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A New Generalization of Pranav Distribution with Application to Model Real Life Data

Maryam Mohiuddin[1,], R. Kannan[1], Atheer Ahmed Alrashed[2] and M. Fasil[3]*

[1]Department of Statistics, Annamalai University, Annamalai nagar, Tamil Nadu-608002, India
[2]Department of Business Administration, Applied Science University, Kingdom of Bahrain
[3]Research Scholar CSIR- Structural Engineering Research Centre, Taramani, Chennai, India

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Abstract: In this study, a two-parameter Pranav distribution was considered for generalization and the proposed model is known as Alpha Power two-parameter Pranav distribution. The new distribution is obtained by using the technique Alpha power transformation which was developed by Mahdavi and Kundu. Some statistical properties along with reliability measures of the model are derived and discussed. Model parameters are estimated with the help of maximum likelihood estimation approach. The proposed model is subjected to real-life data and compared to two-parameter Pranav distribution, two-parameter Lindley distribution, Power Lindley distribution, Pranav distribution, Akash distribution, Ishita distribution, Sujatha distribution, Shanker distribution, Lindley distribution and exponential distribution. The results obtained reveal that proposed model provides the best fit than other competing distributions.

Keywords: Alpha Power Transformation, Two-Parameter Pranav Distribution, Moments, Order Statistics, Entropies, Maximum Likelihood Estimation.

1 Introduction

Over the last few years, the researchers have proposed various methods for generating new continuous distributions to model lifetime data. In practice, however, many classical statistical distributions were found to be inadequate in modeling certain types of datasets. Some of these provide good results for different data sets which are considered lifetime distributions. But some of them does not provide good results for the considered datasets. Applications in various fields of science indicate that classical distributions are not enough to model data sets. So it is necessary to expand some models to model real-life data sets. Several studies have been studied that the extensions to existing distributions provide better flexibility in modeling such types of data in practice by introducing additional parameter(s). The purpose of the additional parameter is to vary the tail weight to the existing distributions, thereby inducing it by skewness. Many authors have extended classical distribution to apply in many fields. A detailed survey of methods for generating distributions has been studied by Lee et al. Most of these distributions are special cases of the T-X class studied by Alzaatreh et al. [1]. This class of distributions extends to some recent families such as the beta-G pioneered by Eugene et al. [2], the gamma-G defined by Zografos and Balakrishnan [3], the Kw-G family proposed by Cordeiro and Castro [4] and the Weibull-G introduced by Bourguignon et al. [5] and so on.

Kus [6] introduced the two-parameter lifetime distribution with decreasing failure rates. The parameters of the distribution were obtained by EM algorithm using maximum likelihood estimation and their asymptotic variances, covariance was also obtained. Dey [7] discussed a model that provides the extension to generalization of the exponential distribution with its application to Ozone data. Peng et al. [8] introduced a new three-parameter lifetime distribution, the exponentiated Lindley geometric distribution, which exhibits increasing, decreasing, uni-modal, and bathtub-shaped hazard rates. Corderio et al.

*Corresponding author e-mail: masmariam7@gmail.com

[9] obtained a lifetime model known as the Lindley-Weibull model for modeling the lifetime data. The model comprises three parameters that accommodates uni-model, bathtub and monotone failure rates. Ibrahim [10] proposed a four-parameter Topp Leone exponentiated Weibull distribution to model survival data. Some properties of the model are obtained. The method of MLE is established for the estimation of the unknown parameters. The goodness of fit was demonstrated by using the survival times of patients suffering from acute myelogenous leukaemia. Rasheed [11] obtained Topp-Leone Dagum distribution to model survival times of patients and failure times of air conditioning system. Rasheed [12] introduced a Log component Rayleigh distribution intending to transform the reliability properties of the distribution. The basic Log transformation technique was applied to the component Rayleigh distribution. The properties of the distribution were studied. The performance of the model is verified by fitting reliability data sets and found superiority over Gompertz, Nadarajah Haghighi, Weibull, Type II Topp Leone Inverse Rayleigh, Lomax and Log Dagum distributions.

In this work, a new distribution using the APT method based on Pranav distribution has been proposed. Alpha power transformed Lindley distribution established by Dey et al. to model earthquake data. Aldahlan [13] introduced the Alpha power log-logistic distribution with application to real-life data. Nassar et al. [14] defined the Alpha power Weibull distribution using the APT method. Elbatal et al. [15] discussed the new Alpha power transformed family of distributions, its properties and applications to the Weibull model. Ceren and Selen [16] obtained the Alpha power inverted Exponential distribution, to model real-life data. Alpha power transformed inverse Lomax distribution was suggested by Hashmi et al. [17] and studied the properties and different methods for estimating the parameters. Alpha power transformed Pareto distribution was introduced by Sakthivel [18] and studied various properties of the distribution with the application. Alpha power transformed Weibull-G was proposed by Golam Kibria [19] with application to failure data. The distribution is generated by combining the two families of distributions APT-G family and the Weibull-G family. The statistical properties of the APTW-G are derived and discussed. The APTW-G reduces to Alpha power transformed Weibull exponential distribution, Alpha power transformed Weibull Rayleigh distribution and Alpha power transformed Weibull Lindley distribution. Nasiru et al. [20] discussed the Alpha power transformed Frechet distribution and its properties with the application. Alpha power transformed Aradhana distribution was introduced by Maryam and Kannan [21] for modeling a real lifetime data. Some properties of the model were investigated. The maximum likelihood estimation was employed for parameter estimation. Alpha power transformed Garima distribution was proposed by Maryam and Kannan [22] with application to real-life data. Some properties of the model were investigated. These including survival function, hazard rate, moments, entropy, order statistics etc. The maximum likelihood estimation and least square estimation was investigated for the estimation of parameters.

Pranav distribution is a one-parameter distribution obtained by Shukla [23] its properties and application in biological and engineering fields. The properties of the distribution include moments, hazard function, mean deviations, Bonferroni and Lorenz curves and order statistics are obtained. A simulation study of the proposed distribution was also performed to know the behaviour of the parameters. For estimating the parameters of the model, the method of moments and the method of maximum likelihood estimation has been discussed. The goodness of fit of the proposed distribution has been investigated and compared with other lifetime distributions of one parameter. A modification of Pranav distribution (TPD) was introduced by Odom et al. [24] by using the quadratic rank transmutation map approach (QRT). The properties of said distribution are derived and discussed. The parameters of the said distribution are determined by the method of maximum likelihood estimation. The two-parameter Pranav distribution with properties and application was introduced by Edith Umeh and Ibenegbu [25]. The two-parameter Pranav distribution is the combination of exponential (θ) and gamma ($4, \theta$). The two-parameter Pranav distribution can be applied to numerous lifetime data such as medical, biosciences, engineering and so many others. A discrete Pranav distribution and its applications were proposed by Berhane Abebe and Shukla [26].

