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Discrete Exponentiated Exponential Distribution

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Abstract: In this paper, we attempt to introduce a new lifetime model as a discrete version of the continuous exponentiated exponential distribution which is called discrete exponentiated exponential distribution (DEE). The introduced model contains geometric distribution as a special case. Some basic distributional properties, moments, reliability indices, probability function, characteristic function, and order statistics of the new model are discussed. Estimation of the parameters is illustrated using the maximum likelihood method and moment method. The model with three real data set is also examined.

Keywords: Discrete Exponentiated Exponential Distribution; Failure rate function; Order Statistics; Maximum likelihood estimator; characteristic function.

1 Introduction

In the field of reliability analysis, continuous distribution is more widely used for modelling the lifetime of a component or system. However, the discrete distributions would be better choices for modelling the lifetime of an on/off switch or lifetime of a device that is exposed to shocks, etc. There are many new models that have been widely used in reliability analysis and other related applications to see Meeker and Escobar [1]. The common way to construct discrete distribution has been recognized by Roy and Gupta [2].

One of the simplest ways to implement discretization is briefly explained here. We assume a continuous random variable X has the survival function (SF) $S_X(x) = P(X \geq x)$, and a random variable Y is defined as $Y = [X]$. The probability mass function (PMF) of Y is then given by [3]

$$P(Y{=}y) = P(y \leq X < y+1) = P(X \geq y) - P(X \geq y+1)$$

$$= S_X(y) - S_X(y+1), \text{ for } y = 0, 1, 2, \ldots \tag{1}$$

Indeed, this method has been widely applied to generate new discrete distribution. See for example Roy [3] and [4], Krishna and Pundir [5], Jazi et al. [6], Para and Jan [7,8], Gomez-Deniz and Caderio-Ojeda [9], Nekoukhou et al. [10], Bakouch et al. [11], Abebe [12], Munindra, et al.[13], Nooghabi et al. [14], Gomez-Deniz and Caderio-Ojeda [15], El-Morshedey et al. [16], Chakroborty and Chakroborty [17], and references cited therein.

Gupta [18] introduced exponentiated exponential distribution (denoted by $EE(\alpha,\lambda)$ with the following probability density function (PDF) and SF:

$$f_E(x,\alpha,\lambda) = \alpha\lambda\left(1-e^{-\lambda x}\right)^{\alpha-1} e^{-\lambda x},\ \lambda, x>0,\ S_E(x,\alpha,\lambda) = 1-\left(1-e^{-\lambda x}\right)^{\alpha-1}, \tag{2}$$

Respectively. Where α is the shape parameter and λ is the scale parameter. It should be noted that the exponential family can be derived from EE (α,λ) by setting $\alpha = 1$.

In this paper, a new two-parameter lifetime distribution is introduced, so-called discrete exponentiated exponential distribution, from exponentiated exponential distribution presented in Equation (2).

This new distribution also contains the geometric distribution as a special case.

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The paper is organized as follows: Section 2 introduces the DEE (α, θ) distribution and discusses some of its important features and properties, including the cumulative and failure function, moments, moment generating function, quantile, survival function, entropy, stress-strength, mean residual lifetime and order statistics. In section 3, we provide the maximum likelihood estimation of unknown parameters. We analyse three real data set using the DEED in section 4. Finally, some concluding remarks are given in section 5.

2 Discrete Exponentiated Exponential distribution

Definition. A random variable X is said to have a discrete exponentiated exponential distribution with parameter $\alpha(\alpha > 0)$ and

$\theta = e^{-\lambda},\ 0 < \theta < 1$, if its pmf has the form:

$$P(X=x) = \left(1-\theta^{(x+1)}\right)^{\alpha} - (1-\theta^{x})^{\alpha};\quad x \in \mathbb{N}_0 \tag{3}$$

We denote this distribution as DEE(α,θ).

It is observed that at $\alpha = 1$, we have the geometric distribution as a special case.

Figure 1 illustrates several examples of the probability mass function of DEE (α, θ) distribution for different values of α and θ.

2.1 Cumulative distribution function

The cumulative distribution function CDF of DEE(α,θ) is given by

$$F(x,\alpha,\theta) = 1 - S(x,\alpha,\theta) + P(X=x)$$

$$= \left(1-\theta^{(x+1)}\right)^{\alpha} \tag{4}$$

Where $\alpha(\alpha>0)$ and $\theta=e^{-\lambda},\ 0<\theta<1$.

Monotonic property

It is easy to verify that

$$\frac{f_X(x+1;\,\alpha,\,\theta)}{f_X(x;\,\alpha,\,\theta)} = \frac{\left(1-\theta^{(x+2)}\right)^{\alpha} - \left(1-\theta^{(x+1)}\right)^{\alpha}}{\left(1-\theta^{(x+1)}\right)^{\alpha} - (1-\theta^{x})^{\alpha}};\quad x \in N_0\ ,\ \alpha(\alpha>0)\, and\ \theta = e^{-\lambda},\ ,\ 0<\theta<1$$

.

Is a decreasing function of X. this implies that

$\{f_X(x;\,\alpha,\,\theta)\}^2 > f_X(x+1;\,\alpha,\,\theta)\, f_X(x-1;\,\alpha,\,\theta);\quad x \in N_0 \quad \alpha(\alpha>0)$ and $\theta=e^{-\lambda},\ 0<\theta<1$.

Hence the distribution is log-concave. As a direct consequence of log concavity, the proposed DEE(α,θ) distribution is strongly unimodal; it has an increasing failure rate distribution, and it all its moments.

Furthermore, the quantile function of DEE(α,θ) distribution, say $Q(p)$, from $F(x_p) = p$, is given by

$$x_p = \left\lceil \frac{1}{\ln\theta}\left(1-\frac{p}{\alpha}\right) - 1 \right\rceil \tag{5}$$

Where $\alpha(\alpha>0)$, $\theta=e^{-\lambda},\ 0<\theta<1$ and $0<p<1$.

Where $\lceil n \rceil$ denotes the greatest integer function.

