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Bayesian and Non-Bayesian Estimation of Extended Exponential Distribution under Type-I Progressive Hybrid Censoring

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Abstract: The estimating problems of the model parameters, reliability and hazard functions of extended exponential distribution when sample is available from Type-I progressive hybrid censoring scheme will be considered. The maximum likelihood estimation has been obtained for any function of the model parameters. Based on the normality property of the classical estimators, approximate confidence intervals for the unknown parameters and any function of them are constructed. Further, to construct the asymptotic confidence interval of the reliability and hazard rate function. Using independent gamma priors, the Bayes estimators of the unknown parameters are derived based on both the symmetric squared error and asymmetric LINEX loss functions. Since the Bayes estimators are obtained in a complex form therefore, Markov Chain Monte Carlo using Metropolis-Hastings algorithm has been used to carry out the Bayes estimates and also to construct the associate highest posterior density credible intervals. To evaluate the performance of the proposed methods, a Monte Carlo simulation study is carried out. Finally, we consider medical data to illustrate the applicability of the methods covered in the paper.

Keywords: Extended exponential distribution, Reliability and hazard rate functions, Bayesian and non-Bayesian estimation, Markov chain Monte Carlo, Type-I progressive hybrid censoring

1 Introduction

A new generalization of the exponential distribution as an alternative to gamma, Weibull and generalized-exponential lifetime models has been introduced by [1]. The extension of the exponential distribution was named NHD by [2] as an abbreviation for the name authors Nadarajah and Haghighi. Also, many properties of extended exponential distribution are discussed by [1]. Suppose that the lifetime X of a testing unit follows two-parameter extended exponential distribution (α,λ). The probability density function $f(\cdot)$, cumulative distribution function $F(\cdot)$, reliability function $S(\cdot)$ and hazard rate function $H(\cdot)$, for given mission time t, are given respectively by

$$f(x;\alpha,\lambda)=\alpha\lambda(1+\lambda x)^{\alpha-1}\exp\left[1-(1+\lambda x)^{\alpha}\right]; \qquad x>0,\ \alpha,\lambda>0, \tag{1}$$

$$F(x;\alpha,\lambda)=1-\exp\left[1-(1+\lambda x)^{\alpha}\right]; \qquad x>0,\ \alpha,\lambda>0, \tag{2}$$

$$S(t;\alpha,\lambda)=\exp\left[1-(1+\lambda t)^{\alpha}\right]; \qquad t>0,\ \alpha,\lambda>0, \tag{3}$$

and

$$H(t;\alpha,\lambda)=\alpha\lambda(1+\lambda t)^{\alpha-1}; \qquad t>0,\ \alpha,\lambda>0, \tag{4}$$

where α and λ are the shape and scale parameters, respectively.
Recently, many studies on estimating the unknown parameters of extended exponential distribution based on different life-testing experiments have been carried out by many authors. [3] obtained the maximum likelihood estimation (MLE) and Bayes estimators of the extended exponential distribution under Type-II progressive censoring scheme (PCS). [4] discussed the MLEs and Bayes estimators of the unknown parameters and reliability characteristics of the extended

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exponential distribution based on complete sampling. [5] introduced a comparisons between several methods for estimating the unknown parameters of extended exponential distribution. [6] discussed MLE and Bayes estimation of the two unknown parameters of extended exponential distribution based on record values. [7] obtained The MLE and Bayes inferential approaches for estimating the unknown two parameters and some lifetime parameters such as reliability and hazard rate functions of extended exponential distribution in presence of progressive first-failure censored sampling. [8] obtained MLE and Bayes estimators of the parameters and acceleration factor for extended exponential distribution based on constant-stress partially accelerated life tests using progressive Type-II censoring. [9] discussed the estimation and prediction problems for the extended exponential distribution using progressive type-II censored samples.
In conventional Type-I and Type-II censoring, a life test is terminated at a prescribed time span or at a predefined number of failures. The main drawback of these censoring schemes is, the units cannot be removed from the test at any time point except the final closure point. However, the Type-II PCS gives the flexibility of eliminating the test units before the final termination. On other hand, the major drawback of the Type-II PCS is that, it can take a lot of time to reach the final termination point. [10] discussed Type-I progressive hybrid censoring scheme (PHCS) which is the result of the mixture of Type-II PCS and conventional Type-I (time censoring) schemes. In Type-I PHCS, the life tests stops when a predefined number of failures occurred or when a prescribed time on the test has reached. Numerous authors have investigated several lifetime models using the hybrid censoring scheme. Due to the importance of Type-I PHCS, many authors have studied estimation of the parameters of various lifetime distributions based on this scheme. [11] obtained some useful classical estimates of the unknown parameters of Weibull distribution under Type-I PHCS. They obtained MLEs and approximate MLEs of the unknown parameters, and then compared the performance of proposed estimates using numerical simulations. [12] obtained various estimates for log-normal distribution under Type-I PHCS. [13] discussed some inferences for Burr Type-XII distribution under Type-I PHCS. [14] studied estimation of parameters of the generalized half-normal distribution under Type-I PHCS. They obtained MLE and Bayes estimates. Further, they computed Bayes estimates based on different approximation techniques. One may refer to [15,16,17,18,19] and the references therein for some more references in this topic.
Let us suppose Type-I PHCS, n identical items are put on a test and the life time distributions of the n items are denotes by $x_1, x_2, ... x_n$. The integer $r < n$ is fixed at the beginning of the experiment, and $(R_1, R_2, ..., R_r)$ are r per-fixed integers satisfying $R_1 + R_2 + \cdots + R_r + r = n$, let time point $T$is also fixed beforehand. At the time of the first failure $x_{(1)}$, R_1 of the remaining $(n-1)$ surviving units are randomly removed. Similarly, at the time of the second failure $x_{(2)}$, R_2 of the $n - R_1 - R_2 - 2$ surviving units are removed, and so on. Finally at the time of the r^{th} failure all $R_r = n - R_1 - \cdots - R_{r-1} - r$ surviving units are removed from the life-test. In progressive hybrid type I, the experiment would terminate at the random time $T^* = \min(x_{(r)}, T)$. In case-I, the r^{th} failure $x_{(r)}$occurs before the time T, the experiment stops at the time point $x_{(r)}$. In case-II, the r^{th} failure $x_{(r)}$ doesn't occurs before the time $T$and only D failures occurs before the time T, then, at the time point T all remaining R_D^* units are removed and the experiment stops. The likelihood function of the observed data (without constant term) is given by

$$L(\theta) \propto \begin{cases} \left[\prod_{i=1}^{r} f(x_{(i)})(1 - F(x_{(i)}))^{R_i} \right], & \text{for case-I;} \\ \left[\prod_{i=1}^{D} f(x_{(i)}) \left[1 - F(x_{(i)})\right]^{R_i} [1 - F(T)]^{R_D^*} \right], & \text{for case-II.} \end{cases} \tag{5}$$

At the present time, the reliability tests should be performed with severe time limitations because of the short product development times, which make the usual progressive type-II censoring scheme no longer appropriate in many field products. Therefore, here, PHCS has been proposed by [10] to overcome the drawback of Type-II progressively censoring scheme is that the experimental time can be very long if the units are highly reliable. So, the main advantage of PHCS is that guarantee terminates the life-test rapidly and guarantee that the experimental time cannot exceed T.
The aim of this paper is the estimation of the unknown parameters, hazard rate and reliability functions of extended exponential distribution under Type-I PHCS .In section 2, The MLEs and the information matrix will be discussed to obtain asymptotic confidence intervals for the parameters and estimate reliability and hazard rate functions. Further, Bayesian estimation under the assumption of independent gamma priors using squared error (SE) and LINEX loss functions will be discussed in section 3. Numerically proposed methods using Monte Carlo simulations and a real data set is compared in Section 4. Finally a conclusion is given in Section 5.

