}{(p(n+1)-1)!(n-p(n+1))!}\right] + mnlog(b) + mnlog(s) + mnslog(\beta) + b\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}x_{(i;p(n+1))j}$$
$$-(s+1)\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}log(\beta-1+e^{bx_{(i;p(n+1))j}}) + s(n-p(n+1))\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}log\left(\frac{\beta}{\beta-1+e^{bx_{(i;p(n+1))j}}}\right) - (s+1)\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}log(\beta-1+e^{bx_{(i;q(n+1))j}})$$
$$+(p(n+1)-1)\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}log\left[1-\frac{\beta^s}{(\beta-1+e^{bx_{(i;p(n+1))j}})^s}\right] + log\left[\prod_{j=1}^{m}\prod_{i=\frac{n}{2}+1}^{n}\frac{n!}{(q(n+1)-1)!(n-q(n+1))!}\right] + b\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}x_{(i;q(n+1))j}$$
$$+s(n-q(n+1))\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}log\left(\frac{\beta}{\beta-1+e^{bx_{(i;q(n+1))j}}}\right) + (q(n+1)-1)\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}log\left[1-\frac{\beta^s}{(\beta-1+e^{bx_{(i;q(n+1))j}})^s}\right]. \quad (31)$$

First partial derivatives of the $logL_{PRSS_{even}}$ for b, s and β, respectively are as follows:

$$\frac{\partial logL_{PRSS_{even}}}{\partial b} = \frac{mn}{b} + \sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}x_{(i;p(n+1))j} - (s+1)\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}\frac{x_{(i;p(n+1))j}e^{bx_{(i;p(n+1))j}}}{\beta-1+e^{bx_{(i;p(n+1))j}}} - s(n-p(n+1))\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}\frac{x_{(i;p(n+1))j}e^{bx_{(i;p(n+1))j}}}{\beta-1+e^{bx_{(i;p(n+1))j}}}$$
$$+(p(n+1)-1)\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}\frac{s\beta^s x_{(i;p(n+1))j}e^{bx_{(i;p(n+1))j}}}{(\beta-1+e^{bx_{(i;p(n+1))j}})^{s+1}-\beta^s\left(\beta-1+e^{bx_{(i;p(n+1))j}}\right)} + \sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}x_{(i;q(n+1))j} - (s+1)\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}\frac{x_{(i;q(n+1))j}e^{bx_{(i;q(n+1))j}}}{\beta-1+e^{bx_{(i;q(n+1))j}}}$$
$$-s(n-q(n+1))\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}\frac{x_{(i;q(n+1))j}e^{bx_{(i;q(n+1))j}}}{\beta-1+e^{bx_{(i;q(n+1))j}}} + (q(n+1)-1)\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}\frac{s\beta^s x_{(i;q(n+1))j}e^{bx_{(i;q(n+1))j}}}{(\beta-1+e^{bx_{(i;q(n+1))j}})^{s+1}-\beta^s\left(\beta-1+e^{bx_{(i;q(n+1))j}}\right)}, \quad (32)$$

$$\frac{\partial logL_{PRSS_{even}}}{\partial s} = \frac{mn}{s} + mnlog(\beta) - \sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}log(\beta-1+e^{bx_{(i;p(n+1))j}}) + (n-p(n+1))\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}log\left(\frac{\beta}{\beta-1+e^{bx_{(i;p(n+1))j}}}\right)$$
$$-(p(n+1)-1)\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}\frac{\beta^s}{(\beta-1+e^{bx_{(i;p(n+1))j}})^s-\beta^s}ln\left(\frac{\beta}{\beta-1+e^{bx_{(i;p(n+1))j}}}\right) - \sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}log(\beta-1+e^{bx_{(i;q(n+1))j}})$$
$$-(q(n+1)-1)\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}\frac{\beta^s}{(\beta-1+e^{bx_{(i;q(n+1))j}})^s-\beta^s}ln\left(\frac{\beta}{\beta-1+e^{bx_{(i;q(n+1))j}}}\right)$$
$$+(n-q(n+1))\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}log\left(\frac{\beta}{\beta-1+e^{bx_{(i;q(n+1))j}}}\right), \quad (33)$$

and

$$\frac{\partial logL_{PRSS_{even}}}{\partial \beta} = \frac{mns}{\beta} - s(p(n+1)-1)\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}\frac{\beta^{s-1}\left(-1+e^{bx_{(i;p(n+1))j}}\right)}{(\beta-1+e^{bx_{(i;p(n+1))j}})^{s+1}-\beta^s\left(\beta-1+e^{bx_{(i;p(n+1))j}}\right)}$$
$$+s(n-p(n+1))\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}\frac{\left(-1+e^{bx_{(i;p(n+1))j}}\right)}{\beta\left(\beta-1+e^{bx_{(i;p(n+1))j}}\right)} - (q(n+1)-1)\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}\frac{s\beta^{s-1}\left(-1+e^{bx_{(i;q(n+1))j}}\right)}{(\beta-1+e^{bx_{(i;q(n+1))j}})^{s+1}-\beta^s\left(\beta-1+e^{bx_{(i;q(n+1))j}}\right)}$$
$$+s(n-q(n+1))\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}\frac{\left(-1+e^{bx_{(i;q(n+1))j}}\right)}{\beta\left(\beta-1+e^{bx_{(i;q(n+1))j}}\right)} - (s+1)\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n}\frac{1}{\left(\beta-1+e^{bx_{(i;q(n+1))j}}\right)}$$
$$-(s+1)\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}\frac{1}{\left(\beta-1+e^{bx_{(i;p(n+1))j}}\right)}. \quad (34)$$