Hence the median can be obtained by putting $p=\frac{1}{2}$ in (5)

$$\text{Med}(X) = \left\lceil \frac{1}{\ln\theta}\left(1-\frac{1}{2\alpha}\right) - 1 \right\rceil$$

Survival function: The survival function of DEE(α,θ) distribution is given by the following

$$S(x,\alpha,\theta) = P(X \geq x) = 1-(1-\theta^{x})^{\alpha};\quad x \in \mathbb{N}_0 \tag{6}$$

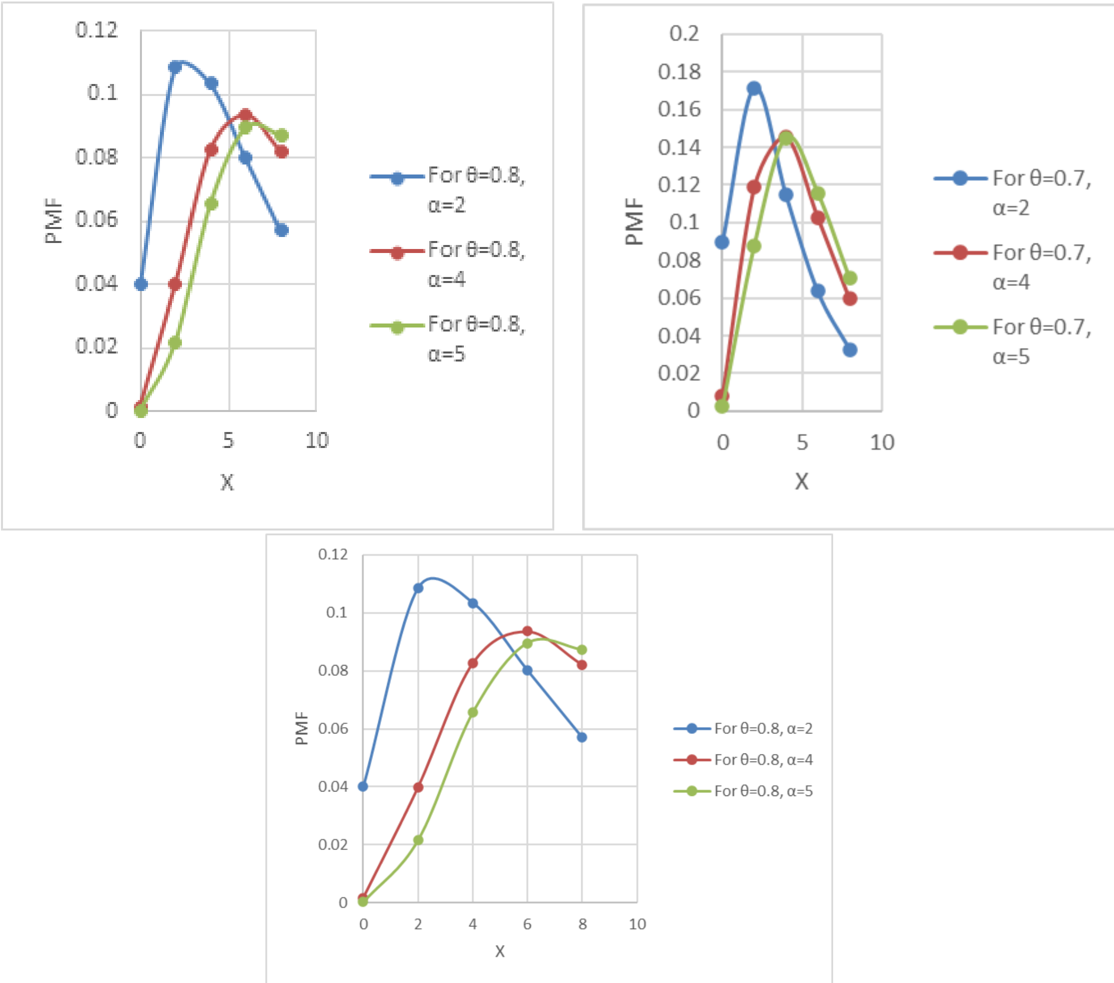

Fig. 1: The PMF of the DEE distribution

Hazard rate, $r(x)$ is given by

$$h(x,\alpha,\theta)=\frac{p(x)}{s(x)}=\frac{\left(1-\theta^{(x+1)}\right)^{\alpha}-(1-\theta^{x})^{\alpha}}{1-(1-\theta^{x})^{\alpha}};\quad x\in\mathbb{N}_0 \tag{7}$$

Figure 2 shows the HRF plots of DEE(α,θ) distribution for different values of α and θ.
Also, the reversed hazard rate function (RHRF) of the DEE(α,θ) distribution can be expressed as follows

$$r(x,\alpha,\theta)=1-(1-\theta^{x})^{\alpha};\quad x\in\mathbb{N}_0 \tag{8}$$

Figure 3 shows the RHRF plots of DEE(α,θ) distribution for various values of α and θ.

3 Different Properties

3.1 Moments and index of dispersion

The r^{th} moment $\mu_r^{'}$ of a discrete exponetiated exponential distribution DEE(α,θ) about the origin is obtained as follows

$$\mu_r^{'}=E[X^r]=\sum_{x=0}^{\infty}x^rP(X=x)$$

$$\mu_r' = \sum_{x=0}^{\infty} x^r [\left(1-\theta^{(x+1)}\right)^{\alpha} - (1-\theta^x)^{\alpha}] \tag{9}$$

The moment generating function (MGF) $M_X(t)$of DEE(α,θ) distribution is computed as follows

$$M_X(t) = E\left[e^{tx}\right] = \sum_{x=0}^{\infty} e^{tx} P(X=x)$$

$$= \sum_{x=0}^{\infty} e^{tx} [\left(1-\theta^{(x+1)}\right)^{\alpha} - (1-\theta^x)^{\alpha}] \tag{10}$$

The r^{th} moment about the origin can also be obtained from the moment generating function. The corresponding moments, means, variance, skewness, and the kurtosis can also be obtained using (10)

The mean (μ) of DEE(α,θ) distribution is as follows

$$\mu_1' = \mu = E[X] = \sum_{x=0}^{\infty} x [\left(1-\theta^{(x+1)}\right)^{\alpha} - (1-\theta^x)^{\alpha}]$$

$$\mu_2' = \mu = E\left[X^2\right] = \sum_{x=0}^{\infty} x^2 [\left(1-\theta^{(x+1)}\right)^{\alpha} - (1-\theta^x)^{\alpha}]$$