2 Maximum likelihood estimation

Suppose n units whose lifetimes are independent and identically distributed extended exponential distribution random variables with the probability density function (1) are placed on a life test without replacement.

Substituting equations (1) and (2) into (5), the likelihood function based on Type-I PHCS from extended exponential distribution can be written as

$$L(\underline{x};\alpha,\lambda)=\begin{cases}\prod_{i=1}^{r}\alpha\lambda(1+\lambda x_i)^{\alpha-1}e^{\left[1-(1+\lambda x_i)^{\alpha}\right]}\left[1-(1-e^{(1-(1+\lambda x_i)^{\alpha})})\right]^{R_i}, & \text{for case-I;}\\ \prod_{i=1}^{D}\alpha\lambda(1+\lambda x_i)^{\alpha-1}e^{\left[1-(1+\lambda x_i)^{\alpha}\right]}\left[1-(1-e^{(1-(1+\lambda x_i)^{\alpha})})\right]^{R_i}\left[1-(1-e^{(1-(1+\lambda T)^{\alpha})})\right]^{R_D^*}, & \text{for case-II.}\end{cases} \tag{6}$$

Taking the natural logarithm for (6), we get the following equation

$$\ln L(\underline{x};\alpha,\lambda)=\begin{cases} r\ln\alpha+r\ln\lambda+(\alpha-1)\sum_{i=1}^{r}\ln(1+\lambda x_i)+\sum_{i=1}^{r}\left[1-(1+\lambda x_i)^{\alpha}\right] & \\ +R_i\left[1-(1+\lambda x_i)^{\alpha}\right], & \text{for case-I;}\\ D\ln\alpha+D\ln\lambda+(\alpha-1)\sum_{i=1}^{D}\ln(1+\lambda x_i)+\sum_{i=1}^{D}\left[1-(1+\lambda x_i)\right]^{\alpha} & \\ +R_i\left[1-(1+\lambda x_i)^{\alpha}\right]+R_D^*\left[1-(1+\lambda T)^{\alpha}\right], & \text{for case-II.}\end{cases} \tag{7}$$

The first-partial derivatives of (7) with respect to α and λ are given, respectively, by

$$\frac{\partial}{\partial\alpha}\ln L(\underline{x};\alpha,\lambda)=\begin{cases}\frac{r}{\alpha}+\sum_{i=1}^{r}\ln(1+\lambda x_i)-\sum_{i=1}^{r}(1+\lambda x_i)^{\alpha}\ln(1+\lambda x_i)(1+R_i), & \text{for case-I;}\\ \frac{D}{\alpha}+\sum_{i=1}^{D}\ln(1+\lambda x_i)-\sum_{i=1}^{D}(1+\lambda x_i)^{\alpha}\ln(1+\lambda x_i)(1+R_i) & \\ -R_D^*(1+\lambda T)^{\alpha}\ln(1+\lambda T), & \text{for case-II.}\end{cases}$$

and

$$\frac{\partial}{\partial\lambda}\ln L(\underline{x};\alpha,\lambda)=\begin{cases}\frac{r}{\lambda}+(\alpha-1)\sum_{i=1}^{r}\frac{x_i}{1+\lambda x_i}-\alpha\sum_{i=1}^{r}x_i(1+\lambda x_i)^{\alpha-1}(1+R_i), & \text{for case-I;}\\ \frac{D}{\lambda}+(\alpha-1)\sum_{i=1}^{D}\frac{x_i}{1+\lambda x_i}-\alpha\sum_{i=1}^{D}x_i(1+\lambda x_i)^{\alpha-1}(1+R_i)-R_D^*\alpha T(1+\lambda T)^{\alpha-1}, & \text{for case-II.}\end{cases}$$

Since these equations after equating them to zero are clearly transcendental equations in $\hat{\alpha}$ and $\hat{\lambda}$ that is, no closed form solutions are known they must be solved by iterative numerical techniques to provide parameter estimates, $\hat{\alpha}$ and $\hat{\lambda}$, in the desired degree of accuracy. The MLEs of reliability function and hazard rate function can be obtained after replacing α and λ by their MLEs $\hat{\alpha}$ and $\hat{\lambda}$ as

$$\hat{S}_{ML}(t)=e^{(1-(1+\hat{\lambda}t)^{\hat{\alpha}})},\ t>0,\quad\text{and}\quad \hat{H}_{ML}(t)=\hat{\alpha}\hat{\lambda}(1+\hat{\lambda}t)^{\hat{\alpha}-1}\ t>0,$$

respectively.

To study the variation of the MLEs $\hat{\alpha}$ and $\hat{\lambda}$, the asymptotic variance of these estimators are obtained. The asymptotic variance covariance matrix of $\hat{\alpha}$ and $\hat{\lambda}$ is obtained by inverting the information matrix with elements that are negative expected values of the second order derivatives of natural logarithms of the likelihood function, for sufficiently large samples, a reasonable approximation to the asymptotic variance covariance matrix of the estimators can be obtained as

$$\mathbf{I}^{-1}(\hat{\alpha},\hat{\lambda})\cong\begin{bmatrix}\mathscr{L}_{11} & \mathscr{L}_{12}\\ \mathscr{L}_{21} & \mathscr{L}_{22}\end{bmatrix}^{-1}_{(\alpha=\hat{\alpha},\lambda=\hat{\lambda})}=\begin{bmatrix}\mathrm{Var}(\hat{\alpha}) & \mathrm{Cov}(\hat{\alpha},\hat{\lambda})\\ \mathrm{Cov}(\hat{\lambda},\hat{\alpha}) & \mathrm{Var}(\hat{\lambda})\end{bmatrix}. \tag{8}$$

From (7), the Fisher's elements of (8) are obtained and reported in Appendix. The matrix can be inverted to obtain the estimate of the asymptotic variance-covariance matrix of the maximum likelihood estimators. The diagonal elements of $\mathbf{I}^{-1}(\hat{\alpha},\hat{\lambda})$ provide the asymptotic variance of $\hat{\alpha}$ and $\hat{\lambda}$ respectively. Then by using large sample theory a two-sided $100(1-\beta)\%$ approximate confidence intervals (ACIs) for α and λ can be constructed, respectively, as

$$\hat{\alpha}\pm z_{1-\beta/2}\sqrt{\widehat{\mathrm{Var}}(\hat{\alpha})}\quad\text{and}\quad\hat{\lambda}\pm z_{1-\beta/2}\sqrt{\widehat{\mathrm{Var}}(\hat{\lambda})}.$$

To construct the ACIs of $S(t)$ and $H(t)$, The variances of them is needed Therefore, the delta method is considered to obtain the approximate estimates of the variance of $\hat{S}(t)$ and $\hat{H}(t)$. Delta method is a general approach for computing ACIs for any function of the MLEs $\hat{\alpha}$ and $\hat{\lambda}$, (see [20]). According to this method, the variance of $\hat{S}(t)$ and $\hat{H}(t)$, can be approximated, by

$$\hat{\sigma}^2_{\hat{S}(t)} = [\nabla\hat{S}(t)]^{\mathbf{T}}\mathbf{I}^{-1}(\hat{\alpha},\hat{\lambda})[\nabla\hat{S}(t)] \quad \text{and} \quad \hat{\sigma}^2_{\hat{H}(t)} = [\nabla\hat{H}(t)]^{\mathbf{T}}\mathbf{I}^{-1}(\hat{\alpha},\hat{\lambda})[\nabla\hat{H}(t)],$$

where $\nabla\hat{S}(t)$ and $\nabla\hat{H}(t)$ are respectively, the gradient (vector of first partial derivatives) of $S(t)$ and $H(t)$ with respect to α and λ obtained at $\hat{\alpha}$ and $\hat{\lambda}$.