The aim of the study is to introduce a new distribution from Pranav distribution by using α-power transformation technique which was developed by Mahdavi and Kundu [27]. The newly generated model is called Alpha power two-parameter Pranav (AP2PP) distribution. The introduced model encompasses the behaviour of and provides better fits than some well-known lifetime distributions, such as exponential, Weibull, gamma, lognormal distributions. We are motivated to introduce the AP2PP distribution as

1. It is capable of modeling decreasing and upside-down bathtub-shaped hazard rates.

2. It can be viewed as a suitable model for fitting skewed data which may not be properly fitted by other common distributions and can be used in a variety of problems in different areas such as biomedicine, failure data, insurance and earthquake data.

2 Proposed Model: Definition and Properties

The probability density function (pdf) of two-parameter Pranav distribution is given by

$$g(x;\theta,\beta)=\left(\frac{\theta^4}{\beta\theta^4+6}\right)\left(\beta\theta+x^3\right)e^{-\theta x} \quad ; \quad x>0,\ \theta>0,\beta\geq 0 \tag{1}$$

and its cumulative distribution function (cdf) is given by

$$G(x;\theta,\beta)=1-\left(1+\frac{\theta x(\theta^2x^2+3\theta x+6)}{\beta\theta^4+6}\right)e^{-\theta x} \quad ; \quad x>0,\theta>0,\ \beta\geq 0 \tag{2}$$

The probability density function (pdf) of the Alpha power transformed model proposed by Mahdavi and Kundu [27] is defined by

$$f(x)=\left(\frac{\log\alpha}{\alpha-1}\right)g(x)\alpha^{G(x)} \quad ; \quad \alpha\neq 1,\alpha>0 \tag{3}$$

The corresponding cumulative distribution function (cdf) is defined as

$$F(x)=\frac{\alpha^{G(x)}-1}{\alpha-1} \quad ; \ \alpha\neq 1,\alpha>0 \tag{4}$$

By using the APT method, substituting the equations (1) and (2) in equation (3) The probability density function (pdf) of the Alpha power two-parameter Pranav (AP2PP) distribution with parameters $\theta>0,\ \alpha>0,\beta\geq 0$ is obtained as

$$f(x;\theta,\alpha,\beta)=\frac{\log\alpha}{\alpha-1}\left(\frac{\theta^4}{\beta\theta^4+6}\right)\left(\beta\theta+x^3\right)\alpha^{1-\left(1+\frac{\theta x(\theta^2x^2+3\theta x+6)}{\beta\theta^4+6}\right)e^{-\theta x}}e^{-\theta x};x>0,\alpha,\theta>0,\beta\geq 0,\alpha\neq 1 \tag{5}$$

Substituting the equation (2) in equation (4), the cumulative distribution function (cdf) of Alpha power two-parameter Pranav (AP2PP) Distribution is obtained as

$$F(x;\theta,\alpha,\beta)=\frac{\alpha^{1-\left(1+\frac{\theta x(\theta^2x^2+3\theta x+6)}{\beta\theta^4+6}\right)e^{-\theta x}}-1}{\alpha-1} \quad ; \ x>0,\alpha,\theta>0,\beta\geq 0,\alpha\neq 1 \tag{6}$$

The behaviour of pdf and cdf of the distribution for different values of the parameter (θ, α, β) are shown graphically in figure 1 and figure 2.