Subsequently, the variance(σ^2) is obtained, as follows

$$\operatorname{var}(X) = \sum_{x=0}^{\infty} x^2 [\left(1-\theta^{(x+1)}\right)^{\alpha} - (1-\theta^x)^{\alpha}] - \left(\sum_{x=0}^{\infty} x [\left(1-\theta^{(x+1)}\right)^{\alpha} - (1-\theta^x)^{\alpha}]\right)^2 \tag{11}$$

The 3rd and 4th moments are, respectively are obtained as

$$\mu_3' = E\left[X^3\right] = \sum_{x=0}^{\infty} x^3 [\left(1-\theta^{(x+1)}\right)^{\alpha} - (1-\theta^x)^{\alpha}]$$

$$\mu_4' = E\left[X^4\right] = \sum_{x=0}^{\infty} x^4 [\left(1-\theta^{(x+1)}\right)^{\alpha} - (1-\theta^x)^{\alpha}]$$

The measure of skewness α_3 of DEE(α,θ) distribution is obtained as follows

$$\alpha_3 = \frac{\mu_3' - 2\mu_2'\mu + \mu^3}{\sigma^3}$$

$$= \frac{1}{\sigma^3}\{\left[\sum_{x=0}^{\infty} x^3 [\left(1-\theta^{(x+1)}\right)^{\alpha} - (1-\theta^x)^{\alpha}\right] - 2\mu \sum_{x=0}^{\infty} x^2 [\left(1-\theta^{(x+1)}\right)^{\alpha} - (1-\theta^x)^{\alpha}]\} + \frac{\mu^3}{\sigma^3} \tag{12}$$

The measure of kurtosis α_4 of DEE(α,θ) distribution is obtained as follows

$$\alpha_4 = \frac{\mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4}{\sigma^4}$$

$$\alpha_4 = \frac{1}{\sigma^4}\{\sum_{x=0}^{\infty} x^4 [\left(1-\theta^{(x+1)}\right)^{\alpha} - (1-\theta^x)^{\alpha}]$$

$$-4\mu \sum_{x=0}^{\infty} x^3 [\left(1-\theta^{(x+1)}\right)^{\alpha} - (1-\theta^x)^{\alpha}] + 6\mu^2 \sum_{x=0}^{\infty} x^2 [\left(1-\theta^{(x+1)}\right)^{\alpha} - (1-\theta^x)^{\alpha}] - 3\mu^4\} \tag{13}$$

The probability generating function (PGF), $G(t)$, of DEE(α,θ) distribution is obtained as follows

$$G(t) = E[t^x] = \sum_{x=0}^{\infty} t^x P(X=x)$$

Table 1: The mean of the DDE distribution

α/θ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
2	0.212	0.458	0.758	1.142	1.666	2.4375	3.705	6.222
3	0.304	0.633	1.016	1.496	2.142	3.088	4.639	7.715
4	0.387	0.78	1.223	1.771	2.504	3.578	5.34	8.836
5	0.464	0.906	1.392	1.991	2.794	3.969	5.901	9.732

Table 2: The variance of the DDE distribution

α/θ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
2	0.231	0.608	1.287	2.544	4.973	10.076	22.473	61.856
3	0.316	0.821	1.748	3.464	6.774	13.714	30.54	83.886
4	0.387	1.004	2.15	4.266	8.335	16.856	37.482	102.777
5	0.447	1.166	2.51	4.979	9.72	19.636	43.608	119.408

$$= \sum_{x=0}^{\infty} t^x [\left(1-\theta^{(x+1)}\right)^{\alpha} - (1-\theta^x)^{\alpha}] \tag{14}$$

It is difficult to obtain a closed-form expression for PGF; however, we can compute it numerically. In general, the r^{th} the factorial moment is given by the following

$$\mu_{[r]} = G^r(1) = \frac{\alpha}{\alpha-1} \sum_{x=0}^{\infty} x(x-1)\ldots\ldots\ldots.(x-r+1)[\left(1-\theta^{(x+1)}\right)^{\alpha} - (1-\theta^x)^{\alpha}]$$

The mean μ (also the first factorial moment of DEE(α,θ) distribution can be obtained by calculating the first derivative of probability generating function at t=1 as follows

$$\mu = \mu_{[1]} = \grave{G}(1) = \frac{\alpha}{\alpha-1} \sum_{x=0}^{\infty} x[\left(1-\theta^{(x+1)}\right)^{\alpha} - (1-\theta^x)^{\alpha}]$$

The second factorial moment can be calculated by taking the second derivative of the probability moment generating function at t=1 as follows

$$\mu_{[2]} = G^{[?]}(1) = \frac{\alpha}{\alpha-1} \sum_{x=0}^{\infty} x(x-1) \left[\left(1-\theta^{(x+1)}\right)^{\alpha} - (1-\theta^x)^{\alpha}\right]$$

The variance, the variance (σ^2) of DEE(α,θ)distribution is given by the following

$$var(X) = \sigma^2 = G''(1) + G'(1) - \left(\grave{G''}(1)\right)^2$$

$$var = \frac{\alpha}{\alpha-1} \sum_{x=0}^{\infty} x^2 \left[\left(1-\theta^{(x+1)}\right)^{\alpha} - (1-\theta^x)^{\alpha}\right] - \left(\frac{\alpha}{\alpha-1} \sum_{x=0}^{\infty} x \left[\left(1-\theta^{(x+1)}\right)^{\alpha} - (1-\theta^x)^{\alpha}\right]\right)^2 \tag{15}$$

Characteristic function: The characteristic function (CF), $\phi_X(w)$ of DEE(α,θ) distribution is of the form

$$\phi_X(w) = E\left[e^{\text{iwx}}\right] = \sum_{x=0}^{\infty} e^{\text{iwx}} P(X=x) = \frac{\alpha}{\alpha-1} \sum_{x=0}^{\infty} e^{\text{iwx}} \left[\left(1-\theta^{(x+1)}\right)^{\alpha} - (1-\theta^x)^{\alpha}\right]$$