$$[\nabla\hat{S}(t)]^T = \left[\frac{\partial\nabla\hat{S}(t)}{\partial\alpha}, \frac{\partial\nabla\hat{S}(t)}{\partial\lambda}\right]_{(\hat{\alpha},\hat{\lambda})} \quad \text{and} \quad [\nabla\hat{H}(t)]^T = \left[\frac{\partial\nabla\hat{H}(t)}{\partial\alpha}, \frac{\partial\nabla\hat{H}(t)}{\partial\lambda}\right]_{(\hat{\alpha},\hat{\lambda})}.$$

Hence, the $100(1-\beta)\%$ ACIs of $S(t)$ and $H(t)$, are given, respectively, by

$$\hat{S}(t) \pm z_{1-\beta/2}\sqrt{\hat{\sigma}^2_{\hat{S}(t)}} \quad \text{and} \quad \hat{H}(t) \pm z_{1-\beta/2}\sqrt{\hat{\sigma}^2_{\hat{H}(t)}}.$$

3 Bayesian estimation

In this section, the Bayesian method is used to obtain the estimators for unknown parameters of extended exponential distribution using SE loss function and LINEX loss functions. We consider independent gamma priors for the parameters α and λ, respectively, as $\pi(\alpha) \propto \alpha^{a-1}e^{-b\alpha},\ \alpha > 0,\ a,b > 0$, and $\pi(\lambda) \propto \lambda^{c-1}e^{-d\lambda},\ \lambda > 0,\ c,d > 0$. Hence, the joint prior density function of α and λ becomes

$$\pi(\alpha,\lambda) \propto \alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)},\ \lambda,\alpha > 0,\ a,b,c,d > 0, \tag{9}$$

where the hyper-parameters a, b, c and d are assumed to be known and non-negative. Combining equation (9) with (6) and using Bayes theorem, the joint posterior distribution can be obtained as

$$\pi(\alpha,\lambda|\underline{x}) = \begin{cases} \dfrac{1}{\psi_1}\alpha^{r+a-1}\lambda^{r+c-1}e^{-(b\alpha+d\lambda)}\displaystyle\prod_{i=1}^{r}(1+\lambda x_i)^{\alpha-1}e^{\left[1-(1+\lambda x_i)^{\alpha}\right]}\left\{1-(1-e^{\left[1-(1+\lambda x_i)^{\alpha}\right]})\right\}^{R_i}, & \text{for case-I;} \\ \dfrac{1}{\psi_2}\alpha^{D+a-1}\lambda^{D+c-1}e^{-(b\alpha+d\lambda)}\displaystyle\prod_{i=1}^{D}(1+\lambda x_i)^{\alpha-1}e^{\left[1-(1+\lambda x_i)^{\alpha}\right]}\left\{1-(1-e^{\left[1-(1+\lambda x_i)^{\alpha}\right]})\right\}^{R_i} & \\ \times\left\{1-(1-e^{\left[1-(1+\lambda T)^{\alpha}\right]})\right\}^{R_i^*}, & \text{for case-II.} \end{cases} \tag{10}$$

where

$$\psi_1 = \int_{\alpha}\int_{\lambda}\alpha^{r+a-1}\lambda^{r+c-1}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{r}(1+\lambda x_i)^{\alpha-1}e^{\left[1-(1+\lambda x_i)^{\alpha}\right]}\left\{1-(1-e^{\left[1-(1+\lambda x_i)^{\alpha}\right]})\right\}^{R_i}d\lambda d\alpha,$$

and

$$\psi_2 = \int_{\alpha}\int_{\lambda}\alpha^{D+a-1}\lambda^{D+c-1}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{D}(1+\lambda x_i)^{\alpha-1}e^{\left[1-(1+\lambda x_i)^{\alpha}\right]}\left\{1-(1-e^{\left[1-(1+\lambda x_i)^{\alpha}\right]})\right\}^{R_i}\left\{1-(1-e^{\left[1-(1+\lambda T)^{\alpha}\right]})\right\}^{R_i^*}d\lambda d\alpha.$$

Bayes estimators $\tilde{\alpha}$ and $\tilde{\lambda}$ of α and λ, respectively, of the extended exponential distribution under the SE loss function are given by the mean of the marginal posterior density function (10) as

$$\tilde{\alpha}_{SE} = \int_{\alpha}\alpha\cdot\pi(\alpha,\lambda|\underline{x}), \quad \text{and} \quad \tilde{\lambda}_{SE} = \int_{\lambda}\lambda\cdot\pi(\alpha,\lambda|\underline{x})d\lambda,$$

respectively. These estimators $\tilde{\alpha}$ and $\tilde{\lambda}$ can be also expressed as

$$\tilde{\alpha}_{SE}=\begin{cases}\int_{\alpha}\alpha\frac{1}{\psi_1}\alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{r}\alpha\lambda(1+\lambda x_i)^{\alpha-1}e^{\left[1-(1+\lambda x_i)^{\alpha}\right]}\left\{1-(1-e^{\left[1-(1+\lambda x_i)^{\alpha}\right]})\right\}^{R_i}d\alpha, & \text{for case-I;}\\ \int_{\alpha}\alpha\frac{1}{\psi_2}\alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{D}\alpha\lambda(1+\lambda x_i)^{\alpha-1}e^{\left[1-(1+\lambda x_i)^{\alpha}\right]}\left\{1-(1-e^{\left[1-(1+\lambda x_i)^{\alpha}\right]})\right\}^{R_i} & \\ \times\left\{1-(1-e^{\left[1-(1+\lambda T)^{\alpha}\right]})\right\}^{R_i^*}d\alpha, & \text{for case-II,}\end{cases} \tag{11}$$

and

$$\tilde{\lambda}_{SE}=\begin{cases}\int_{\lambda}\lambda\frac{1}{\psi_1}\alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{r}\alpha\lambda(1+\lambda x_i)^{\alpha-1}e^{\left[1-(1+\lambda x_i)^{\alpha}\right]}\left\{1-(1-e^{\left[1-(1+\lambda x_i)^{\alpha}\right]})\right\}^{R_i}d\lambda, & \text{for case-I;}\\ \int_{\lambda}\lambda\frac{1}{\psi_2}\alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{D}\alpha\lambda(1+\lambda x_i)^{\alpha-1}e^{\left[1-(1+\lambda x_i)^{\alpha}\right]}\left\{1-(1-e^{\left[1-(1+\lambda x_i)^{\alpha}\right]})\right\}^{R_i} & \\ \times\left\{1-(1-e^{\left[1-(1+\lambda T)^{\alpha}\right]})\right\}^{R_i^*}d\lambda, & \text{for case-II,}\end{cases} \tag{12}$$

respectively. Also, the Bayes estimates $\tilde{S}(t)$ and $\tilde{H}(t)$ of the reliability and hazard functions are given, respectively, by

$$\tilde{S}_{SE}(t)=\begin{cases}\int_{\alpha}\int_{\lambda}e^{\left[1-(1+\lambda x)^{\alpha}\right]}\pi(\alpha,\lambda|\underline{x})\ d\lambda d\alpha, & \text{for case-I;}\\ \int_{\alpha}\int_{\lambda}e^{\left[1-(1+\lambda x)^{\alpha}\right]}\pi(\alpha,\lambda|\underline{x})\ d\lambda d\alpha, & \text{for case-II,}\end{cases} \tag{13}$$

and

$$\tilde{H}_{SE}(t)=\begin{cases}\int_{\alpha}\int_{\lambda}\alpha\lambda(1+\lambda x)^{\alpha-1}\pi(\alpha,\lambda|\underline{x})\,d\lambda d\alpha, & \text{for case-I;}\\ \int_{\alpha}\int_{\lambda}\alpha\lambda(1+\lambda x)^{\alpha-1}\pi(\alpha,\lambda|\underline{x})\,d\lambda d\alpha, & \text{for case-II.}\end{cases} \tag{14}$$

Following [21], the Bayes estimators of α and λ under LINEX loss function are given, respectively, by

$$\tilde{\alpha}_{LINEX}=\frac{1}{c^*}\ln\left(E\left(\exp(-c^*\alpha)\right)\right) \quad \text{and} \quad \tilde{\lambda}_{LINEX}=\frac{1}{c^*}\ln\left(E\left(\exp(-c^*\lambda)\right)\right),$$

respectively, where $E(\cdot)$ denotes the posterior expectation.