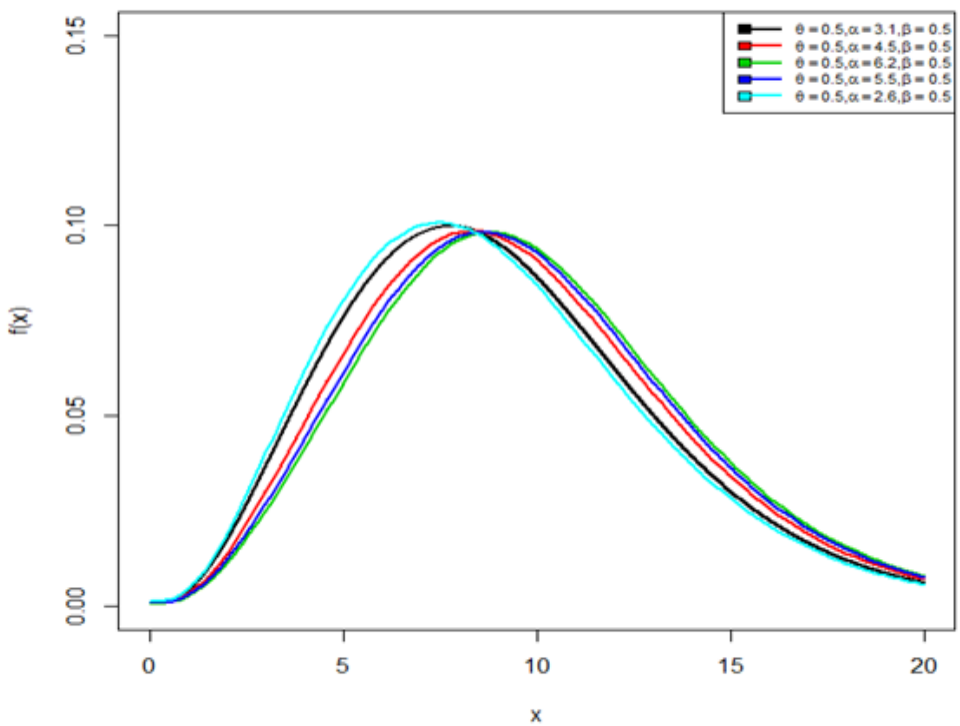


Fig.1: pdf plot of AP2PP distribution

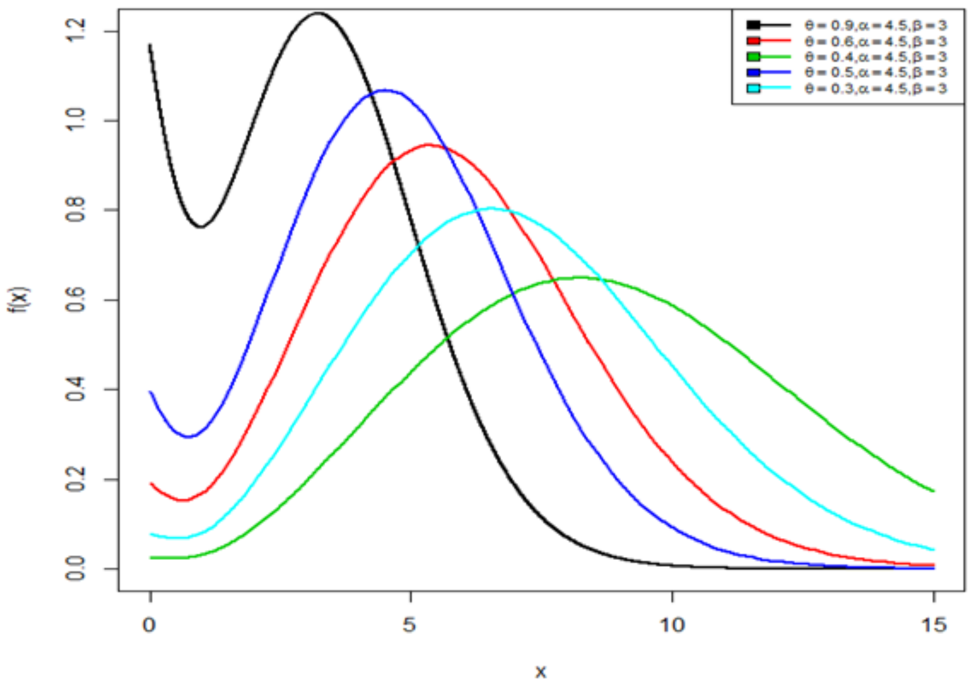


Fig.1b: pdf plot of AP2PP distribution.

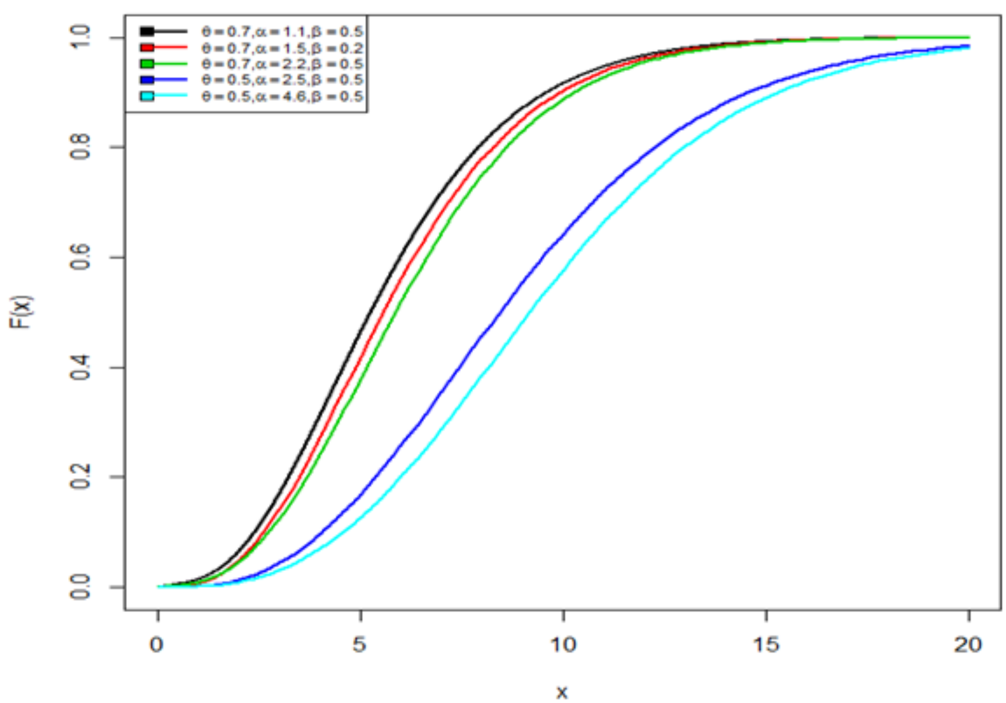


Fig.2: cdf plot of AP2PP distribution.

3 Reliability Measures of Alpha Power Two-Parameter Pranav Distribution

In this section, we will provide survival function and hazard function for AP2PP distribution.

3.1 Reliability Function

The survival function or the reliability function of the AP2PP distribution is given by

$$S(x;\theta,\alpha,\beta)=\frac{\alpha-\alpha^{1-\left(1+\frac{\theta x(\theta^2x^2+3\theta x+6)}{\beta\theta^4+6}\right)e^{-\theta x}}}{\alpha-1};\qquad \alpha\neq 1 \tag{7}$$

3.2 Hazard Function

The hazard function of the AP2PP distribution is specified by

$$h(x;\theta,\alpha,\beta)=\frac{\alpha^{-\left(1+\frac{\theta x(\theta^2x^2+3\theta x+6)}{\beta\theta^4+6}\right)e^{-\theta x}}}{1-\alpha^{-\left(1+\frac{\theta x(\theta^2x^2+3\theta x+6)}{\beta\theta^4+6}\right)e^{-\theta x}}}\left(\frac{\theta^4}{\beta\theta^4+6}\right)\left(\beta\theta+x^3\right)e^{-\theta x}\log\alpha\ ;\quad \alpha,\theta,\beta>0,\alpha\neq 1 \tag{8}$$

The behaviour of survival function and hazard rates of the distribution for different values of the parameter (θ, α, β) are shown graphically in figure 3 and figure 4.

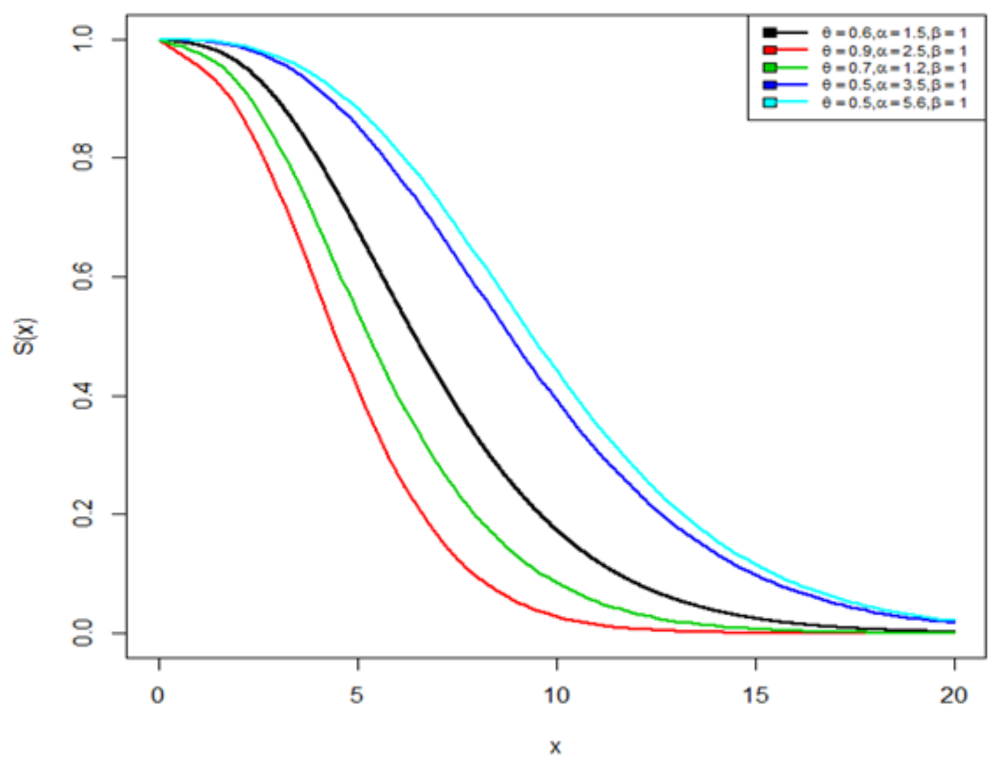


Fig.3: survival plot of AP2PP distribution.