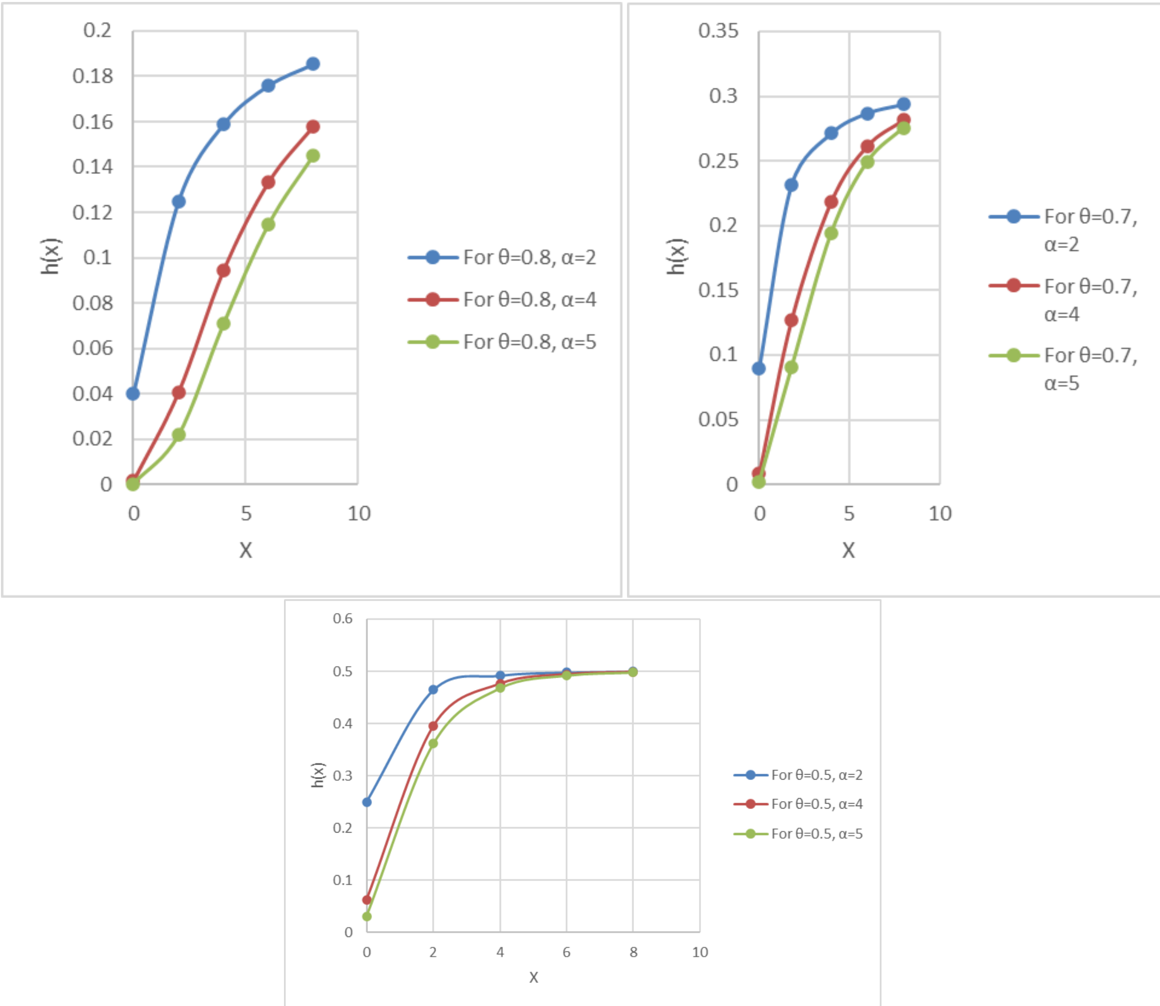

Fig. 2: The HRF of the DDE distribution.

3.2 Stress-Strength (S-S) Analysis*

In the context of reliability, the stress-strength has an essential importance in the technical system. A component fails at the instant when the stress applied to it exceeds the strength, and the component will function effectively if $X_{s^*} > X_s$. Therefore, $R^* = p\,[X_s \leq X_{s^*}]$ is considered to be a measure for component reliability. In this case, the expected reliability(R^*) can be calculated by

$$R^* = p\,[X_s \leq X_{s^*}] = \sum_{x=0}^{\infty} f_{X_s}(x)\, R_{X_{s^*}} \tag{16}$$

If $X_s \sim$ DEE$(\alpha_1,\ \theta_1\ )$and $X_{s^*} \sim$ DEE $(\alpha_2,\ \theta_2)$then R^* can be expressed as follows

$$R^* = \sum_{x=0}^{\infty} 1-(1-\theta_2)^{\alpha_2}[\left(1-\theta_1{}^{(x+1)}\right)^{\alpha_1} - (1-\theta_1^x)^{\alpha_1}] \tag{17}$$