Thus, the Bayes estimates $\tilde{\alpha}_{LINEX}$ and $\tilde{\lambda}_{LINEX}$ of α and λ under LINEX loss function can be obtained, respectively, as

$$\tilde{\alpha}_{LINEX}=\begin{cases}\frac{1}{c^*}\ln\int_{\alpha}e^{-c^*\alpha}\frac{1}{\psi_1}\alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{r}\alpha\lambda(1+\lambda x_i)^{\alpha-1}e^{\left[1-(1+\lambda x_i)^{\alpha}\right]} & \\ \times(1-(1-e^{(1-(1+\lambda x_i)^{\alpha})}))^{R_i}d\alpha, & \text{for case-I;}\\ \frac{1}{c^*}\ln\int_{\alpha}e^{-c^*\alpha}\frac{1}{\psi_2}\alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{D}\alpha\lambda(1+\lambda x_i)^{\alpha-1}e^{\left[1-(1+\lambda x_i)^{\alpha}\right]} & \\ \times\left\{1-(1-e^{\left[1-(1+\lambda x_i)^{\alpha}\right]})\right\}^{R_i}\left\{1-(1-e^{\left[1-(1+\lambda T)^{\alpha}\right]})\right\}^{R_i^*}d\alpha, & \text{for case-II.}\end{cases} \tag{15}$$

and

$$\tilde{\lambda}_{LINEX}=\begin{cases}\frac{1}{c^*}\ln\int_{\lambda}e^{-c^*\lambda}\frac{1}{\psi_1}\alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{r}\alpha\lambda(1+\lambda x_i)^{\alpha-1}e^{\left[1-(1+\lambda x_i)^{\alpha}\right]} & \\ \times\left\{1-(1-e^{\left[1-(1+\lambda x_i)^{\alpha}\right]})\right\}^{R_i}d\lambda, & \text{for case-I;}\\ \frac{1}{c^*}\ln\int_{\lambda}e^{-c^*\lambda}\frac{1}{\psi_2}\alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{D}\alpha\lambda(1+\lambda x_i)^{\alpha-1}e^{\left[1-(1+\lambda x_i)^{\alpha}\right]} & \\ \times\left\{1-(1-e^{\left[1-(1+\lambda x_i)^{\alpha}\right]})\right\}^{R_i}\left\{1-(1-e^{\left[1-(1+\lambda T)^{\alpha}\right]})\right\}^{R_i^*}d\lambda, & \text{for case-II,}\end{cases} \tag{16}$$

respectively, and the Bayes estimators of the reliability and hazard functions are also given by

$$\tilde{S}_{LINEX}(t) = \begin{cases} \frac{1}{c^*} \ln \int_\alpha \int_\lambda e^{-c^* e^{[1-(1+\lambda x)^\alpha]}} \pi(\alpha,\lambda|\underline{x})\, d\lambda d\alpha, & \text{for case-I;} \\ \frac{1}{c^*} \ln \int_\alpha \int_\lambda e^{-c^* e^{[1-(1+\lambda x)^\alpha]}} \pi(\alpha,\lambda|\underline{x})\, d\lambda d\alpha, & \text{for case-II,} \end{cases} \tag{17}$$

and

$$\tilde{H}_{LINEX}(t) = \begin{cases} \frac{1}{c^*} \ln \int_\alpha \int_\lambda e^{-c^* \alpha\lambda(1+\lambda x)^{\alpha-1}} \pi(\alpha,\lambda|\underline{x})\, d\lambda d\alpha, & \text{for case-I;} \\ \frac{1}{c^*} \ln \int_\alpha \int_\lambda e^{-c^* \alpha\lambda(1+\lambda x)^{\alpha-1}} \pi(\alpha,\lambda|\underline{x})\, d\lambda d\alpha, & \text{for case-II,} \end{cases} \tag{18}$$

respectively.

It is clear that the proposed estimators formulated in Equations (11-18) cannot be obtained in a closed form, so the approximate methods is employed. Markov Chain Monte Carlo (MCMC) using Metropolis-Hastings (MH) algorithm has been used to carry out the Bayes estimates and also to construct the associate HPD credible intervals.

4 Simulated study and Real data analysis

The aim of this section is to compare the performance of the different methods of estimation discussed in the previous sections. A Monte Carlo study is employed to check the behavior of the proposed methods as well as to assess the statistical performances of the estimators under Type-I progressive hybrid. Also, a real data set is analyzed for illustrative purpose. $\mathscr{R}$-statistical programming language will be used for calculation.

4.1 Simulated study

In this section, we perform a Monte Carlo simulated study (1000 times) to compare the performance of different estimators of unknown parameters of the extended exponential distribution. We also assess the behavior of predictors of censored observations under the considered censoring scheme. The performance of different estimators is compared in terms of corresponding average estimates and mean square error (MSE) values. For this purpose, we generate Type-I progressive hybrid censored samples using various sampling schemes by considering different combinations of (n,r) and assuming that T is either (0.63, 1.79). We used the $\mathscr{R}$-statistical software for all computations. The MLEs of α and λ are computed and the information matrix will be discussed to obtain asymptotic confidence intervals for the parameters and estimate reliability and hazard rate functions. Bayes estimates of parameters are computed with respect to a gamma prior distribution under symmetric SE and asymmetric LINEX loss functions. Both MLEs and Bayes estimates of parameters are obtained for arbitrarily taken unknown parameters $\alpha = 1.5$ and $\lambda = 0.5$.

For the MLEs, one may generate 1000 data from the extended exponential distribution with the following assumptions:
Step 1: Assume the following selected cases of parameters of the extended exponential distribution: $(\alpha,\lambda) = (1.5, 0.5)$.
Step 2: Sample sizes, are $n = 50, 100, 200$ and number of observed failures $r = 20, 40, 80$, respectively.
Step 3: Censoring times Type-I PHCS are assumed T_q corresponding to the selected $q-th$ quantiles, where $q = (40, 80\%)$. The $q-th$ quantiles of lifetimes distribution is given by $P(X \leq T_q) = q \Rightarrow T_q = Q(q)$, where $Q(\cdot)$ is the inverse of the cdf (quantile) of the given distribution.
Step 4: Removed items $R_i,\ i = 1, 2, ..., r$, are assumed to as follows:
Scheme I: $R_1 = n - r$ and $R_2 = \cdots = R_r = 0$.
Scheme II: $R_1 = \cdots = R_{\frac{r}{2}} = 1$ and $R_{\frac{r}{2}+1} = \cdots = R_r = 2$.
Scheme III: $R_1 = \cdots = R_{r-1} = 0$ and $R_r = n - r$.

The values of hyper-parameters are chosen to satisfy the prior mean become the expected value of the corresponding parameter. These values, hyper parameters, are then plugged-in to calculate the desired estimates. While utilizing MH algorithm, the MLEs are taken into account as initial guess values, and the associated variance-covariance matrix $(\theta^{(0)}) = (ln(\hat{\alpha}), ln(\hat{\lambda}))$. At the end, 2000 burn-in samples are discarded among the overall 10000 samples generated from the posterior density, and subsequently obtained Bayes estimates and HPD interval estimates.