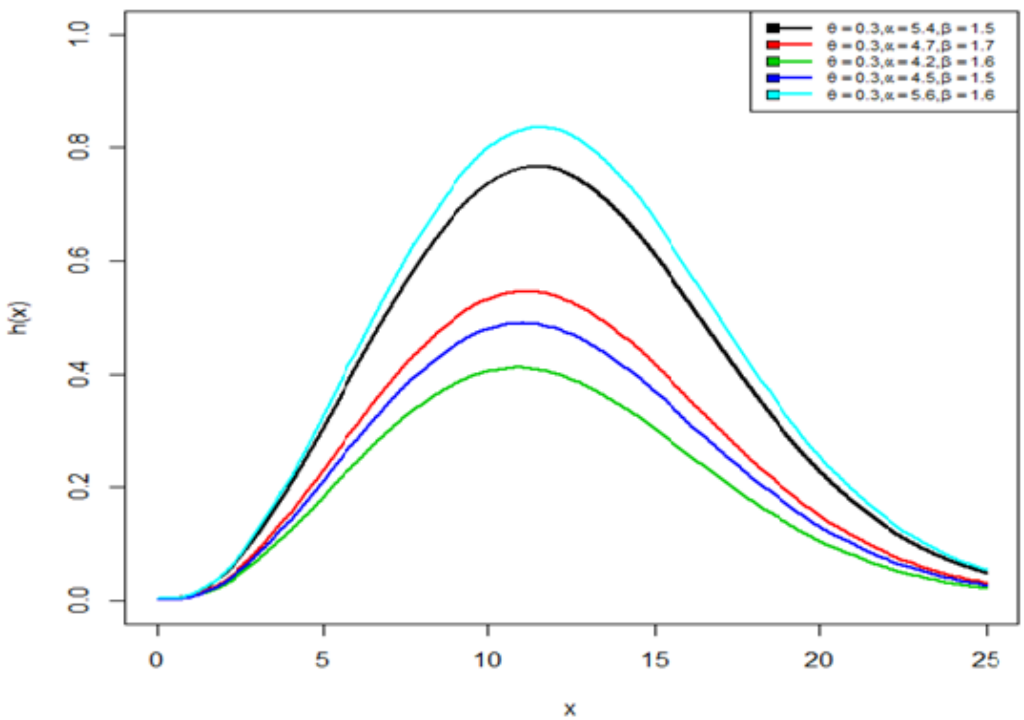


Fig.4: hazard plot of AP2PP distribution

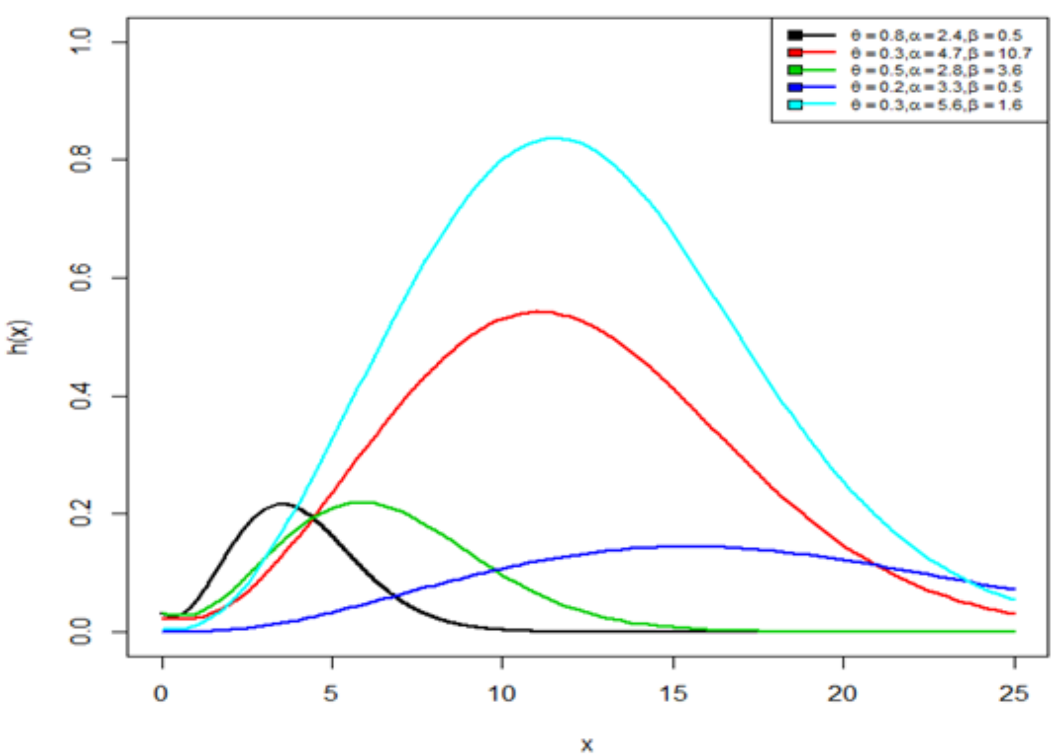


Fig.4b: hazard plot of AP2PP distribution.