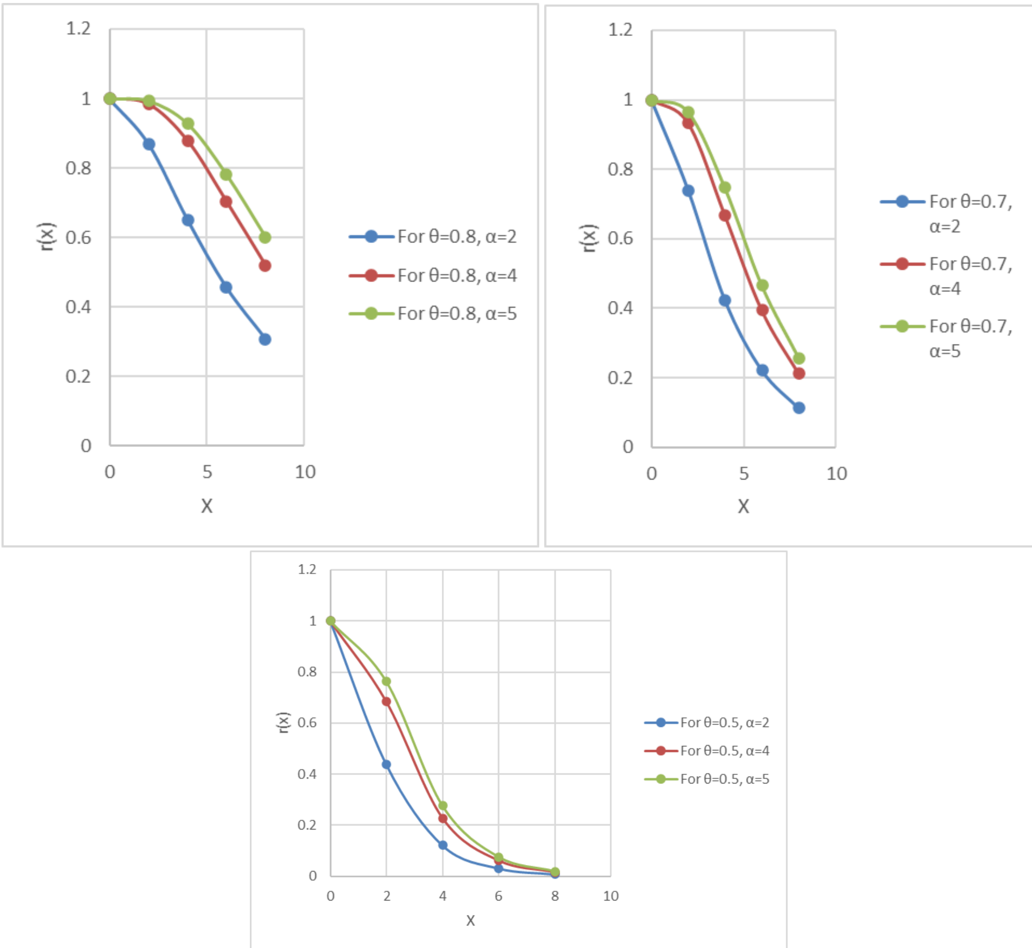

Fig. 3: The RHRF of the EED distribution.

3.3 Mean residual lifetime (MRL) and mean past lifetime (MPL)

Several measures in reliability and survival analysis are presented to study the aging behavior of the components. One of these measures is MPL, say $\varsigma(i)$ which is a helpful tool to model and analyze the burn-in and maintenance policies. In the discrete setting, the MRL is defined as follows

$$\varsigma(i) = E\left[X - \frac{i}{X} \geq i\right] = \frac{1}{R(j)} \sum_{j=i+1}^{l} R(j),\ i \in N_0 \tag{18}$$

Where $0 < l < \infty$. if the RV $X \sim$DEE(α, θ), then the MPL can be expressed as follows

$$\varsigma(i) = \frac{1}{(1-\theta^i)^\alpha} \sum_{j=i+1}^{l} 1 - \left(1-\theta^j\right)^\alpha \tag{19}$$

3.4 Order statistics

Order statistics play an important role in the theoretical and practical aspects of statistics. This importance is shown in statistical inference and non-parametric statistics. let $X_1, X_2, \ldots, X_n$ be a random sample from A DEE(α, θ) distribution, and let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be their corresponding order statistics (OS). Then, the CDF of the i^{th} OS for an integer value of x can be expressed as follows:

$$F_{i:n}(x,\alpha,\theta)=\sum_{k=i}^{n}\binom{n}{k}[F_i(x,\alpha,\theta)]^k[F_i(x,\alpha,\theta)]^{n-k}$$

$$=\sum_{k=i}^{n}\sum_{m=0}^{\alpha k}(-1)^m\binom{n}{k}\binom{\alpha k}{m}\theta^{m(x+1)}[\left(1-\theta^{(x+1)}\right)^{\alpha}]^{n+m-k} \tag{20}$$

Furthermore, the PMF of the k^{th} Os can be expressed as follows

$$f_{k:n}(x,\alpha,\theta,\beta)=\sum_{m=0}^{k-1}\Theta_m^{(n,\alpha(k-1))}\theta^{m(x+1)}[\left(1-\theta^{(x+1)}\right)^{\alpha}]^{n+m-k}$$

$$\left[\left(1-\theta^{(x+1)}\right)^{\alpha}-(1-\theta^{x})^{\alpha}\right] \tag{21}$$

Where $\Theta_m^{(n,k-1)}=(-1)^m\left(\frac{\alpha k}{m}\right)\frac{n!}{(k-1)!(n-k)!}$.
So, the q^{th} moments of $X_{i:n}$ can be written as follows

$$E\left[X_{i:n}^{q}\right]=\sum_{x=0}^{\infty}\sum_{m=0}^{k-1}\Theta_m^{(n,\alpha(k-1))}\theta^{m(x+1)}x^q[\left(1-\theta^{(x+1)}\right)^{\alpha}]^{n+m-k}$$

$$\left[\left(1-\theta^{(x+1)}\right)^{\alpha}-(1-\theta^{x})^{\alpha}\right] \tag{22}$$

Where $\Theta_m^{(n,k-1)}=(-1)^m\left(\frac{\alpha k}{m}\right)\frac{n!}{(k-1)!(n-k)!}$.

3.5 Renyi entropy

The entropy of a random variable X is a measure of uncertainty variation. Renyi entropy plays a vital role in information theory. The Renyi entropy is defined as:

$$I_R(\gamma)=\frac{1}{1-\gamma}\log\sum_x(P(X=x))^{\gamma} \tag{23}$$

Where $\gamma>0$ and $\gamma\neq1$ [19]. For the DEE distribution and when γ is an integer number, we can write

$$\sum_{x=0}^{\infty}(P(X=x))^{\gamma}=\sum_{j=0}^{\gamma}\left[\left(1-\theta^{(j+1)}\right)^{\alpha}-\left(1-\theta^{j}\right)^{\alpha}\right]^{\gamma}$$

$$I_R(\gamma)=\frac{1}{1-\gamma}\log\sum_{j=0}^{\gamma}\left[\left(1-\theta^{(j+1)}\right)^{\alpha}-\left(1-\theta^{j}\right)^{\alpha}\right]^{\gamma} \tag{24}$$

4 Estimation of unknown parameters

In this section, we derive estimates of unknown parameters of DAPW (x,α,θ,β) distribution using

4.1 Maximum likelihood method

Let $X_1,X_2,...,X_n$ denote lifetimes of n independent test units following $DEE(x;\alpha,\theta)$ distribution. Then, the corresponding log-likelihood function is given by the following;

$$p(x)=\left(1-\theta^{(x+1)}\right)^{\alpha}-(1-\theta^{x})^{\alpha}$$

$$L[P(X=x)]=\prod_{i=1}^{n}p(x_i)=\prod_{i=1}^{n}\left(1-\theta^{(x_i+1)}\right)^{\alpha}-(1-\theta^{x_i})^{\alpha} \tag{25}$$

$$l(x,\alpha,\,\theta)=\sum_{i=1}^{n}\ln\left(\left(1-\theta^{(x_i+1)}\right)^{\alpha}-\left(1-\theta^{x_i}\right)^{\alpha}\right) \tag{26}$$