Further, we have also obtained the MLEs and Bayesian estimates of the reliability function and hazard function where the true values of $\hat{S}(t)$ and $\hat{H}(t)$ are taken form the specified time censoring, termination point of the test $T^* = \min(T, x_r))$, of Type-I progressive hybrid scheme. The true values of hazard function are $h(t = 0.63, \alpha, \lambda) = 0.8606$ and

Table 1: Average estimated values (first-line) and MSEs (second-line) of α and λ based on Type-I progressive hybrid censoring schemes at different time censoring and different values of (n,r) for $(\alpha,\lambda)=(1.5,0.5)$.

(n,r)	Method	$q_i=40\%$			$q_i=80\%$		
		I	II	III	I	II	III
(50,20)	MLE$_\alpha$	1.8701	0.9812	2.3246	0.3153	2.1020	2.3129
		19.797	7.9061	16.768	3.8529	13.502	15.153
	MLE$_\lambda$	1.8565	2.2510	1.3418	1.9124	1.5140	1.3501
		5.8899	7.5381	3.3664	4.7880	4.2194	2.9253
	Bayes-SE$_\alpha$	0.4618	0.3708	0.9756	0.1534	1.3546	1.3571
		1.4665	1.7550	1.2348	1.9387	1.7292	0.7334
	Bayes-SE$_\lambda$	1.1972	1.6984	1.0123	1.2456	1.1852	0.9769
		1.3426	2.9776	1.2685	1.1979	1.6298	1.0696
	Bayes-LINEX$_\alpha$	0.4205	0.3132	0.9239	0.1367	1.2001	1.3272
		1.5288	1.7554	1.3719	1.9505	1.1051	1.0557
	Bayes-LINEX$_\lambda$	0.9305	1.3754	0.8507	1.0384	0.9647	0.8201
		0.6531	1.6701	0.7659	0.6711	0.8978	0.6419
(100,40)	MLE$_\alpha$	1.7768	0.3800	1.8935	0.2879	2.2504	2.5576
		13.021	2.4797	9.7939	1.7922	13.787	14.919
	MLE$_\lambda$	1.4151	2.3569	1.0457	1.4405	1.3741	1.0770
		3.9575	7.3426	2.0825	2.7538	4.7931	1.6173
	Bayes-SE$_\alpha$	0.6378	0.2540	1.0690	0.2060	1.3407	1.6491
		1.6674	1.8236	1.2818	1.8271	1.8035	1.9933
	Bayes-SE$_\lambda$	1.1573	1.9598	0.8868	1.1588	1.5541	0.8314
		1.7300	4.1772	1.2027	1.2554	3.9712	0.6658
	Bayes-LINEX$_\alpha$	0.5568	0.2302	1.0200	0.1859	1.3183	1.6094
		1.5873	1.8388	1.3126	1.8540	1.8250	1.8682
	Bayes-LINEX$_\lambda$	0.9586	1.6865	0.7863	1.0240	1.2838	0.7365
		1.0120	2.7636	0.8624	0.8536	2.4275	0.4745
(200,80)	MLE$_\alpha$	1.3040	0.4590	1.8679	1.1737	2.0121	2.4964
		6.2076	1.4828	8.4083	0.4340	8.9683	12.345
	MLE$_\lambda$	1.1673	2.0616	0.5836	0.2898	0.9604	0.8316
		2.8800	6.8371	0.7150	0.5457	4.0482	0.6982
	Bayes-SE$_\alpha$	0.6388	0.3728	1.2274	0.4355	1.1729	1.8803
		1.4678	1.5443	1.3973	1.3264	2.6708	2.6449
	Bayes-SE$_\lambda$	1.0127	1.7671	0.5434	0.4602	1.6200	0.7234
		1.6752	4.2872	0.5074	0.3130	3.5280	0.4520
	Bayes-LINEX$_\alpha$	0.5642	0.3421	1.1847	0.2979	1.1584	1.8480
		1.4795	1.5835	1.6415	1.5320	2.6386	2.6909
	Bayes-LINEX$_\lambda$	0.8957	1.5891	0.5031	0.3731	1.0812	0.6619
		1.1703	3.2711	0.4311	0.2618	1.5361	0.3554

$h(t=1.79,\alpha,\lambda)=1.0325$ and the true values of reliability function are $S(t=0.63,\alpha,\lambda)=0.6$ and $h(t=1.79,\alpha,\lambda)=0.2$. It should be noted that, when $q=(40,80)\%$, one gets $T_{40\%}=Q(40\%,\alpha=1.5,\lambda=0.5)=0.6334$ and $T_{80\%}=Q(80\%,\alpha=1.5,\lambda=0.5)=1.7908$. Further, the corresponding true value of hazard function $H(t)$ at $q=(40,80)\%$ becomes $H(t=0.6334,\alpha=1.5,\lambda=0.5)=0.8606$ and $H(t=1.7908,\alpha=1.5,\lambda=0.5)=1.0325$, respectively. Similarly, the corresponding true value of reliability function $S(t)$ at $q=(40,80)\%$ becomes $S(t=0.6334,\alpha=1.5,\lambda=0.5)=0.6000$ and $S(t=1.7908,\alpha=1.5,\lambda=0.5)=0.2000$, respectively. All the average estimates and associated MSEs for both methods of all unknown parameters α, λ, $H(t)$ and $S(t)$ are reported in Tables 1 and 2. Further, the corresponding average interval lengths (AILs) and coverage probabilities (CPs) for 95% asymptotic/HPD credible intervals are listed in Tables 3 and 4.

4.2 Real data analysis

A real data set is analyzed for illustrative purpose as well as to assess the statistical performances of the MLEs and Bayes estimators for the extended exponential distribution under Type-I progressive hybrid censoring schemes. The following

Table 2: Average estimated values (first-line) and MSEs (second-line) of $H(t)$ and $S(t)$ based on Type-I progressive hybrid censoring schemes at different time censoring and different values of (n,r) for $(\alpha,\lambda)=(1.5,0.5)$.