Setting the derivatives to zero, the three non-linear equations can be solved by numerical methods. The LF and $logL$ functions of PRSS technique for n odd are given by

$$L_{PRSS_{odd}}(x_{(i;i)j};b,s,\beta)=\prod_{j=1}^{m}\prod_{i}^{\frac{n}{2}}[\frac{n!}{(p(n+1)-1)!(n-p(n+1))!}\frac{bse^{bx_{(i;p(n+1))j}}\beta^s}{(\beta-1+e^{bx_{(i;p(n+1))j}})^{s+1}}\left[1-\frac{\beta^s}{(\beta-1+e^{bx_{(i;p(n+1))j}})^s}\right]^{p(n+1)-1}$$
$$\times\left[\frac{\beta}{(\beta-1+e^{bx_{(i;p(n+1))j}})}\right]^{s(n-p(n+1))}]\times\prod_{j=1}^{m}\prod_{i=\frac{n}{2}+1}^{n-1}[\frac{n!}{(q(n+1)-1)!(n-q(n+1))!}\frac{bse^{bx_{(i;q(n+1))j}}\beta^s}{(\beta-1+e^{bx_{(i;q(n+1))j}})^{s+1}}$$
$$\times\left[1-\frac{\beta^s}{(\beta-1+e^{bx_{(i;q(n+1))j}})^s}\right]^{q(n+1)-1}\left[\frac{\beta}{(\beta-1+e^{bx_{(i;q(n+1))j}})}\right]^{s(n-q(n+1))}]$$
$$\times\left[\frac{n!}{[(\frac{n-1}{2})!]^2}\frac{bse^{bx_{(i;\frac{n+1}{2})j}}\beta^s}{(\beta-1+e^{bx_{(i;\frac{n+1}{2})j}})^{s+1}}\left[1-\frac{\beta^s}{(\beta-1+e^{bx_{(i;\frac{n+1}{2})j}})^s}\right]^{\frac{n-1}{2}}\left[\frac{\beta}{(\beta-1+e^{bx_{(i;\frac{n+1}{2})j}})}\right]^{\frac{s(n-1)}{2}}\right], \tag{35}$$

and

$$logL_{PRSS_{odd}}=log\left[\prod_{j=1}^{m}\prod_{i=1}^{\frac{n}{2}}\frac{n!}{(p(n+1)-1)!(n-p(n+1))!}\right]+mn\,log(b)+mn\,log(s)+mns\,log(\beta)+b\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}x_{(i;p(n+1))j}$$
$$-(s+1)\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}log(\beta-1+e^{bx_{(i;p(n+1))j}})+s(n-p(n+1))\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}log\left(\frac{\beta}{\beta-1+e^{bx_{(i;p(n+1))j}}}\right)+b\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n-1}x_{(i;q(n+1))j}$$
$$+log\left[\prod_{j=1}^{m}\prod_{i=\frac{n}{2}+1}^{n-1}\frac{n!}{(q(n+1)-1)!(n-q(n+1))!}\right]-(s+1)\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n-1}log(\beta-1+e^{bx_{(i;q(n+1))j}})$$
$$+(q(n+1)-1)\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n-1}log\left[1-\frac{\beta^s}{(\beta-1+e^{bx_{(i;q(n+1))j}})^s}\right]+s(n-q(n+1))\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n-1}log\left(\frac{\beta}{\beta-1+e^{bx_{(i;q(n+1))j}}}\right)+log\left[\frac{n!}{[(\frac{n-1}{2})!]^2}\right]$$
$$-(s+1)\,log(\beta-1+e^{bx_{(i;\frac{n+1}{2})j}})+\frac{(n-1)}{2}log\left(\frac{\beta}{\beta-1+e^{bx_{(i;\frac{n+1}{2})j}}}\right)+(p(n+1)-1)\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}log\left[1-\frac{\beta^s}{(\beta-1+e^{bx_{(i;p(n+1))j}})^s}\right]$$
$$+\frac{s(n-1)}{2}log\left[1-\frac{\beta^s}{(\beta-1+e^{bx_{(i;\frac{n+1}{2})j}})^s}\right]. \tag{36}$$

First partial derivatives of the $logL_{PRSS_{odd}}$ with respect to b, s and β, respectively, are

$$\frac{\partial logL_{PRSS_{odd}}}{\partial b}=\frac{mn}{b}+\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}x_{(i;p(n+1))j}-(s+1)\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}\frac{x_{(i;p(n+1))j}e^{bx_{(i;p(n+1))j}}}{\beta-1+e^{bx_{(i;p(n+1))j}}}-s(n-p(n+1))\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}\frac{x_{(i;p(n+1))j}e^{bx_{(i;p(n+1))j}}}{\beta-1+e^{bx_{(i;p(n+1))j}}}$$
$$+(p(n+1)-1)\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}}\frac{s\beta^s x_{(i;p(n+1))j}e^{bx_{(i;p(n+1))j}}}{(\beta-1+e^{bx_{(i;p(n+1))j}})^{s+1}-\beta^s\left(\beta-1+e^{bx_{(i;p(n+1))j}}\right)}+\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n-1}x_{(i;q(n+1))j}-\frac{s(n-1)}{2}\frac{x_{(i;\frac{n+1}{2})j}e^{bx_{(i;\frac{n+1}{2})j}}}{\beta-1+e^{bx_{(i;\frac{n+1}{2})j}}}$$
$$-(s+1)\left(\frac{x_{(i;\frac{n+1}{2})j}e^{bx_{(i;\frac{n+1}{2})j}}}{\beta-1+e^{bx_{(i;\frac{n+1}{2})j}}}\right)-(s+1)\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n-1}\frac{x_{(i;q(n+1))j}e^{bx_{(i;q(n+1))j}}}{\beta-1+e^{bx_{(i;q(n+1))j}}}-s(n-q(n+1))\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n-1}\frac{x_{(i;q(n+1))j}e^{bx_{(i;q(n+1))j}}}{\beta-1+e^{bx_{(i;q(n+1))j}}}$$
$$+(q(n+1)-1)\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n-1}\frac{s\beta^s x_{(i;q(n+1))j}e^{bx_{(i;q(n+1))j}}}{(\beta-1+e^{bx_{(i;q(n+1))j}})^{s+1}-\beta^s\left(\beta-1+e^{bx_{(i;q(n+1))j}}\right)}-s(n-q(n+1))\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n-1}\frac{x_{(i;q(n+1))j}e^{bx_{(i;q(n+1))j}}}{\beta-1+e^{bx_{(i;q(n+1))j}}}$$
$$+(q(n+1)-1)\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n-1}\frac{s\beta^s x_{(i;q(n+1))j}e^{bx_{(i;q(n+1))j}}}{(\beta-1+e^{bx_{(i;q(n+1))j}})^{s+1}-\beta^s\left(\beta-1+e^{bx_{(i;q(n+1))j}}\right)}$$
$$+\frac{(n-1)}{2}\frac{s\beta^s x_{(i;\frac{n+1}{2})j}e^{bx_{(i;\frac{n+1}{2})j}}}{(\beta-1+e^{bx_{(i;\frac{n+1}{2})j}})^{s+1}-\beta^s\left(\beta-1+e^{bx_{(i;\frac{n+1}{2})j}}\right)}, \tag{37}$$