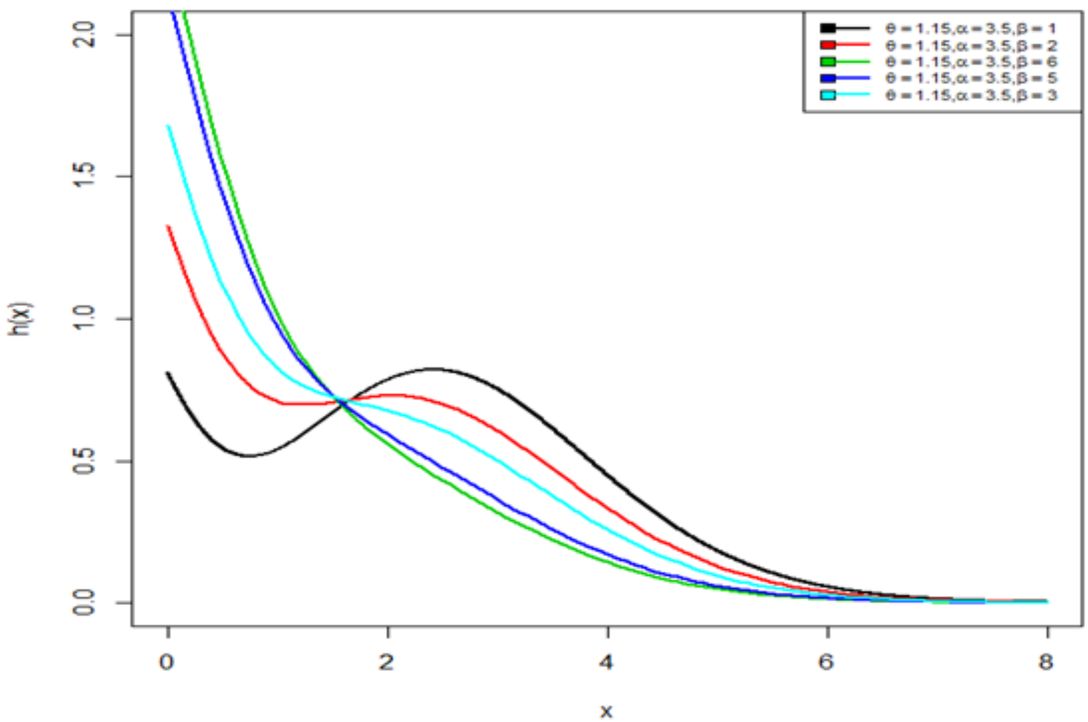


Fig.4c: hazard plot of AP2PP distribution.

The figure 4a shows that the hazard rate of the proposed distribution tends to normal distribution when, $\theta < 1$, β, α > 1.

The figure 4c reveals that the failure rate of the distribution is decreasing for,$\beta > 1$, keeping $\theta.\alpha$ constant.

4 Statistical Properties of Alpha Power Two Parameter Pranav Distribution

In this section, moments and moment generating function of Alpha power two-parameter Pranav distribution is obtained.

4.1 Moments

Let X denotes the random variable of defied on AP2PP distribution with parameters θ, α, β then r^{th} order moment $E(X^r)$ of the AP2PP distribution can be obtained as

$$E(X^r) = \mu_r' = \int_0^\infty x^r f(x)dx$$

$$= \int_0^\infty x^r \left(\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta^4}{\beta\theta^4+6}\right)\left(\beta\theta+x^3\right)e^{-\theta x}\,\alpha^{1-\left(1+\frac{\theta x(\theta^2x^2+3\theta x+6)}{\beta\theta^4+6}\right)e^{-\theta x}} \right)dx$$

$$= \left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta^4}{\beta\theta^4+6}\right)\int_0^\infty x^r\left(\beta\theta+x^3\right)e^{-\theta x}\,\alpha^{1-\left(1+\frac{\theta x(\theta^2x^2+3\theta x+6)}{\beta\theta^4+6}\right)e^{-\theta x}}dx \quad (9)$$

To obtain moments, we use following series representation as

$$\alpha^{-z} = \sum_{i=0}^{\infty} \frac{(-\log\alpha)^i}{i!} z^i \tag{10}$$

After simplification, the r^{th}moment of the AP2PP distribution can be obtained as

$$E(X^r) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\left(\frac{i!}{(i-j)!(j-k)!(k-l)!l!}\right)\left(\frac{1}{\beta\theta^4+6}\right)^{j+1}\frac{(-\log\alpha)^i}{i!}\left(\frac{\alpha\log\alpha}{(\alpha-1)}\right)2^l 3^k \tag{11}$$

$$\left(\frac{\theta^3\,\Gamma(r+3j-k-l+1)}{\theta^r\,(i+1)^{(r+3j-k-l+1)}}+\frac{\Gamma(r+3j-k-l+4)}{\theta^{r+1}\,(i+1)^{(r+3j-k-l+4)}}\right)$$

Putting $r = 1$ in equation (11), we get mean of AP2PP distribution which is given by

$$E(X) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\left(\frac{i!}{(i-j)!(j-k)!(k-l)!l!}\right)\left(\frac{1}{\beta\theta^4+6}\right)^{j+1}\frac{(-\log\alpha)^i}{i!}\left(\frac{\alpha\log\alpha}{(\alpha-1)}\right)2^l 3^k$$

$$\left(\frac{\theta^3\,\Gamma(3j-k-l+2)}{\theta(i+1)^{(3j-k-l+2)}}+\frac{\Gamma(3j-k-l+5)}{\theta^2\,(i+1)^{(3j-k-l+5)}}\right)$$

Putting $r = 2$, we obtain the second moment as

$$E(X^2) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\left(\frac{i!}{(i-j)!(j-k)!(k-l)!l!}\right)\left(\frac{1}{\beta\theta^4+6}\right)^{j+1}\frac{(-\log\alpha)^i}{i!}\left(\frac{\alpha\log\alpha}{(\alpha-1)}\right)2^l 3^k$$

$$\left(\frac{\theta^3\,\Gamma(3j-k-l+3)}{\theta^2\,(i+1)^{(3j-k-l+3)}}+\frac{\Gamma(3j-k-l+6)}{\theta^3\,(i+1)^{(3j-k-l+6)}}\right)$$

Therefore

$$\text{Variance} = \mu_2' - \left(\mu_1'\right)^2$$

4.2 *Moment Generating Function and Characteristic Function of AP2PP Distribution*

Let X have an Alpha power two-parameter Pranav distribution, then the MGF of X is obtained as

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x)dx$$

We use Taylor's series,

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} \left(1 + tx + \frac{(tx)^2}{2!} + \dots \right) f(x)dx$$

$$= \int_0^{\infty} \sum_{r=0}^{\infty} \frac{t^r}{r!} x^r f(x)dx$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r f(x)dx$$

$$= \sum_{r=0}^{\infty} \mu_r' \frac{t^r}{r!}$$

$$M_X(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{j} \sum_{l=0}^{k} \sum_{r=0}^{\infty} \frac{t^r}{r!} \left(\frac{i!}{(i-j)!(j-k)!(k-l)!l!} \right) \left(\frac{1}{\beta\theta^4 + 6} \right)^{j+1} \frac{(-\log\alpha)^i}{i!} \left(\frac{\alpha \log \alpha}{(\alpha - 1)} \right) 2^l 3^k$$

$$\left(\frac{\theta^3 \Gamma(r + 3j - k - l + 1)}{\theta^r (i+1)^{(r+3j-k-l+1)}} + \frac{\Gamma(r + 3j - k - l + 4)}{\theta^{r+1} (i+1)^{(r+3j-k-l+4)}} \right)$$