(n,r)	Method	$q_i=40\%$			$q_i=80\%$		
		I	II	III	I	II	III
(50,20)	$\text{MLE}_{H(t)}$	0.1124	0.0658	0.4584	0.0258	0.9559	1.0673
		0.5613	0.6355	0.3933	1.0300	0.6746	0.8393
	$\text{MLE}_{S(t)}$	0.9072	0.9435	0.7714	0.9402	0.3642	0.2401
		0.0945	0.1183	0.0702	0.5496	0.1406	0.0140
	$\text{Bayes-SE}_{H(t)}$	0.1256	0.1405	0.6332	0.0272	1.1100	1.1091
		0.5500	5.0160	14.335	1.0268	0.9101	0.4845
	$\text{Bayes-SE}_{S(t)}$	0.9161	0.9464	0.7732	0.9445	0.3553	0.2240
		0.1022	0.1220	0.0756	0.5565	0.1570	0.0202
	$\text{Bayes-LINEX}_{H(t)}$	0.0861	0.0558	0.3224	0.0229	0.6089	0.6058
		0.6015	0.6491	0.3959	1.0272	0.3689	0.2685
	$\text{Bayes-LINEX}_{S(t)}$	0.9403	0.9584	0.8269	0.9510	0.4588	0.3713
		0.1162	0.1288	0.0761	0.5656	0.1643	0.0517
(100,40)	$\text{MLE}_{H(t)}$	0.0695	0.0342	0.4117	0.0114	0.8585	1.0638
		0.6262	0.6830	0.4066	1.0428	0.6603	0.4743
	$\text{MLE}_{S(t)}$	0.9464	0.9699	0.7983	0.9699	0.4645	0.2285
		0.1200	0.1368	0.0811	0.5928	0.2250	0.0103
	$\text{Bayes-SE}_{H(t)}$	0.0811	0.0360	0.4477	0.0140	0.8801	1.1456
		0.6207	0.6806	0.4578	1.0378	0.6717	0.4000
	$\text{Bayes-SE}_{S(t)}$	0.9452	0.9713	0.7937	0.9669	0.4705	0.2010
		0.1203	0.1380	0.0862	0.5889	0.2413	0.0113
	$\text{Bayes-LINEX}_{H(t)}$	0.0604	0.0316	0.3190	0.0116	0.5884	0.7218
		0.6408	0.6873	0.4184	1.0424	0.4656	0.1991
	$\text{Bayes-LINEX}_{S(t)}$	0.9572	0.9747	0.8350	0.9716	0.5304	0.3213
		0.1277	0.1404	0.0844	0.5954	0.2406	0.0325
(200,80)	$\text{MLE}_{H(t)}$	0.0388	0.0191	0.4084	0.0081	0.6130	1.0526
		0.6755	0.7082	0.4143	1.0496	0.7278	0.2877
	$\text{MLE}_{S(t)}$	0.9712	0.9843	0.8084	0.9846	0.6245	0.2179
		0.1378	0.1477	0.0859	0.6156	0.3505	0.0061
	$\text{Bayes-SE}_{H(t)}$	0.0468	0.0237	0.4484	0.0302	0.6633	1.1974
		0.6631	0.7010	0.4445	1.0063	1.7224	0.4356
	$\text{Bayes-SE}_{S(t)}$	0.9672	0.9820	0.7991	0.9425	0.5840	0.1867
		0.1351	0.1461	0.0880	0.5556	0.3049	0.0068
	$\text{Bayes-LINEX}_{H(t)}$	0.0379	0.0195	0.3493	0.0145	0.4871	0.8578
		0.6770	0.7074	0.4149	1.0364	0.6355	0.1435
	$\text{Bayes-LINEX}_{S(t)}$	0.9730	0.9849	0.8290	0.9707	0.6447	0.2691
		0.1392	0.1481	0.0864	0.5943	0.3422	0.0133

original data set which an uncensored data set corresponding to remission times (in months) of a random sample of 128 bladder cancer patients is considered. These data were previously studied by [22] [23] and [2]. The remission times of the bladder cancer are as follows: 0.08, 0.20, 0.40, 0.50, 0.51, 0.81, 0.90, 1.05, 1.19, 1.26, 1.35, 1.40, 1.46, 1.76, 2.02, 2.02, 2.07, 2.09, 2.23, 2.26, 2.46, 2.54, 2.62, 2.64, 2.69, 2.69, 2.75, 2.83, 2.87, 3.02, 3.25, 3.31, 3.36, 3.36, 3.48, 3.52, 3.57, 3.64, 3.70, 3.82, 3.88, 4.18, 4.23, 4.26, 4.33, 4.34, 4.40, 4.50, 4.51, 4.87, 4.98, 5.06, 5.09, 5.17, 5.32, 5.32, 5.34, 5.41, 5.41, 5.49, 5.62, 5.71, 5.85, 6.25, 6.54, 6.76, 6.93, 6.94, 6.97, 7.09, 7.26, 7.28, 7.32, 7.39, 7.59, 7.62, 7.63, 7.66, 7.87, 7.93, 8.26, 8.37, 8.53, 8.65, 8.66, 9.02, 9.22, 9.47, 9.74, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05.

We first check whether the extended exponential distribution is suitable for analyzing this data set or not. The value of Kolmogorov–Smirnov (K–S) test statistic is calculated to judge the goodness of fit. The calculated Kolmogorov-Smirnov (K-S) distance between the empirical and the fitted extended exponential distribution is 0.0935 and its p-value is 0.2131. Which indicate that this distribution can be considered as an adequate model for the given data

Table 3: The AILs (first-line) and CPs% (second-line) of $S(t)$ and $H(t)$ based on hybrid progressive Type-I censoring schemes at different time censoring and different values of (n,r) for $(\alpha,\lambda)=(1.5,0.5)$.

(n,r)	Method	$q_i=40\%$			$q_i=80\%$		
		I	II	III	I	II	III
(50,20)	$\text{MLE}_{H(t)}$	10.5658	6.4004	10.1919	3.3843	9.2125	9.7817
		93.0	94.9	93.7	98.2	93.8	94.2
	$\text{MLE}_{S(t)}$	5.8036	6.3984	4.5395	5.1898	5.0185	4.2616
		93.7	94.5	92.6	94.0	91.6	93.4
	$\text{Bayes-SE}_{H(t)}$	1.5110	1.5971	2.7170	0.7729	3.1116	2.5530
		95.1	95.0	95.1	95.2	95.0	95.1
	$\text{Bayes-SE}_{S(t)}$	2.7739	3.7096	3.0927	2.6173	3.4539	2.9560
		95.0	95.0	95.1	95.2	95.1	95.1
	$\text{Bayes-LINEX}_{H(t)}$	1.7279	1.4182	2.8872	0.6177	3.0849	3.0551
		95.1	95.0	95.0	95.1	95.1	95.1
	$\text{Bayes-LINEX}_{S(t)}$	2.0595	2.8091	2.4190	2.0281	2.6694	2.4160
		95.0	95.0	95.1	95.1	95.6	95.1
(100,40)	$\text{MLE}_{H(t)}$	8.8324	2.5507	7.9825	1.4024	9.3831	9.8450
		93.8	97.5	94.6	91.2	94.2	93.8
	$\text{MLE}_{S(t)}$	4.8791	6.2269	3.6658	4.1216	5.3115	3.3000
		93.3	95.9	94.2	95.5	90.9	94.8
	$\text{Bayes-SE}_{H(t)}$	2.8590	1.3305	2.8847	1.1419	3.7048	3.5673
		95.1	95.1	95.0	95.1	95.1	95.1
	$\text{Bayes-SE}_{S(t)}$	3.3174	4.4264	3.0509	2.7325	4.8497	2.4660
		95.2	95.1	95.0	95.1	95.2	95.3
	$\text{Bayes-LINEX}_{H(t)}$	2.6606	1.2604	3.1927	1.0507	3.9142	3.8401
		95.1	95.1	95.1	95.1	95.1	95.1
	$\text{Bayes-LINEX}_{S(t)}$	2.5890	3.5871	2.6622	2.3056	3.8490	2.1228
		95.2	95.2	95.0	95.1	95.0	95.3
(200,80)	$\text{MLE}_{H(t)}$	6.1746	1.6980	7.5087	2.2444	7.7986	9.1048
		95.7	99.7	95.1	99.0	95.4	94.5
	$\text{MLE}_{S(t)}$	4.2271	6.1745	2.2338	1.6785	4.8018	2.3358
		94.4	95.2	94.2	92.8	89.0	94.8
	$\text{Bayes-SE}_{H(t)}$	2.5447	1.5721	2.7880	1.2888	4.0728	4.3208
		95.2	95.1	95.0	95.1	95.0	95.2
	$\text{Bayes-SE}_{S(t)}$	3.3952	4.5673	2.0841	1.6710	4.2202	2.0585
		95.1	95.1	95.0	95.1	95.1	95.2
	$\text{Bayes-LINEX}_{H(t)}$	2.3801	1.5310	3.1424	0.8781	4.2040	4.7707
		95.1	95.1	95.0	95.1	95.0	95.2
	$\text{Bayes-LINEX}_{S(t)}$	2.8657	3.9892	1.9713	1.5576	3.1120	1.8937
		95.1	95.1	95.0	95.1	95.1	95.0

set. The MLEs of the parameters are obtained where $\hat{\alpha}=0.7539$ and $\hat{\lambda}=1.6622$. From the original data, one can generate four Type-I progressive hybrid censoring samples with number of stages $r=40$ at time censoring $T=5$ and removed items R_i are assumed to as follows:
Scheme I: $R_1=n-r$ and $R_2=\cdots=R_r=0$.
Scheme II: $R_1=\cdots=R_{\frac{r}{2}}=1$ and $R_{\frac{r}{2}+1}=\cdots=R_r=2$.
Scheme III: $R_1=\cdots=R_{r-1}=0$ and $R_r=n-r$.
Scheme IV: $T=80$ and $R_1=n-r$ and $R_2=\cdots=R_r=0$.