$$\frac{\partial logL_{PRSS_{odd}}}{\partial s} = \frac{mn}{s} + mn\,log(\beta) - \sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}} log(\beta - 1 + e^{bx_{(i;p(n+1))j}}) + (n - p(n+1))\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}} log\left(\frac{\beta}{\beta - 1 + e^{bx_{(i;p(n+1))j}}}\right)$$
$$-(p(n+1)-1)\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}} \frac{\beta^s}{(\beta - 1 + e^{bx_{(i;p(n+1))j}})^s - \beta^s}\, ln\left(\frac{\beta}{\beta - 1 + e^{bx_{(i;p(n+1))j}}}\right) - \sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n-1} log(\beta - 1 + e^{bx_{(i;q(n+1))j}})$$
$$+(n - q(n+1))\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n-1} log\left(\frac{\beta}{\beta - 1 + e^{bx_{(i;q(n+1))j}}}\right) - (q(n+1)-1)\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n-1} \frac{\beta^s}{(\beta - 1 + e^{bx_{(i;q(n+1))j}})^s - \beta^s}\, ln\left(\frac{\beta}{\beta - 1 + e^{bx_{(i;q(n+1))j}}}\right)$$
$$+\frac{(n-1)}{2} log\left(\frac{\beta}{\beta - 1 + e^{bx_{(i;\frac{n+1}{2})j}}}\right) - \frac{(n-1)}{2}\left[\frac{\beta^s}{(\beta - 1 + e^{bx_{(i;\frac{n+1}{2})j}})^s - \beta^s}\, ln\left(\frac{\beta}{\beta - 1 + e^{bx_{(i;\frac{n+1}{2})j}}}\right)\right] - log(\beta - 1 + e^{bx_{(i;\frac{n+1}{2})j}}), \quad (38)$$

and

$$\frac{\partial logL_{PRSS_{odd}}}{\partial \beta} = \frac{mns}{\beta} - s(p(n+1)-1)\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}} \frac{\beta^{s-1}\left(-1 + e^{bx_{(i;p(n+1))j}}\right)}{(\beta - 1 + e^{bx_{(i;p(n+1))j}})^{s+1} - \beta^s\left(\beta - 1 + e^{bx_{(i;p(n+1))j}}\right)}$$
$$+s(n - p(n+1))\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}} \frac{\left(-1 + e^{bx_{(i;p(n+1))j}}\right)}{\beta\left(\beta - 1 + e^{bx_{(i;p(n+1))j}}\right)} - (q(n+1)-1)\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n-1} \frac{s\beta^{s-1}\left(-1 + e^{bx_{(i;q(n+1))j}}\right)}{(\beta - 1 + e^{bx_{(i;q(n+1))j}})^{s+1} - \beta^s\left(\beta - 1 + e^{bx_{(i;q(n+1))j}}\right)}$$
$$-\frac{(s+1)}{\beta - 1 + e^{bx_{(i;\frac{n+1}{2})j}}} + \frac{s(n-1)}{2}\left[\frac{e^{bx_{(i;\frac{n+1}{2})j}} - 1}{\beta(\beta - 1 + e^{bx_{(i;\frac{n+1}{2})j}})^s}\right] + s(n - q(n+1))\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n-1} \frac{\left(-1 + e^{bx_{(i;q(n+1))j}}\right)}{\beta\left(\beta - 1 + e^{bx_{(i;q(n+1))j}}\right)}$$
$$-(s+1)\sum_{j=1}^{m}\sum_{i=\frac{n}{2}+1}^{n-1} \frac{1}{\left(\beta - 1 + e^{bx_{(i;q(n+1))j}}\right)} - \frac{(n-1)}{2}\left[\frac{s\beta^{s-1}\left(-1 + e^{bx_{(i;\frac{n+1}{2})j}}\right)}{(\beta - 1 + e^{bx_{(i;\frac{n+1}{2})j}})^{s+1} - \beta^s\left(\beta - 1 + e^{bx_{(i;\frac{n+1}{2})j}}\right)}\right]$$
$$-(s+1)\sum_{j=1}^{m}\sum_{i=1}^{\frac{n}{2}} \frac{1}{\left(\beta - 1 + e^{bx_{(i;p(n+1))j}}\right)}. \quad (39)$$

Equating the derivatives to zero, the three non-linear equations can be solved by numerical methods.