Similarly, the characteristic function of AP2PP distribution can be obtained as

$$\Phi_X(t) = M_X(it)$$

$$M_X(it) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{j} \sum_{l=0}^{k} \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \left(\frac{i!}{(i-j)!(j-k)!(k-l)!l!} \right) \left(\frac{1}{\beta\theta^4 + 6} \right)^{j+1} \frac{(-\log\alpha)^i}{i!} \left(\frac{\alpha \log \alpha}{(\alpha - 1)} \right) 2^l 3^k$$

$$\left(\frac{\theta^3 \Gamma(r + 3j - k - l + 1)}{\theta^r (i+1)^{(r+3j-k-l+1)}} + \frac{\Gamma(r + 3j - k - l + 4)}{\theta^{r+1} (i+1)^{(r+3j-k-l+4)}} \right)$$

5 Mean Residual Life and Mean Waiting Time

5.1 Mean Residual Life

we suppose that X is a continuous random variable with survival function (7), then the mean residual life is the expected additional lifetime that the component has survived after a fixed time point t. The mean residual life function is given by

$$\mu(t) = \frac{1}{S(t)} \left(E(t) - \int_0^t x f(x; \alpha, \theta) dx \right) - t \tag{12}$$

where,

$$\int_0^t x f(x)dx = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\left(\frac{i!}{(i-j)!(j-k)!(k-l)!l!}\right)\left(\frac{1}{\beta\theta^4+6}\right)^{j+1}\frac{(-\log\alpha)^i}{i!}\left(\frac{\alpha\log\alpha}{(\alpha-1)}\right)2^l 3^k$$

(13)

$$\left(\left(\gamma(\theta(i+1))t,(3j-k-l+2)\right)+\left(\frac{1}{\theta(i+1)}\right)^3\gamma((\theta(i+1))t,(3j-k-l+5))\right)$$

$$E(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\left(\frac{i!}{(i-j)!(j-k)!(k-l)!l!}\right)\left(\frac{1}{\beta\theta^4+6}\right)^{j+1}\frac{(-\log\alpha)^i}{i!}\left(\frac{\alpha\log\alpha}{(\alpha-1)}\right)2^l 3^k$$

(14)

$$\left(\frac{\theta^3\,\Gamma(3j-k-l+2)}{\theta(i+1)^{(3j-k-l+2)}}+\frac{\Gamma(3j-k-l+5)}{\theta^2\,(i+1)^{(3j-k-l+5)}}\right)$$

Substituting equations (7), (11), (13) and (14), in equation (12), we obtain mean residual life of Alpha power two-parameter Pranav distribution as

$$\mu(t)=\frac{(\alpha-1)\left(\begin{array}{l}E(t)-\displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\left(\frac{i!}{(i-j)!(j-k)!(k-l)!l!}\right)\left(\frac{1}{\beta\theta^4+6}\right)^{j+1}\frac{(-\log\alpha)^i}{i!}\left(\frac{\alpha\log\alpha}{(\alpha-1)}\right)2^l 3^k\\ \left(\left(\gamma(\theta(i+1))t,(3j-k-l+2)\right)+\left(\frac{1}{\theta(i+1)}\right)^3\gamma((\theta(i+1))t,(3j-k-l+5))\right)\end{array}\right)}{\alpha-\alpha^{1-\left(1+\frac{\theta x(\theta^2x^2+3\theta x+6)}{\beta\theta^4+6}\right)e^{-\theta x}}}-t$$

5.2 Mean Waiting Time

The mean waiting time represents the waiting time elapsed since the failure on an object on condition that this failure have occurred in the interval $[0.t]$. The mean waiting time of X.

$$\bar{\mu}(t)=t-\frac{1}{F(t)}\int_0^t x\,f(x;\theta,\alpha,\beta)dx \tag{15}$$

Using equations (4) and (13), in equation (15), we obtain the mean waiting time of AP2PP distribution as follows

$$\bar{\mu}(t) = t - \frac{\alpha - 1}{\alpha^{1-\left(1+\frac{\theta x(\theta^2 x^2+3\theta x+6)}{\beta\theta^4+6}\right)e^{-\theta x}} - 1} \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\left(\frac{i!}{(i-j)!(j-k)!(k-l)!l!}\right)\left(\frac{1}{\beta\theta^4+6}\right)^{j+1}\frac{(-\log\alpha)^i}{i!}\left(\frac{\alpha\log\alpha}{(\alpha-1)}\right)$$

$$2^l 3^k \left(\left(\gamma\left(\theta(i+1)\right)t, (3j-k-l+2)\right) + \left(\frac{1}{\theta(i+1)}\right)^3 \gamma\left(\left(\theta(i+1)\right)t, (3j-k-l+5)\right)\right)$$

6 Entropy Measures of Alpha Power Two-Parameter Pranav Distribution

The concept of entropy is important in different areas such as probability and statistics, physics, communication theory and economics. Entropy or information theory deals with the study of transmission, processing, utilization, and extraction of information. Entropy measures quantify the diversity, uncertainty or randomness of a system. Entropy of a random variable X is a measure of variation of uncertainty.

6.1 Renyi Entropy

The Renyi entropy is important in ecology and statistics as the index of diversity. It was proposed by Renyi [28]. The Renyi entropy of order ξ for a random variable X is given by

$$RE_X(\xi) = \frac{1}{1-\xi}\log\left(\int_0^{\infty} f^{\xi}(x)dx\right)$$

Where $\xi > 0$ and $\xi \neq 0$ then we have,

$$RE_X(\zeta) = \frac{1}{1-\zeta}\log\int_0^{\infty}\left(\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta^4}{\beta\theta^4+6}\right)\left(\beta\theta+x^3\right)e^{-\theta x}\alpha^{1-\left(1+\frac{\theta x(\theta^2x^2+3\theta x+6)}{\beta\theta^4+6}\right)e^{-\theta x}}\right)^{\zeta} dx$$

$$RE_X(\zeta) = \frac{1}{1-\zeta}\log\left(\left(\frac{\log\alpha}{\alpha-1}\right)^{\zeta}\left(\frac{\theta^4}{\beta\theta^4+6}\right)^{\zeta}\int_0^{\infty}\left(\beta\theta+x^3\right)^{\zeta} e^{-\theta\zeta x}\alpha^{-\zeta\left(1+\frac{\theta x(\theta^2x^2+3\theta x+6)}{\beta\theta^4+6}\right)e^{-\theta x}}\right)dx$$