Note that: Scheme IV can be considered as a Type-II progressive censoring scheme, a special case of Type-I progressive hybrid censoring. Tables 5 and 6 give the MLEs of the parameters α, λ, $H(t)$ and $S(t)$ as well as their associated asymptotic confidence interval at proposed schemes for Type-I progressive hybrid censoring samples in the given real data set. Also, Bayes estimates under two loss functions; namely: SE loss function and LINEX loss function, were computed by utilizing the MH algorithm under the non-informative prior, i.e., $a=b=c=d=0$. It is indicated that, while generating samples from the posterior distribution utilizing the MH algorithm, initial values of (α,λ) are

Table 4: The AILs (first-line) and CPs% (second-line) of $S(t)$ and $H(t)$ based on hybrid progressive Type-I censoring schemes at different time censoring and different values of (n,r) for $(\alpha,\lambda) = (1.5,0.5)$.

(n,r)	Method	$q_i = 40\%$			$q_i = 80\%$		
		I	II	III	I	II	III
(50,20)	MLE$_{H(t)}$	0.8602	0.3060	1.9251	0.0877	4.8897	7.4175
		99.0	99.9	99.7	99.8	99.8	99.8
	MLE$_{S(t)}$	0.4933	0.1829	0.5239	0.1307	0.3257	0.9468
		99.0	99.0	99.0	99.0	85.9	99.0
	Bayes-SE$_{H(t)}$	0.2883	0.1652	1.6131	0.0346	2.7145	2.3434
		95.2	95.0	95.1	95.2	95.0	95.1
	Bayes-SE$_{S(t)}$	0.1561	0.0899	0.5768	0.0492	0.9446	0.4657
		99.6	99.8	99.8	99.0	99.5	95.5
	Bayes-LINEX$_{H(t)}$	0.1282	0.0832	0.8398	0.0291	1.2846	1.0312
		95.9	95.3	95.6	95.5	95.1	96.3
	Bayes-LINEX$_{S(t)}$	0.0723	0.0440	0.3848	0.0400	0.8243	0.6106
		99.1	99.8	99.7	99.4	99.8	95.5
(100,40)	MLE$_{H(t)}$	0.4278	0.0718	1.1036	0.0359	3.5805	5.7294
		99.0	99.0	97.7	99.0	99.4	99.0
	MLE$_{S(t)}$	0.2707	0.0544	0.3070	0.0990	0.2710	0.6674
		99.0	99.0	99.0	99.0	83.0	99.0
	Bayes-SE$_{H(t)}$	0.1674	0.0593	1.3809	0.0248	2.1196	1.9885
		95.1	95.2	95.0	95.1	95.1	95.1
	Bayes-SE$_{S(t)}$	0.0883	0.0294	0.5302	0.0338	0.9269	0.3908
		99.8	99.9	99.9	99.9	99.0	97.1
	Bayes-LINEX$_{H(t)}$	0.0779	0.0392	0.8769	0.0177	1.4066	1.2388
		96.7	95.4	95.1	95.2	95.1	95.1
	Bayes-LINEX$_{S(t)}$	0.0405	0.0184	0.4022	0.0226	0.8328	0.5019
		98.3	99.8	99.0	99.0	99.0	95.1
(200,80)	MLE$_{H(t)}$	0.1983	0.0439	0.7497	0.1995	1.9412	3.9325
		99.0	99.0	85.7	99.0	96.2	99.9
	MLE$_{S(t)}$	0.1627	0.0459	0.2127	0.6875	0.4736	0.4816
		99.0	99.0	83.0	99.0	43.7	99.0
	Bayes-SE$_{H(t)}$	0.0928	0.0501	1.3061	0.0985	1.8689	1.7371
		95.4	95.1	95.0	95.1	95.0	95.2
	Bayes-SE$_{S(t)}$	0.0504	0.0261	0.5221	0.1841	0.9039	0.3052
		99.8	99.9	99.0	99.0	99.0	97.4
	Bayes-LINEX$_{H(t)}$	0.0444	0.0255	0.9443	0.0305	1.4610	1.2089
		96.8	95.1	95.0	95.1	95.0	96.1
	Bayes-LINEX$_{S(t)}$	0.0217	0.0119	0.4106	0.0547	0.8446	0.3066
		98.5	99.0	99.9	99.0	99.0	97.1

considered as $(\alpha^{(0)},\lambda^{(0)}) = (\hat{\alpha},\hat{\lambda})$ where $\hat{\alpha},\hat{\lambda}$ are the MLEs of the parameters (α,λ), respectively. Finally, discarded 2000 burn-in samples among the total 10000 samples created from the posterior density, and subsequently obtained Bayes estimates and HPD interval. Further, the estimates of α, λ, $H(t)$ and $S(t)$ are obtained in case of MLEs and Bayesian estimates with their standard errors (St.Es) at a specified time censoring $T = 5$. The convergence of MCMC estimation in case of scheme I of hybrid progressive Type-I censoring can be showed for α and λ in Figure 1.

5 Concluding remarks

In this article, the estimation of the unknown parameters and reliability and hazard functions of an extended exponential distribution under Type-I PHCS is considered. Different estimates for the unknown parameters using ML and Bayesian approaches are computed. The asymptotic confidence intervals are also constructed. Bayes estimates of unknown parameters are developed using MH algorithm with respect to gamma prior distributions under SE and LINEX loss

Table 5: The ML and Bayesian estimates of α, λ, $H(t)$ and $S(t)$ with their St.Es for real data set based on Type-I progressive hybrid censoring under various censoring schemes.

Scheme	Parameter	MLE		Bayes-SE		Bayes-LINEX	
		Estimate	St.E	Estimate	St.E	Estimate	St.E
I	α	0.4170	8.90×10^{-1}	0.3716	9.01×10^{-3}	0.3629	9.11×10^{-3}
	λ	0.1288	2.83×10^{-1}	0.1486	2.02×10^{-4}	0.1483	3.02×10^{-4}
	$H(t)$	0.0157	—	0.0528	—	0.0514	—
	$S(t)$	0.9919	—	0.9733	—	0.9740	—
II	α	0.0765	1.17×10^{-1}	0.0802	8.28×10^{-6}	0.0802	8.28×10^{-6}
	λ	0.7525	$0.13\times10^{+1}$	0.6867	2.21×10^{-3}	0.6656	2.50×10^{-3}
	$H(t)$	0.5233	—	0.0420	—	0.0410	—
	$S(t)$	0.7620	—	0.9763	—	0.9769	—
III	α	3.0887	1.25×10^{-5}	2.8309	8.58×10^{-8}	2.8127	8.60×10^{-2}
	λ	0.0155	2.47×10^{-3}	0.0154	9.13×10^{-8}	0.0153	9.01×10^{-5}
	$H(t)$	0.0002	—	0.0443	—	0.0438	—
	$S(t)$	0.9998	—	0.9782	—	0.9784	—
IV	α	5.8029	1.46×10^{-6}	5.7834	9.20×10^{-2}	5.7012	9.88×10^{-2}
	λ	0.0073	1.14×10^{-3}	0.0074	1.49×10^{-6}	0.0074	1.50×10^{-6}
	$H(t)$	0.0001	—	0.0563	—	0.0552	—
	$S(t)$	0.9995	—	0.6738	—	0.6783	—

Table 6: The 95% two-sided asymptotic/HPD credible intervals of α, λ, $H(t)$ and $S(t)$ for real data set based on Type-I progressive hybrid censoring under various censoring schemes .