3.5 Estimation Based on NRSS

By substituting in Eq. (9). The LF and $logL$ functions the NRSS method are followed by

$$L_{NRSS}(x_{(i)j};\theta) = \prod_{j=1}^{m}\left[\prod_{i=1}^{n+1}\frac{n^2!}{(k(i) - k(i-1) - 1)!}\prod_{i=1}^{n}\frac{bse^{bx_{k(i)j}}\beta^s}{(\beta - 1 + e^{bx_{k(i)j}})^{s+1}}\right]$$
$$\times\prod_{i=1}^{n+1}\left[\frac{\beta^s}{(\beta - 1 + e^{bx_{k(i-1)j}})^s} - \frac{\beta^s}{(\beta - 1 + e^{bx_{k(i)j}})^s}\right]^{k(i)-k(i-1)-1}, \quad (40)$$

$$logL_{NRSS}(x_{(i)j};\theta) = log\left[\prod_{j=1}^{m}\prod_{i=1}^{n+1}\frac{n^2!}{(k(i) - k(i-1) - 1)!}\right] + mn\,log(b) + mn\,log(s) + mns\,log(\beta) + b\sum_{j=1}^{m}\sum_{i=1}^{n} x_{k(i)j}$$
$$-(s+1)\sum_{j=1}^{m}\sum_{i=1}^{n} log(\beta - 1 + e^{bx_{k(i)j}}) + \sum_{j=1}^{m}\sum_{i=1}^{n+1}(k(i) - k(i-1) - 1)\,log\left[\frac{\beta^s}{(\beta - 1 + e^{bx_{k(i-1)j}})^s} - \frac{\beta^s}{(\beta - 1 + e^{bx_{k(i)j}})^s}\right]. \quad (41)$$

First partial derivatives of $logL_{NRSS}$ with respect to b, s and β, respectively, are

$$\frac{\partial logL_{NRSS}}{\partial b} = \frac{mn}{b} + \sum_{j=1}^{m}\sum_{i=1}^{n} x_{k(i)j} - (s+1)\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{x_{k(i)j}e^{bx_{k(i)j}}}{\beta - 1 + e^{bx_{k(i)j}}}$$
$$+s\sum_{j=1}^{m}\sum_{i=1}^{n+1}(k(i) - k(i-1) - 1)\left[\frac{\frac{x_{k(i)j}e^{bx_{k(i)j}}}{[\beta - 1 + e^{bx_{k(i)j}}]^{s+1}} - \frac{x_{k(i-1)j}e^{bx_{k(i-1)j}}}{[\beta - 1 + e^{bx_{k(i-1)j}}]^{s+1}}}{\left[\frac{1}{\beta - 1 + e^{bx_{k(i-1)j}}}\right]^s - \left[\frac{1}{\beta - 1 + e^{bx_{k(i)j}}}\right]^s}\right], \quad (42)$$

$$\frac{\partial logL_{NRSS}}{\partial s} = \frac{mn}{s} + mn\,log(\beta) - \sum_{j=1}^{m}\sum_{i=1}^{n} log(\beta - 1 + e^{bx_{k(i)j}})$$
$$+ \sum_{j=1}^{m}\sum_{i=1}^{n+1}(k(i) - k(i-1) - 1)\left[\frac{\left[\frac{1}{\beta-1+e^{bx_{k(i-1)j}}}\right]^{s} ln\left[\frac{\beta}{\beta-1+e^{bx_{k(i-1)j}}}\right] - \left[\frac{1}{\beta-1+e^{bx_{k(i)j}}}\right]^{s} ln\left[\frac{\beta}{\beta-1+e^{bx_{k(i)j}}}\right]}{\left[\frac{1}{\beta-1+e^{bx_{k(i-1)j}}}\right]^{s} - \left[\frac{1}{\beta-1+e^{bx_{k(i)j}}}\right]^{s}}\right], \tag{43}$$

and

$$\frac{\partial logL_{NRSS}}{\partial \beta} = \frac{mns}{\beta} - (s+1)\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{1}{\left(\beta - 1 + e^{bx_{(i;i)j}}\right)} + \frac{s}{\beta}\sum_{j=1}^{m}\sum_{i=1}^{n+1}(k(i) - k(i-1) - 1)\left[\frac{\frac{-1+e^{bx_{k(i-1)j}}}{[\beta-1+e^{bx_{k(i-1)j}}]^{s+1}} - \frac{e^{bx_{k(i)j}}-1}{[\beta-1+e^{bx_{k(i)j}}]^{s+1}}}{\left[\frac{1}{\beta-1+e^{bx_{k(i-1)j}}}\right]^{s} - \left[\frac{1}{\beta-1+e^{bx_{k(i)j}}}\right]^{s}}\right]. \tag{44}$$

Setting the derivatives to zero, the three non-linear equations can be solved by numerical methods.

4 Simulation Study

In this section, a simulation study is performed to compare the ML estimators based on the shape and scale parameters of the G/G distribution of different sampling schemes. The application of the G/G distribution depends on its ability to correspond to statistical inferences of the parameters of the distribution. Starting from 1000 repetitions, use RSS, MRSS, PRSS ($p = 0.2$), and NRSS (relative to SRS) to perform computer simulations to study the efficiency performance of the sample mean. The Monte Carlo simulation is made for the G/G distribution with a different sample sizes ($n = 15, 20$ *and* 25), for the period ($m = 1$ *and* 3), and different true values of the parameters $G/G(3,3,3)$ and $G/G(2,2,4)$. All simulations are performed using routines for statistical calculations developed by the authors in the R environment. Let $\hat{\theta}_h$ be the h^{th} sample estimator generated by the sampling design $h = 1,2,...,1000$ based on the specific RSS. Comparisons using RE were made using two standards, namely the bias and mean square error (MSE), of which are calculated as follows: [9]

$$MSE(\hat{\theta}) = \frac{1}{1000}\sum_{h=1}^{1000}(\hat{\theta}_h - \theta_h)^2 \text{ where } E(\hat{\theta} - \theta) \text{ is called bias,}$$

and

$$RE(\hat{\theta}_{SRS}, \hat{\theta}_{RSS_{methods}}) = \frac{MSE(\hat{\theta}_{SRS})}{MSE(\hat{\theta}_{RSS_{methods}})}. \tag{45}$$

if $RE(\hat{\theta}_{SRS}\ ,\ \hat{\theta}_{RSS_{methods}}) \geq 1$, then $\hat{\theta}_{RSS_{methods}}$ is better than $\hat{\theta}_{SRS}$. Simulation results were summarized in a list of tables, as shown in Table 1, Table 2, and Table 3 below. Also in the list of figures, as shown in Figure 1 and Figure 2 below. The results in these tables and these figures show the performance of different RSS designs for different parameters associated with SRS and can be noticed that:

Table 1: Estimators and Relative biases for different RSS designs under perfect ranking

Distribution	n			SRS	RSS m= 1	RSS m= 3	MRSS m= 1	MRSS m= 3	PRSS m= 1	PRSS m= 3	NRSS m= 1	NRSS m= 3
G/G(3,3,3)	15	$\hat{b}$	Estimator	4.2753	3.6522	3.6207	4.1372	3.7290	3.5566	3.4657	3.0525	3.0801
			bias	−1.2753	−0.6522	−0.6207	−1.1372	−0.7290	−0.5566	−0.4657	−0.0525	−0.0801
	20		Estimator	4.0190	3.6645	3.4549	3.7854	3.4028	3.6186	3.1799	2.9057	3.0471
			bias	−1.0190	−0.6645	−0.4549	−0.7854	−0.4028	−0.5280	−0.1799	0.0943	−0.0471
	25		Estimator	4.0234	3.5579	3.7016	3.8473	3.4876	3.3069	3.0767	2.9224	2.6420
			bias	−1.0234	−0.5579	−0.7016	−0.8473	−0.4876	−0.3069	−0.0767	0.0776	0.3580
	15	$\hat{s}$	Estimator	3.2610	2.2065	2.1813	2.3081	2.1900	2.6424	2.5753	3.7891	3.6373
			bias	−0.2610	0.7935	0.8187	0.6919	0.8100	0.3576	0.4247	−0.7891	−0.6373
	20		Estimator	3.5890	2.1885	2.1700	2.5725	2.5109	2.8928	2.8859	3.1609	3.3888
			bias	−0.5890	0.8115	0.8300	0.4275	0.4891	0.0948	0.1141	−0.1609	−0.3888
	25		Estimator	3.0024	2.1725	2.2018	2.2253	2.1693	2.6561	2.7543	3.3085	3.1868
			bias	−0.0024	0.8275	0.7982	0.7747	0.8307	0.3439	0.2457	−0.3085	−0.1868
	15	$\hat{\beta}$	Estimator	5.3744	2.8097	2.7058	3.8789	2.9427	3.5680	3.2478	3.2481	3.5357
			bias	−2.3744	0.1903	0.2942	−0.8789	0.0573	−0.5680	−0.2478	−0.2481	−0.5357
	20		Estimator	5.1863	2.8300	2.7122	3.8604	2.9717	4.1049	3.1673	3.2261	3.0783
			bias	−2.1863	0.1700	0.2878	−0.8604	0.0283	−0.9410	−0.1673	−0.2261	−0.0783
	25		Estimator	4.5573	2.7574	2.8879	3.1519	2.7283	3.0392	3.5633	3.2924	3.3028
			bias	−1.5573	0.2426	0.1121	−0.1519	0.2717	−0.0392	0.4367	−0.2924	−0.3028
G/G(2,2,4)	15	$\hat{b}$	Estimator	2.5547	2.0834	1.9046	2.3991	2.3527	2.2515	2.1381	1.9082	1.9293
			bias	−0.5547	−0.0834	0.0954	−0.3991	−0.3527	−0.2515	−0.1381	0.0918	0.0707
	20		Estimator	2.5135	1.9622	1.9233	2.2848	2.3105	2.1058	2.0657	1.9779	1.7127
			bias	−0.5135	0.0378	0.0767	−0.2848	−0.3105	−0.1058	−0.0657	0.0221	0.2873
	25		Estimator	2.5614	2.0593	2.0838	2.3135	2.1481	2.2365	2.0694	1.9858	1.8378
			bias	−0.5614	−0.0593	−0.0838	−0.3135	−0.1481	−0.2365	−0.0694	0.0142	0.1622
	15	$\hat{s}$	Estimator	2.6538	1.4653	1.4528	1.5387	1.4711	1.7486	1.7060	2.6611	2.4553
			bias	−0.6538	0.5347	0.5472	0.4613	0.5289	0.2514	0.2940	−0.6611	−0.4553
	20		Estimator	2.5860	1.4601	1.4931	1.7722	1.7305	1.9322	1.9099	2.1936	2.3029
			bias	−0.5860	0.5399	0.5069	0.2278	0.2695	0.0678	0.0901	−0.1936	−0.3029
	25		Estimator	2.0768	1.4506	1.9363	1.5054	1.4667	1.9382	1.7491	2.2028	2.5548
			bias	−0.0768	0.5494	0.0637	0.4946	0.5333	0.0618	0.2509	−0.2028	−0.5548
	15	$\hat{\beta}$	Estimator	6.2838	2.7294	2.1832	3.9510	3.6133	4.3227	3.7398	4.0752	4.7798
			bias	−2.2838	1.2706	1.8168	0.0490	0.3867	−0.3227	0.2602	−0.0752	−0.7798
	20		Estimator	5.5259	2.2844	2.2455	4.1775	4.1790	4.2367	4.0183	4.2811	4.5183
			bias	−1.5259	1.7156	1.7545	−0.1775	−0.1790	−0.2367	−0.0183	−0.2811	−0.5183
	25		Estimator	4.9918	2.5741	2.7777	3.5137	2.8331	4.1170	3.5479	4.3503	4.6643
			bias	−0.9918	1.4259	1.2223	0.4863	1.1669	−0.1170	0.4521	−0.3503	−0.6643

It is clear from Table 1 that:

- The SRS estimators for $(b, s\ and\ \beta)$ are far away from the true values of $b, s\ and\ \beta$.
- In almost all cases, the biases of RSS techniques are very small compared to the biases of SRS.
- The values of the estimators of $(b, s\ and\ \beta)$ are very close to the true values of $(b, s\ and\ \beta)$ in different RSS techniques.
- The biases of the sampling techniques increase as b decreases.
- The accuracy of the RSS estimators to the true values increase as m increases.
- The biases of the sampling techniques decrease as β increases.
- As m increases, the MRSS estimators become more accurate and close to the true values.
- As m increases, the PRSS estimators get increasingly accurate and nearer to the true values.
- The PRSS, and NRSS estimators for $(b, s\ and\ \beta)$ have a smaller values of bias than the other estimators.
- The biases of the MRSS estimators are small when compared tothe biases of the RSS estimators, except some cases.
- As n increases, the changes in NRSS estimators in methods of estimation in the research are little.