Likelihood equations are then obtained as follows:

$$\frac{\delta l}{\delta\alpha}=\sum_{i=1}^{n}\frac{\left(1-\theta^{(x_i+1)}\right)^{\alpha}\ln\left(1-\theta^{(x_i+1)}\right)-\left(1-\theta^{x_i}\right)^{\alpha}\ln\left(1-\theta^{x_i}\right)}{\left(1-\theta^{(x_i+1)}\right)^{\alpha}-\left(1-\theta^{x_i}\right)^{\alpha}} \tag{27}$$

$$\frac{\delta l}{\delta\theta}=\sum_{i=0}^{n}\frac{-\alpha(x_i+1)\,\theta^{x_i}\left(1-\theta^{(x_i+1)}\right)^{\alpha-1}+\alpha x_i\theta^{x_i-1}\left(1-\theta^{x_i}\right)^{\alpha-1}}{\left(1-\theta^{(x_i+1)}\right)^{\alpha}-\left(1-\theta^{x_i}\right)^{\alpha}}=0; \tag{28}$$

The solution of the above normal equations cannot obtain in closed form, and then it can be solved using a numerical solution. We can compute the second partial derivatives, which are useful to obtain the Fisher's information matrix as follows

$$I_x(\alpha,\theta)=\begin{bmatrix}-E\left[\frac{\partial^2 l}{\partial\alpha^2}\right] & -E\left[\frac{\partial^2 l}{\partial\alpha\partial\theta}\right]\\ -E\left[\frac{\partial^2 l}{\partial\theta\partial\alpha}\right] & -E\left[\frac{\partial^2 l}{\partial\theta^2}\right]\end{bmatrix} \tag{29}$$

One can show that the DEE$(x;\alpha,\theta)$ distribution satisfies the regularity conditions (see, e.g., Ferguson, [20]). Hence, the MLE vector $\left(\hat{\alpha},\hat{\theta}\right)^T$ is consistent and asymptotically normal. That is $I_x^{\frac{1}{2}}(\alpha,\theta)\left(\left(\hat{\alpha},\hat{\theta}\right)^T-(\alpha,\theta)^T\right)$ converges in distribution to a bivariate normal distribution with (vector) mean zero and the identity covariance matrix. The Fisher's information matrix given in Equation (29) can be approximate as follows

$$I_x(\alpha,\theta)=\begin{bmatrix}-\left.\frac{\partial^2 l}{\partial\alpha^2}\right|_{(\hat{\alpha},\hat{\theta})} & -\left.\frac{\partial^2 l}{\partial\alpha\partial\theta}\right|_{(\hat{\alpha},\hat{\theta})}\\ -\left.\frac{\partial^2 l}{\partial\theta\partial\alpha}\right|_{(\hat{\alpha},\hat{\theta})} & -\left.\frac{\partial^2 l}{\partial\theta^2}\right|_{(\hat{\alpha},\hat{\theta})}\end{bmatrix} \tag{30}$$

Where $\hat{\alpha}$ and $\hat{\theta}$ are the MLEs of α and θ, respectively (see Gomez-Deniz, [21]).

4.2 Method of Moments Estimation

The moments' estimates (MME$_s$) of $(\alpha,\,\theta)$are obtained by solving the following equations

$$\sum_{i=1}^{\infty}x_i\left[\left(1-\theta^{(x_i+1)}\right)^{\alpha}-\left(1-\theta^{x_i}\right)^{\alpha}\right]\ =\mu_1{}^{[1]},$$

And

$$\sum_{i=1}^{\infty}x_i^2\left[\left(1-\theta^{(x_i+1)}\right)^{\alpha}-\left(1-\theta^{x_i}\right)^{\alpha}\right]\ =\mu_2{}^{[2]},$$

Where $\mu_1{}^{[1]}$ and $\mu_2{}^{[2]}$ denote the first, the second, and the third sample moments, respectively

5 A simulation study

In this section, we assess the performance of the maximum-likelihood estimate with respect to sample size n. The assessment is based on a simulation study:

I.. Generate 10000 samples of size n from Equation (3). The inversion method is used to generate samples; that is, varites of the discrete exponentiated exponential distribution are generated using

$$X=\left\{\frac{\ln\left(1-u^{\frac{1}{\alpha}}\right)}{\ln\theta}-1\right\};\qquad 0<u<1$$

Where $U\sim U(0,1)$ is a uniform variable on the unit interval;
II. Compute the maximum-likelihood estimates for 10000 samples, say $\hat{\theta}_i$ for $i=1,2,\ldots,10000$,
III. Compute the biases and mean-squared errors given by

$$\text{bias}(n)=\frac{1}{10000}\sum_{i=1}^{10000}(\hat{\theta}_i-\theta_i)$$

And

$$\text{MSE}(n)=\frac{1}{10000}\sum_{i=1}^{10000}(\hat{\theta}_i-\theta_i)^2$$

Table 3: The averages bias and averages MSE for simulated results of ML estimates

sample Size	$\alpha=2.73$					
	$\theta=0.93$		$\theta=0.95$		$\theta=0.99$	
	Bias	MSE	Bias	MSE	Bias	MSE
10	$5.762*10^{-5}$	$3.32*10^{-9}$	$-1.285*10^{-4}$	$1.651*10^{-8}$	$-8.95*10^{-5}$	$-8.01*10^{-9}$
70	$-8.37*10^{-4}$	$7.005*10^{-7}$	$-8.287*10^{-4}$	$6.867*10^{-7}$	$-2.588*10^{-4}$	$6.698*10^{-8}$
130	$-4.019*10^{-3}$	$1.615*10^{-5}$	$-3.129*10^{-3}$	$9.788*10^{-6}$	$-7.456*10^{-4}$	$5.559*10^{-7}$
210	$-4.245*10^{-3}$	$1.802*10^{-5}$	$-3.313*10^{-3}$	$1.098*10^{-5}$	$-7.815*10^{-4}$	$6.108*10^{-7}$
sample Size	$\theta=0.93$					
	$\alpha=2.69$		$\alpha=2.70$		$\alpha=2.73$	
	Bias	MSE	Bias	MSE	Bias	MSE
10	-0.39	0.152	-0.378	0.143	$5.762*10^{-5}$	$3.32*10^{-9}$
70	0.078	$6.011*10^{-3}$	0.055	$3*10^{-3}$	$-8.37*10^{-4}$	$7.005*10^{-7}$
130	0.279	0.078	0.234	0.055	$-4.019*10^{-3}$	$1.615*10^{-5}$
210	0.015	$2.258*10^{-4}$	$-6.695*10^{-3}$	$4.851*10^{-5}$	$-4.245*10^{-3}$	$1.802*10^{-5}$

IV.. The empirical results are given in Table 3.
From Table 3, the following observations can be noted:

– The magnitude of the bias always decreases to zero as $n\longrightarrow\infty$.
– The MSEs always decrease to zero as $n\longrightarrow\infty$. This shows the consistency of the estimators.