After simplifying the expression, we get

$$RE_X(\zeta) = \frac{1}{1-\zeta}\log\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{m=0}^{\infty}\binom{i}{j}\binom{j}{k}\binom{k}{l}\binom{\zeta}{m}\left(\frac{\alpha\log\alpha}{\alpha-1}\right)^{\zeta}\left(\frac{(-\zeta\log\alpha)^i}{i!}\right)$$

$$\left(\frac{\theta}{\beta\theta^4+6}\right)^{j+\zeta} 2^l 3^k \frac{\theta^{2j+4\zeta-k-l-m}\Gamma(3j-k-l-3m+1)}{\left(\theta(\zeta+i)\right)^{(3j-k-l-3m+1)}}$$

6.2 Tsallis Entropy

A generalization of Boltzmann-Gibbs a statistical mechanics initiated by Tsallis has focused a great deal of attention. This generalization of B-G statistics was first developed by introducing the mathematical expression of Tsallis entropy [29] for a continuous random variable is defined as follows

$$S_\lambda = \frac{1}{\lambda-1}\left(1-\int_0^\infty f^\lambda(x)dx\right)$$

$$S_\lambda = \frac{1}{\lambda-1}\left(1-\int_0^\infty\left(\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta^4}{\beta\theta^4+6}\right)\left(\beta\theta+x^3\right)e^{-\theta x}\alpha^{1-\left(1+\frac{\theta x(\theta^2x^2+3\theta x+6)}{\beta\theta^4+6}\right)e^{-\theta x}}\right)^\lambda dx\right)$$

After simplifying the expression, we get

$$S_\lambda = \frac{1}{\lambda-1}\left(1-\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{m=0}^{\infty}\binom{i}{j}\binom{j}{k}\binom{k}{l}\binom{\lambda}{m}\left(\frac{\alpha\log\alpha}{\alpha-1}\right)^\lambda\left(\frac{\left(-\lambda\log\alpha\right)^i}{i!}\right)2^l 3^k\left(\frac{\theta}{\beta\theta^4+6}\right)^{j+\lambda}\frac{\theta^{2j+4\lambda-k-l-m}\Gamma(3j-k-l-3m+1)}{\left(\theta(\lambda+i)\right)^{(3j-k-l-3m+1)}}\right)$$

7 Bonferroni and Lorenz Curves

The Bonferroni curve was proposed by Bonferroni [30]. The Bonferroni curve have wide range of applicability in various fields includes economics to study income and poverty, reliability, demography, insurance, and medicine. The Bonferroni curve is defined as

$$B(p) = \frac{1}{\mu_1' p}\int_0^q x f(x)dx$$

and

$$L(p) = p\,B(p) = \frac{1}{\mu_1'}\int_0^q x\, f(x)\, dx$$

Here, we define the first raw moment as

$$\mu_1' = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\left(\frac{i!}{(i-j)!(j-k)!(k-l)!l!}\right)\left(\frac{1}{\beta\theta^4+6}\right)^{j+1}\frac{(-\log\alpha)^i}{i!}\left(\frac{\alpha\log\alpha}{(\alpha-1)}\right)2^l 3^k\left(\frac{\theta^3\,\Gamma(3j-k-l+2)}{\theta(i+1)^{(3j-k-l+2)}}+\frac{\Gamma(3j-k-l+5)}{\theta^2\,(i+1)^{(3j-k-l+5)}}\right)$$

and $q = F^{-1}(p)$.

We have

$$B(p) = \frac{1}{\mu_1' p}\int_0^q x\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta^4}{\beta\theta^4+6}\right)\left(\beta\theta+x^3\right)e^{-\theta x}\alpha^{1-\left(1+\frac{\theta x(\theta^2x^2+3\theta x+6)}{\beta\theta^4+6}\right)e^{-\theta x}}dx$$

After simplification, we get

$$B(p) = \frac{1}{\mu_1' p}\left(\frac{i!}{(i-j)!(j-k)!(k-l)!l!}\right)\left(\frac{\theta}{\beta\theta^4+6}\right)^{j+1}\frac{\left(-\log\alpha\right)^i}{i!}\left(\frac{\alpha\log\alpha}{(\alpha-1)}\right)2^l\,3^k\,\theta^{2j-k-l+3}$$

$$\left(\frac{1}{\theta(i+1)}\right)^{3j-k-l+2}\left(\left(\gamma\left(\theta(i+1)\right)q,(3j-k-l+2)\right)+\left(\frac{1}{\theta(i+1)}\right)^3\gamma\left(\left(\theta(i+1)\right)q,(3j-k-l+5)\right)\right)$$

Lorenz curve was developed by Lorenz [31] and can be obtained for the Alpha power two-parameter Pranav distribution as

$$L(p)=\frac{1}{\mu_1'}\left(\frac{i!}{(i-j)!(j-k)!(k-l)!l!}\right)\left(\frac{\theta}{\beta\theta^4+6}\right)^{j+1}\frac{(-\log\alpha)^i}{i!}\left(\frac{\alpha\log\alpha}{(\alpha-1)}\right)2^l\,3^k\,\theta^{2j-k-l+3}$$

$$\left(\frac{1}{\theta(i+1)}\right)^{3j-k-l+2}\left(\left(\gamma\left(\theta(i+1)\right)q,(3j-k-l+2)\right)+\left(\frac{1}{\theta\left(i+1\right)}\right)^3\gamma\left(\left(\theta(i+1)\right)q,(3j-k-l+5)\right)\right)$$

8 Order Statistics

Let $X_{(1)},X_{(2)},X_{(3)}...X_{(n)}$ be the order statistics of a random variable $X_1,X_2,X_3...X_n$ drawn from the continuous population with pdf $f_X(x)$ and cdf $F_X(x)$, then the pdf of r^{th} order statistic $X_{(r)}$ is given by

$$f_{X(r)}(x)=\frac{n!}{(r-1)!\,(n-r)!}f_X(x)\left[F_X(x)\right]^{r-1}\left[1-F_X(x)\right]^{n-r} \tag{16}$$

Using equations (5) and (6) in equation (16), the pdf of r^{th} order statistic $X_{(r)}$ of the AP2PP distribution is given by

$$f_{X(r)}(x)=\frac{n!}{(r-1)!\,(n-r)!}\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta^4}{\beta\theta^4+6}\right)\left(\beta\theta+x^3\right)e^{-\theta x}\,\alpha^{1-\left(1+\frac{\theta x(\theta^2x^2+3\theta x+6)}{\beta\theta^4+6}\right)e^{-\theta x}}$$

$$\left(\frac{\alpha^{1-\left(1+\frac{\theta x(\theta^2x^2+3\theta x+6)}{\beta\theta^4+6}\right)e^{-\theta x}}-1}{\alpha-1}\right)^{r-1}\left(1-\frac{\alpha^{1-\left(1+\frac{\theta x(\theta^2x^2+3\theta x+6)}{\beta\theta^4+6}\right)e^{-\theta x}}-1}{\alpha-1}\right)^{n-r}$$