Table2 : Mean square errors of the different RSS schemes.

Distribution	n		SRS	RSS		MRSS		PRSS		NRSS	
				m = 1	m = 3	m = 1	m = 3	m = 1	m = 3	m = 1	m = 3
G/G(3,3,3)	15	$\hat{b}$	1.3217	0.2032	0.2520	0.2854	0.1709	0.0300	0.1244	0.0442	0.0617
	20		0.9119	0.2945	0.4002	0.0569	0.7254	0.0255	0.1279	0.0469	0.0440
	25		0.6794	0.2884	0.3137	0.2082	0.2125	0.1161	0.1438	0.0456	0.0941
	15	$\hat{s}$	1.5265	0.5880	0.9853	0.5808	0.8984	0.1465	0.4325	0.1271	0.0543
	20		1.7062	0.5923	0.6464	0.4574	0.1553	0.1746	0.1337	0.0655	0.0783
	25		2.0648	0.4748	0.8508	0.0964	0.6699	0.4495	0.0919	0.0394	0.3722
	15	$\hat{\beta}$	0.6584	0.3016	0.2240	0.2412	0.3858	0.2181	0.2001	0.1794	0.0445
	20		6.4715	0.1753	0.5007	0.9486	0.3512	0.1356	0.4526	0.0650	0.2905
	25		0.6316	0.1772	0.1362	0.2955	0.1028	0.0264	0.0563	0.0200	0.0275
G/G(2,2,4)	15	$\hat{b}$	2.2787	0.3409	0.1976	0.1899	0.1511	0.1629	0.0917	0.1034	0.0553
	20		1.0046	0.5166	0.6963	0.2499	0.0681	0.0442	0.1199	0.0333	0.0451
	25		0.9080	0.1388	0.7888	0.1195	0.0902	0.0546	0.0170	0.0177	0.0101
	15	$\hat{s}$	2.1453	0.3851	0.3485	0.3803	0.3723	0.3140	0.0683	0.2310	0.0637
	20		1.9125	0.4320	0.4296	0.4135	0.1274	0.0643	0.1029	0.0307	0.0372
	25		2.1640	0.2769	0.8964	0.4368	0.2430	0.1787	0.0752	0.0255	0.0546
	15	$\hat{\beta}$	1.1936	0.5139	0.4190	0.1531	0.2662	0.0748	0.0704	0.0943	0.0673
	20		1.2303	0.6588	0.8435	0.5357	0.6911	0.1570	0.5059	0.0590	0.2678
	25		1.7096	0.6736	1.4257	0.4326	0.6629	0.0259	0.0413	0.0194	0.0563

Table 3: Exact relative eff ciencies of the RSS-based estimators compared to SRS-based estimators for the different parameters.

Distribution	n		RSS		MRSS		PRSS		NRSS	
			m = 1	m = 3	m = 1	m = 3	m = 1	m = 3	m = 1	m = 3
G/G(3,3,3)	15	$\hat{b}$	6.5045	5.2457	4.6312	7.7338	43.9839	10.6254	29.8731	21.4190
	20		3.0965	2.2785	16.0218	1.2570	35.7869	7.1324	19.4351	20.7410
	25		2.3558	2.1658	3.2637	3.1964	5.8531	4.7237	14.9041	7.2202
	15	$\hat{s}$	2.5961	1.5493	2.6282	1.6992	10.4188	3.5295	12.0148	28.0882
	20		2.8808	2.6395	3.7305	10.9882	9.7733	12.7617	26.0353	21.8045
	25		4.3489	2.4269	21.4130	3.0824	4.5938	22.4659	52.3689	5.5472
	15	$\hat{\beta}$	2.1830	2.9387	2.7292	1.7064	3.0186	3.2895	3.6705	14.7893
	20		36.9118	12.9252	6.8222	18.4283	47.7381	14.2976	99.5871	22.2802
	25		3.5653	4.6386	2.1371	6.1431	23.9508	11.2155	31.6433	22.9436
G/G(2,2,4)	15	$\hat{b}$	6.6853	11.5316	11.9995	15.0761	13.9875	24.8520	22.0290	41.2173
	20		1.9446	1.4427	4.0207	14.7567	22.7348	8.3785	30.1969	22.2859
	25		6.5426	1.1510	7.5960	10.0697	16.6292	53.3853	51.2984	90.1401
	15	$\hat{s}$	5.5705	6.1555	5.6404	5.7627	6.8330	31.4276	9.2874	33.6817
	20		4.4268	4.4517	4.6249	15.0124	29.7498	18.5904	62.3926	51.3450
	25		7.8157	2.4141	4.9540	8.9059	12.1088	28.7597	84.8453	39.6386
	15	$\hat{\beta}$	2.3229	2.8490	7.7940	4.4840	15.9586	16.9566	12.6589	17.7491
	20		1.8676	1.4586	2.2967	1.7801	7.8367	2.4319	20.8663	4.5947
	25		2.5380	1.1991	3.9520	2.5791	65.8976	41.4156	88.0066	30.3508

Table 2 and Table 3 show that:

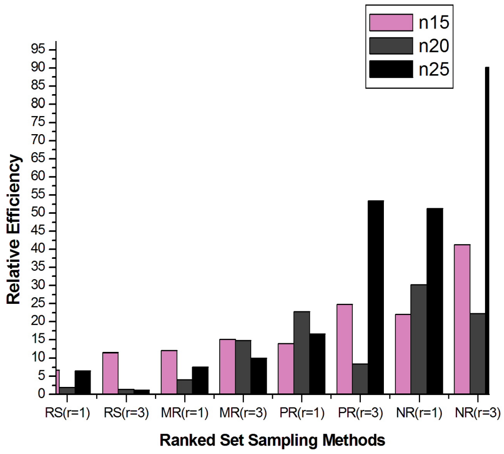
• In almost all cases, the biases of RSS methods are very small compared to the biases of SRS.