Scheme	Parameter	ACI	HPD	
			Bayes SE	Bayes LINEX
I	α	(0.0000, 0.1604)	(0.1935, 0.5523)	(0.1925, 0.5503)
	λ	(0.0000, 2.6842)	(0.1183, 0.1835)	(0.1171, 0.1840)
	$H(t)$	(0.0006, 0.0307)	(0.0460, 0.0596)	(0.0443, 0.0586)
	$S(t)$	(0.9844, 0.9994)	(0.9703, 0.9763)	(0.9708, 0.9772)
II	α	(0.0000, 0.3052)	(0.1935, 0.5523)	(0.1925, 0.5503)
	λ	(0.0000, 3.3651)	(0.1183, 0.1835)	(0.1171, 0.1840)
	$H(t)$	(0.3735, 0.6730)	(0.0407, 0.0433)	(0.0396, 0.0424)
	$S(t)$	(0.6998, 0.8242)	(0.9750, 0.9776)	(0.9754, 0.9784)
III	α	(0.0000, 7.6866)	(2.5820, 3.0510)	(2.5824, 3.0509)
	λ	(0.0001, 0.1005)	(0.0103, 0.0260)	(0.0101, 0.0265)
	$H(t)$	(0.0000, 0.0045)	(0.0000, 0.6149)	(0.0000, 0.6105)
	$S(t)$	(0.9977, 1.0020)	(0.7029, 1.2534)	(0.7049, 1.2519)
IV	α	(0.0010, 7.5080)	(5.3270, 6.3320)	(5.3289, 6.3305)
	λ	(0.0000, 0.0812)	(0.0054, 0.0095)	(0.0052, 0.0099)
	$H(t)$	(0.0000, 0.0019)	(0.0000, 1.9481)	(0.0000, 1.9044)
	$S(t)$	(0.9838, 1.0153)	(0.0000, 8.7201)	(0.7049, 8.6253)

functions. Also considered HPD intervals based on MH procedure are considered. A real data set and simulation study was conducted to examine and compare the performance of the proposed methods for different; sample sizes, censoring times and censoring schemes. From the results we reported some comments observed from numerical results.

–Depended on MSEs, higher values of n lead to better estimates.
–The decreasing in T the estimate is better.
–The performance of Bayes estimates for the parameters α and λ obtained under LINEX loss function is better than the performance of Bayes estimates obtained under SE loss function and the MLEs.

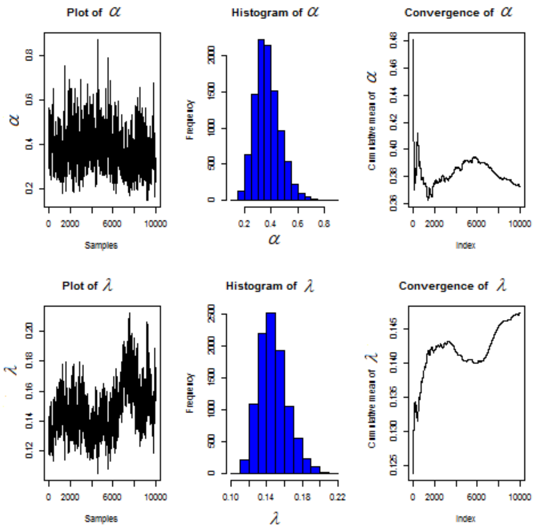

Fig. 1: Convergence of MCMC estimators for α and λ using MH algorithm.

- The Bayes estimates under LINEX loss function average interval lengths (AILs) and associated coverage probabilities (CPs) of HPD intervals are better than those of SE loss function and the MLEs.
- It is notice that the MLEs of $S(t)$ and $H(t)$ are better than the Bayes estimates under both loss functions.
- Furthermore, the performance of the estimates in scheme III is better than other two schemes (I and II).
- The performance of the estimates in censoring times $q = 80\%$ is better than $q = 40\%$.

As a future work, the inferential results discussed in this paper can be performed for some lifetime censoring schemes as the joint progressive Type-I censoring scheme introduced by [24], joint Type-I progressive hybrid censoring scheme proposed by [25] and generalized Type-II hybrid censoring scheme introduced by [26]. Another future work is to investigate the progressive hybrid censored schemes based on maximum product spacing with application to extended exponential distribution as [27].

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Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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Appendix

From (7), the elements of the Fisher information matrix given in (8) will be

$$\mathscr{L}_{11} = \begin{cases} -\dfrac{r}{\hat{\alpha}^2} - \displaystyle\sum_{i=1}^{r} (ln(1+\hat{\lambda}x_i))^2 (1+\hat{\lambda}x_i)^{\hat{\alpha}} [1+R_i], & \text{for case-I;} \\ -\dfrac{D}{\hat{\alpha}^2} - \displaystyle\sum_{i=1}^{D} (ln(1+\hat{\lambda}x_i))^2 (1+\hat{\lambda}x_i)^{\hat{\alpha}} [1+R_i] - R_D^*(1+\hat{\lambda}T)^{\hat{\alpha}} [\ln(1+\hat{\lambda}T)]^2, & \text{for case-II,} \end{cases}$$

$$\mathscr{L}_{22}=\begin{cases}-\dfrac{r}{\hat{\lambda}^2}-(\hat{\alpha}-1)\displaystyle\sum_{i=1}^{r}\dfrac{x_i^2}{(1+\hat{\lambda}x_i)^2}-\hat{\alpha}(\hat{\alpha}-1)\displaystyle\sum_{i=1}^{r}x_i^2(1+\hat{\lambda}x_i)^{\hat{\alpha}-2}[1+R_i], & \text{for case-I;}\\ -\dfrac{D}{\hat{\lambda}^2}-(\hat{\alpha}-1)\displaystyle\sum_{i=1}^{D}\dfrac{x_i^2}{(1+\hat{\lambda}x_i)^2}-\hat{\alpha}(\hat{\alpha}-1)\displaystyle\sum_{i=1}^{D}(x_i^2(1+\hat{\lambda}x_i)^{\hat{\alpha}-2})[1+R_i]-R_D^*[\hat{\alpha}(\hat{\alpha}-1)T^2(1+\hat{\lambda}T)^{\hat{\alpha}-2}], & \text{for case-II,}\end{cases}$$

and

$$\mathscr{L}_{12}=\begin{cases}\displaystyle\sum_{i=1}^{r}\dfrac{x_i}{(1+\hat{\lambda}x_i)}-\displaystyle\sum_{i=1}^{r}x_i(1+\hat{\lambda}x_i)^{\hat{\alpha}-1}(\hat{\alpha}\ln(1+\hat{\lambda}x_i)+1)[1+R_i], & \text{for case-I;}\\ \displaystyle\sum_{i=1}^{D}\dfrac{x_i}{(1+\hat{\lambda}x_i)}-\displaystyle\sum_{i=1}^{D}(x_i(1+\hat{\lambda}x_i)^{\hat{\alpha}-1})(\hat{\alpha}\ln(1+\hat{\lambda}x_i)+1)[1+R_i]-R_D^*(T(1+\hat{\lambda}T)^{\hat{\alpha}-1})[\hat{\alpha}\ln(1+\hat{\lambda}T)+1], & \text{for case-II.}\end{cases}$$