6 Data application

Here, we illustrate the superiority of a discrete exponentiated exponential distribution over traditional distributions (Poisson and Geometric) beside new models (discrete Gamma, discrete Weibull, Discrete Logistic, and Discrete Lindley).

We use three real data sets. The first data are given in Table4 consists of survival times in days of 72 guinea pigs. These data are taken from table 6 in [30]. The data have been analyzed by Alshunnar et al. [22] and Ghitany et al. [23]. The data are discrete by definition.

The MLE of (α,θ) values in all these cases has been computed. The Kolmogorov-Smirnov (K-S) distance between the empirical cumulative distribution function and the fitted distribution function in each case and the associated P-value are computed. The result is reported in table 5.

The data set given in table 6 consists of the 2003 final examination marks of 48 slow space students in mathematics in the Indian Institute of Technology at Kanpur. The data set is taken from Gupta and Kundu [24].

The MLE of (α,θ) values in all these cases have been computed. The Kolmogorov-Smirnov (K-S) distance between the empirical cumulative distribution function and the fitted distribution function in each case and the associated P-value are computed. The result is reported in table 7.

Table 4: Data set 1.

12	15	22	24	24	32	32	33	34	38	38	43	44	48
52	53	54	54	55	56	57	58	58	59	60	60	60	60
61	62	63	65	65	67	68	70	70	72	73	75	76	76
81	83	84	85	87	91	95	96	98	99	109	110	121	127
129	131	143	146	146	175	175	211	233	258	258	263	297	341
341	376												

Table 5: Fitted estimates for data set 1.

Distribution	p(x)	Parameter Estimates	p value	K-S statistics
Discrete Exponentiated Exponential Distribution	(3)	$\alpha = 2.739, \theta = .983$	0.0524	0.1567
Poisson	$\lambda^x e^{-\lambda} x$	$\lambda = 99.8194$	$9.5313x10^{(-22)}$	0.5755
Geometric	$p(1-p)^x$	p=.0099	0.002	0.216
Discrete Weibull	$q^{x^\beta} - q^{(x+1)^\beta}$	$q = .9532, \beta = .9020$	$1.0821x10^{(-24)}$	0.614
Discrete Gamma	$\frac{\gamma(\alpha,\beta(x+1))}{\Gamma(\alpha)} - \frac{\gamma(\alpha,\beta(x))}{\Gamma(\alpha)}$	$\alpha = .9853, \beta = .0125$	$4.1283x10^{(-6)}$	0.2966

Table 6: Data set 2

29	25	50	15	13	27
15	18	7	7	8	19
12	18	5	21	15	86
21	15	14	39	15	14
70	44	6	23	58	19
50	23	11	6	34	18
28	34	12	37	4	60
20	23	40	65	19	31

Table 7: Fitted estimates for data set 2.

Distribution	p(x)	Parameter Estimates	p value	K-S statistics
Discrete Exponentiated Exponential Distribution	(3)	$\alpha = 2.63, \theta = .963$	0.3353	0.1338
Poisson	$\lambda^x e^{-\lambda}/x!$	$\lambda = 25.8958$	$2.4013x10^{(-7)}$	0.3998
Geometric	$p(1-p)^x$	p=.0372	0.0145	0.2223
Discrete Weibull	$q^{x^\beta} - q^{(x+1)^\beta}$	$q = .6488, \beta = .6758$	$2.9221x10^{(-24)}$	0.7419
Discrete Gamma	$\frac{\gamma(\alpha,\beta(x+1))}{\Gamma(\alpha)} - \frac{\gamma(\alpha,\beta(x))}{\Gamma(\alpha)}$	$\alpha = .8098, \beta = .0350$	$2.6082x10^{(-4)}$	0.2993

The third data set given in table 8 consists of remission times in weeks for 20 leukemia patients randomly assigned to a certain treatment. It is taken from pages 346 of Lawless [25]. The data have been analyzed recently by Damien and Walker [26] and Kottas [27].

Table 8: Data set 3

1	3	3	6	7	7	10	12	14	15
18	19	22	26	28	29	34	40	48	49

The MLE of (α, θ) values in all these cases have been computed. The Kolmogorov-Smirnov (K-S) distance between the empirical cumulative distribution function and the fitted distribution function in each case and the associated P-value are computed. The result is reported in table 9.

For the first, second and the third data sets, the discrete exponentiated exponential distribution provides the only acceptable p-values. The distribution plots suggest that the discrete exponentiated exponential distribution produces the best fit among the competitor distributions.

On the basis of the tabulated results, we conclude that the discrete exponentiated exponential distribution distribution provides the best fit as compared to its sub models.

Table 9: Fitted estimates for data set 3.

Distribution	p(x)	Parameter Estimates	p value	K-S statistics
Discrete Exponentiated Exponential Distribution	[3]	α=2.107,θ =.93	0.4888	0.1812
Poisson	$\lambda^x e^{-\lambda}$/x	λ =19.5500	0.01	0.3523
Geometric	$p(1-p)^x$	p=.0487	0.7436	0.1447
Discrete Weibull	$q^{x^\beta} - q^{(x+1)^\beta}$	q=.3682,$\beta = .3603$	$3.5869x10^{(-10)}$	0.7233
Discrete Gamma	$\frac{\gamma(\alpha,\beta(x+1))}{\Gamma(\alpha)}$-$\frac{\gamma(\alpha,\beta(x))}{\Gamma(\alpha)}$	$\alpha = .5623, \beta = .0310$	0.0541	0.2909

Table 10: Some statistical measures for data sets

Data set	Mean	Median	Std. Deviation	Variance	Skewness	Std. Error of Skewness	Kurtosis	Std. Error of Kurtosis	Minimum	Maximum
I	99.82	70	81.118	6580.12	1.835	0.283	2.894	0.559	12	376
II	25.9	19.5	18.605	346.138	1.375	0.343	1.608	0.674	4	86
III	19.55	16.5	14.699	216.05	0.707	0.512	-0.436	0.992	1	49

7 Concluding remarks

In this, paper, a new two-parameter for lifetime modeling which is organized from continuous exponentiated exponential distribution is introduced, so-called discrete exponentiated exponential distribution DEE (α,θ) distribution. The proposed distribution contains the geometric distribution as a special case. The failure rate of the new model is decreasing. Some important probabilistic properties of this distribution are studied in detail. The unknown parameters of the DEE distribution are estimated using two methods, namely, the moments method and the maximum likelihood method. The flexibility of the DEE distribution has been empirically proven by using three real-life data sets. The DEE distribution has proven to show efficiency in fitting data better than some existing distributions. Finally, we believe that the presented distribution will benefit a wide range of applications including reliability, physics and so on.

Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this article.

References

[1] W. Q .Meeker and L.A. Escobar, Statistical methods for reliability data, Wily, New York, (1998).
[2] D. Roy and R. P. Gupta, Classifications of discrete lives, Micro Electronics and Reliability, 32, 1459–1473 (1992).
[3] S. Kotz, C. B. Read, N. Balakrishnan and B. Vidakovic, Wiley Encyclopedia of Statistical Sciences, Wily, New York, (2006).
[4] D. Roy, Reliability measures in the discrete bivariate set up and related characterization results for a bivariate geometric distribution, Multivariate Analysis, 46,362–373 (1993).
[5] H. Krishna and P. Singh Pundir, Discrete burr and discrete pareto distributions, Statistical Methodology, 6(2), 177–188 (2009).
[6] M.A. Jazi, B. A. Para and T. R. Jan, Discretization of burr-type III distribution, Journal of Reliability and Statistical Studies, ISSN, 87–94 (2014).
[7] B. A. Para and T. R. Jan, On discrete three parameter burr type XII and discrete lomax distributions and their applications to model count data from medical science, Biometrics & Biostatistics International Journal (2016).
[8] B. A. Para and T. R. Jan, Discrete inverse weibull minimax distribution: properties and applications, Journal of Statistics applications & Probability, 6(1), 205–218 (2017).
[9] E. Gomez-Deniz and E. Calderın-Ojeda, The discrete lindley distribution: properties and applications, Journal of Statistical Computation and Simulation, 81(11), 1405–1416 (2011).
[10] V. Nekoukhou and H. Bidram, Exponential-Discrete generalized exponential distribution: A New Compound Model,. Journal of Statistical Theory and Applications, 15(2), 169, (2016).
[11] S.Hassan Bakouch, A. Jazi, and S. Nadarajah, A new discrete distribution, Statistics, 48(1), 200-240, (2014).
[12] B. Abebe, A discrete lindely distribution with applications in biological sciences, biometrics and biostatistics international journal, 7(1),.48-52, (2018).
[13] B. Munindra, R.S. Kirshna and J. Junali, A study on two parameters discrete quasi lindely distribution and its derived distribution, International journal of mathematical archive, 6(12), 149-156, (2015).
[14] M. Shafaei Nooghabi, G. Reza Mohtashami Borzadaran and A. Hamid Rezaei Roknabadi, Discrete modified weibull METRON, 69, 207, (2011).

[15] E. Gomez-Deniz and E. Calderın-Ojeda, The compound DGL/erlang distribution in the collective risk model revistada de methods cuanttativos para la econom la y la empresa, (16). 121–142, (2013).
[16] M. El-Morshedy, M. S. Eliwa and H. Nagy, A new two-parameter exponentiated discrete lindley distribution: properties estimation and applications, Journal of Applied Statistics, DOI: 10.1080/02664763.2019.1638893,(2019).
[17] S. Chakraborty and D. Chakraborty, A new discrete probability distribution with integer support on, Communication in Statistics-Theory and Methods, 45(2), 492-505. (2016).
[18] D. R. Gupta and D. Kundu, Exponentiated exponential family: an alternative to gamma and weibull distribution biometrical journal, 43(1), 117-130, (2001).
[19] Rrnyi, On measures of entropy and information. In Fourth Berekely Symposium on Mathematical Statistics and Probability, 1, 547-561, (1961).
[20] T. S. Ferguson, A course in large sample theory. London: Chapman and Hall, (1996).
[21] E. Gomez-Deniz, Another generalization of the geometric distribution, Test 19, 399-415,(2010).
[22] F. S. Alshunnar, M. Z. Raqab and D. Kundu, On the comparison of the fisher information of the log-normal and generalized rayleigh distributions. Appl. Stat, 37, 391-404, (2010).
[23] M. E. Ghitany, F. Alqallaf, D. K. Al-Mutairi and H. A. Hussain, A two-parameter weighted lindely distribution and its applications to survival Data, Math. Comput. Simul, 81, 1190-1201, (2011).
[24] R. D. Gupta and D. Kundu, (2009), A new class of weighted exponential distribution. Statistics, 43, 621-634
[25] J.F. Lawless, Statistical models and methods for lifetime data. John Wiley and Sons, Inc., New York (1982, 2003).
[26] P. Damien and S. Walker, A Bayesian non-parametric comparison of two treatments. Scand. J. Stat, 29, 51-56 (2002).
[27] A. Kottas, Nonparametric Bayesian survival analysis using mixtures of weibull distributions, J. Statist. Plan. Inference, 136, 578-596, (2006).
[28] D. Roy, The discrete normal distribution. Communications in Statistics-Theory and Methods, 32(10), 1871–1883, (2003).
[29] D. Roy, Discrete Rayleigh distribution. IEEE Transactions on Reliability, 53(2), 255–260, (2004).
[30] T. Bjerkedal, Acquisition of resistance in guinea pigs infected with different doses of virulent tubercle baccili Am. J.Hyg, 72, 130-148, (1960).
[31] K. Jayakumar and M. G. Babu, Discrete weibull geometric distribution and its properties, Commun. Statist. –Theory Methods, 47(7), 1767–1783, (2018).
[32] S. Chakraborty and D. Chakravarty, A new discrete probability distribution with integer support on $(-\infty, infty)$. Commun Statist. -Theory Methods, 45(2), 492–505, (2016).
[33] K. Jayakumar and M. G. Babu, Discrete additive weibull geometric distribution. J. Stat. Theory Appl., 18(1), 33–45 (2019).