• The values of all the estimators of $(b, s \; and \; \beta)$ are very close to the true values of $b, s \; and \; \beta$ in RSS modifications.

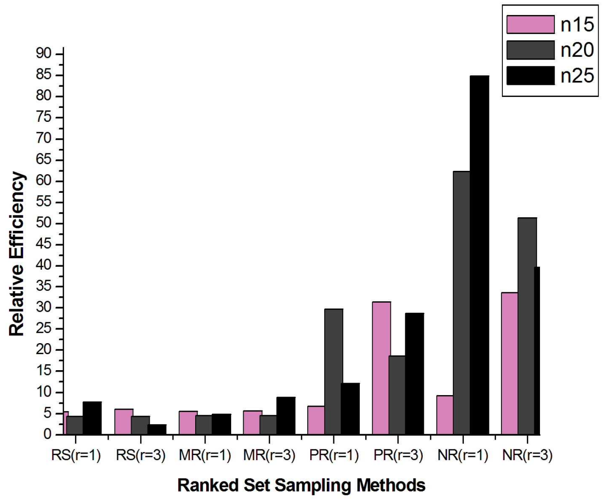
• In almost all cases, MSEs of all estimators based on RSS, MRSS, PRSS and NRSS for $(b, s \; and \; \beta)$ are less than MSEs of estimators based on SRS.

• In almost all cases, with the exception of some cases, the estimators based on NRSS of$(b, s \; and \; \beta)$ are more efficient than those based on RSS, MRSS, and PRSS.

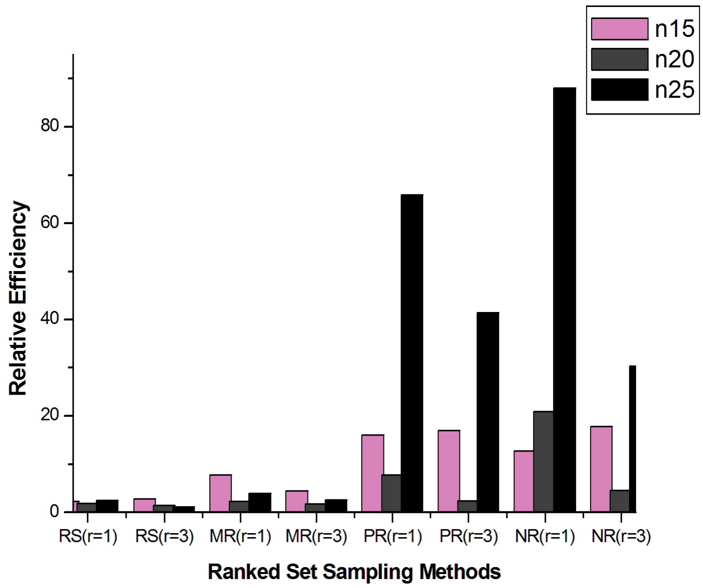
•The efficiency of PRSS-based of $(b, s \; and \; \beta)$ estimators is greater than that of RSS and MRSS-based estimators, except in one case.



a) For $\hat{b}$ estimator



b) For $\hat{s}$ estimator



c) For $\hat{beta}$ estimator

Fig. 1: shows the REs of the estimators for $G/G(2,2,4)$.

Fig. 1 shows that:

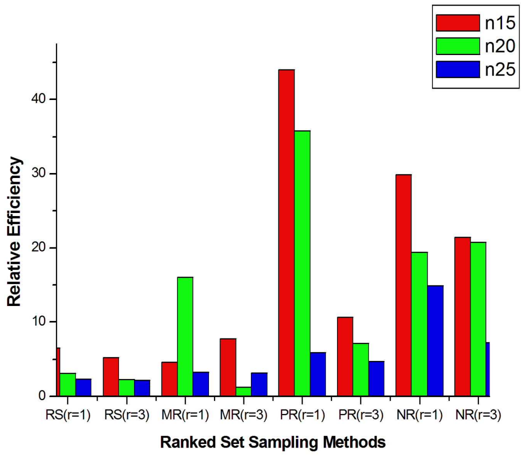
• Efficiencies of the estimators based on NRSS for $(b, s\ and\ \beta)$ have the highest efficiencies in all cases, except in a few.

• Except in a few cases, the efficiency of MRSS-based $(b, s\ and\ \beta)$ estimators is greater than that of RSS-based estimators.

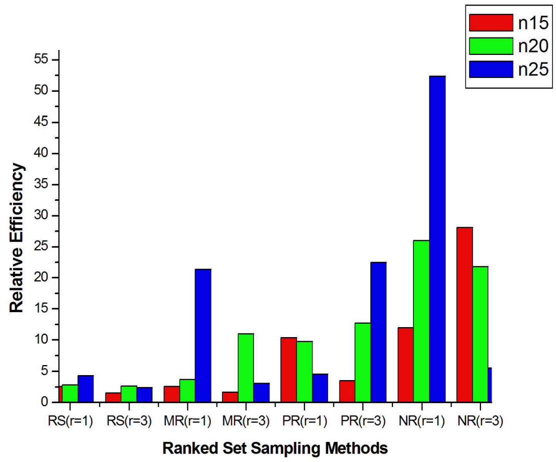
• Except in a few cases, the efficiencies of the estimators for $(b, s\ and\ \beta)$ based on RSS have the lowest efficiencies.

• The efficiency of estimators based on RSS and MRSS decreases as the value of $(b$, s and $\beta)$ increases, except in some cases.

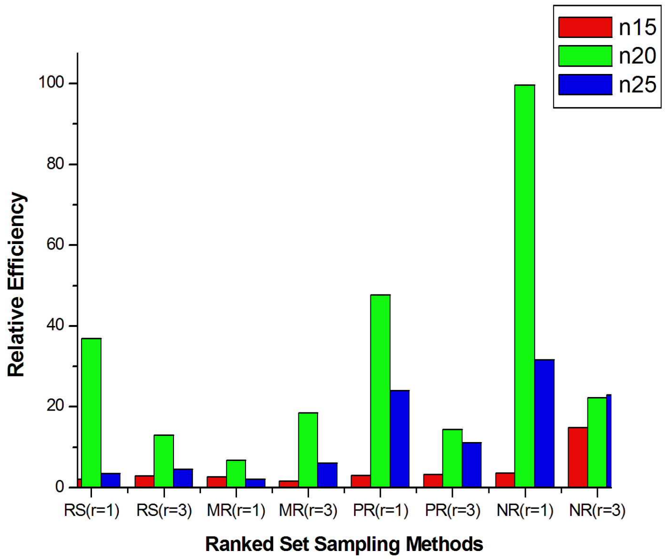
• Except in a few cases, the efficiency of PRSS-based $(b, s\ and\ \beta)$ estimators is greater than that of RSS and MRSS-based estimators.



a) For $\hat{b}$ estimator



b) For $\hat{s}$ estimator



c) For $\hat{beta}$ estimator

Fig. 2: Shows the REs of the estimators for $G/G(3,3,3)$.

Fig. 2 shows that:

- The efficiency of estimators based on RSS, MRSS, PRSS, and NRSS decreases as the value of $(b, s\ and\ \beta)$.
- RSS-based of $(b, s\ and\ \beta)$ estimators have the lowest efficiency in all cases, except in a little cases.

5 Summary and Conclusions

In this paper, the LF and the first partial derivatives of ML estimators are obtained for G/G distribution parameters based on SRS, RSS, MRSS, PRSS and NRSS techniques. To estimate the unknown parameters of the G/G distribution, first derivative equations of ML estimators are numerically solved. From the numerical comparisons between SRS and different RSS schemes, the estimators based on NRSS, PRSS, MRSS, and RSS techniques are more efficient than SRS estimators using different sample sizes. Also, NRSS is more efficient than other SRS-based RSS techniques for different sample sizes and cycle sizes. Furthermore, RSS has proven to be less effective than other methods with large MSEs.

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Conflict of interest

The authors declare that they have no conflict of interest.

References

[1] G.A. McIntyre. *A method of unbiased selective sampling, using ranked sets.* Australian Journal Agricultural Research, **3**, 385-390 (1952).
[2] K. Takahasi and K. Wakimoto. *On unbiased estimates of the population mean based on the sample stratified by means of ordering.* Annals of the Institute of Statistical Mathematics, **20**, 1-31 (1968).
[3] L.K. Halls and T.R. Dell. *Trials of ranked set sampling for forage yields.* Forest Science, **12**, 22-26 (1966).
[4] T.R. Dell and J.L. Clutter. *Ranked set sampling theory with an order statistics background.* Biometrics, **28**, 545-555 (1972).
[5] H.A. Muttlak. *Median ranked set sampling.* Journal of Applied Statistical Sciences, **6**, 245-255 (1997).
[6] H.A. Muttlak. *Modified ranked set sampling methods.* Pakistan Journal of Statistics, **19**, 315-323 (2003).
[7] A. Hassan. *Maximum likelihood and Bayes estimators of the unknown parameters for exponential distribution using ranked set sampling.* International Journal of Engineering Research and Applications ,**3**, 720-725 (2013).
[8] B. Nadjafi. *An Iterated Local Search Algorithm for Estimating The Parameters of the Gamma/Gompertz Distribution.* Hindawi Publishing Corporation, **1**, 1-7 (2014).
[9] M.H. Sameh and A. Qtait. *Estimation for the Parameters of the Exponentiated Exponential Distribution Using a Median Ranked Set Sampling.* Journal of Modern Applied Statistical Methods, **14**, 215-237 (2015).
[10] E. Zamanzade and A.I. Al-Omari. *New ranked set sampling for estimating the population mean and variance.* Hacettepe Journal of Mathematics and Statistics, **45**, 1891-1905 (2016).
[11] T. Missov. *Gamma/Gompertz life expectancy at birth.* Demographic Research, **28**, 259-270 (2013).
[12] A.C. Bemmaor and N. Glady. *Modeling Purchasing Behavior with Sudden "Death" A Flexible Customer Lifetime Model.* Managrment Science, **58**, 1012-1021 (2012).
[13] N. Balakrishnan and W.W. Chen. *CRC Handbook of Tables for Order Statistics from Inverse Gaussian Distributions with Application.*CRC Press, Boca Raton, (1997).
[14] M.A. Sabry and M. Shaaban. *Dependent Ranked Set Sampling Designs for Parametric Estimation with Applications.*, Annals of Data Science **7**, 357-371 (2020).
[15] N.M. Hassan, E.A. Rady and N.I. Rashwan. *Median Double Ranked Set Sampling Method.* Journal of Advances in Mathematics, **14**, 7503-7511 (2018).
[16] N.I. Rashwan. *Ranked set sampling methods :Comparison study. presented at the 21th Int. Conf. Statistics and Modeling in Humanities and Social Sciences.* Faculty of Economics and Political Science, Cairo University, (2009).