Therefore the pdf of the higher order statistic $X_{(n)}$ can be obtained as

$$f_{X(n)}(x)=n\left(\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta^4}{\beta\theta^4+6}\right)\left(\beta\theta+x^3\right)e^{-\theta x}\,\alpha^{1-\left(1+\frac{\theta x(\theta^2x^2+3\theta x+6)}{\beta\theta^4+6}\right)e^{-\theta x}}\right)\left(\frac{\alpha^{1-\left(1+\frac{\theta x(\theta^2x^2+3\theta x+6)}{\beta\theta^4+6}\right)e^{-\theta x}}-1}{\alpha-1}\right)^{n-1}$$

the pdf of the first order statistic $X_{(1)}$ can be obtained as

$$f_{X(1)}(x)=n\left(\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta^4}{\beta\theta^4+6}\right)\left(\beta\theta+x^3\right)e^{-\theta x}\,\alpha^{1-\left(1+\frac{\theta x(\theta^2x^2+3\theta x+6)}{\beta\theta^4+6}\right)e^{-\theta x}}\right)\left(1-\frac{\alpha^{1-\left(1+\frac{\theta x(\theta^2x^2+3\theta x+6)}{\beta\theta^4+6}\right)e^{-\theta x}}-1}{\alpha-1}\right)^{n-1}$$

9 Parameter Estimation of Alpha Power Two-Parameter Pranav Distribution

This section gives the parameter estimation of Alpha power two-parameter Pranav distribution via maximum likelihood estimation method.

9.1 Maximum Likelihood Estimation

Let $x_1, x_2, x_3 \ldots\ldots x_n$ be a random sample taken from AP2PP distribution (α, θ, β) with pdf (3) then we can write the likelihood function as

$$L(x;\theta,\alpha,\beta)=\prod_{i=1}^{n}\left(\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta^4}{\beta\theta^4+6}\right)\left(\beta\theta+x_i^3\right)e^{-\theta x_i}\ \alpha^{1-\left(1+\frac{\theta x_i(\theta^2x_i^2+3\theta x_i+6)}{\beta\theta^4+6}\right)e^{-\theta x_i}}\right)$$

The log likelihood function is obtained as

$$\log L=n\left(\log(\log\alpha)-\log(\alpha-1)+4\log\theta-\log\left(\beta\theta^4+6\right)\right)+\sum_{i=1}^{n}\log\left(\beta\theta+x_i^3\right)-\theta\sum_{i=1}^{n}x_i$$

$$+\log\alpha\sum_{i=1}^{n}\left(1-\left(1+\frac{\theta x_i(\theta^2x_i^2+3\theta x_i+6)}{\beta\theta^4+6}\right)e^{-\theta x_i}\right) \qquad (17)$$

The MLE's of α, θ, β can be obtained by differentiating equation (17) with respect to α , θ ,β and satisfy the normal equations.

$$\frac{\partial\log L}{\partial\alpha}=\frac{n}{\alpha\log\alpha}-\frac{n}{\alpha-1}+\frac{1}{\alpha}\sum_{i=1}^{n}\left(1-\left(1+\frac{\theta x_i(\theta^2x_i^2+3\theta x_i+6)}{\beta\theta^4+6}\right)e^{-\theta x_i}\right) \qquad (18)$$

$$\frac{\partial\log L}{\partial\theta}=\frac{4n}{\theta}-\frac{4\beta\theta^3 n}{\left(\beta\theta^4+6\right)}+\sum_{i=1}^{n}\frac{\beta}{(\beta\theta+x_i^3)}-\sum_{i=1}^{n}x_i+\log\alpha\sum_{i=1}^{n}\left(1-\left(1+\frac{\theta x_i(\theta^2x_i^2+3\theta x_i+6)}{\beta\theta^4+6}\right)e^{-\theta x_i}\right) \qquad (19)$$

$$\frac{\partial\log L}{\partial\beta}=-\frac{n\theta^4}{(\beta\theta^4-6)}+\sum_{i=1}^{n}\frac{\theta}{(\beta\theta+x_i^3)}-\log\alpha\sum_{i=1}^{n}\left(\frac{\theta^5x_i(\theta^2x_i^2+3\theta x_i+6)}{(\beta\theta^4+6)^2}\right)e^{-\theta x_i} \qquad (20)$$

Because of the complicated form of the above likelihood equations, algebraically it is very difficult to solve the system of nonlinear equations. The function optim () BGBS in stat package is used for the estimation. Therefore we use R [32] for estimating the required parameters.

10 Goodness of Fit of Alpha Power Two Parameter Pranav Distribution

In this section, we perform an application of the Alpha power two-parameter Pranav distribution to prove empirically its potentiality. The data set is used for the validation of the proposed model. To compare the fit of Alpha power two-parameter Pranav model with other competing models, we consider model selection criterion, Akaike information criterion (AIC), Bayesian information criterion (BIC) and Akaike information criterion corrected (AICC).

Data set

The data set represents the lifetime data relating to times of 105 patients who were diagnosed with hypertension and received at least one treatment related to hypertension in the hospital where death is the event of interest [33]. The data set is as follows:

45, 37, 14, 64, 67, 58, 67, 55, 64, 62, 9, 65, 65,43, 13, 8, 31, 30, 66, 9, 10, 31, 31, 31, 46, 37, 46, 44,45, 30, 26, 28, 45, 40, 47, 53, 47, 41, 39, 33, 38, 26, 22, 31, 46, 47, 66, 61, 54, 28, 9, 63, 56, 9, 49, 52, 58, 49, 53, 63, 16, 67, 61, 67, 28, 17, 31, 46, 52, 50, 30, 33, 13, 63, 54, 63, 56, 32, 33, 37, 7, 56, 1, 67, 38, 33, 22, 25, 30, 34, 53, 53, 41, 45, 59, 59, 60, 62, 14, 57, 56, 57, 40, 44, 63.

For comparison of distributions, we consider criteria such as AIC (Akaike Information Criterion), AICC (Akaike Information Criterion Corrected) and BIC (Bayesian Information Criterion). The better distribution corresponds to lesser AIC, AICC and BIC values.

$$AIC = 2k \; - 2\log L \qquad BIC = k\log n \; - 2\log L \qquad AICC = AIC + \frac{2k(k+1)}{n-k-1}$$

where, k is the number of parameters in the statistical model, n is the sample size, and -2log$L$ is the maximized value of the log-likelihood function and are shown in Table 1.

10.1 Akaike Information Criterion (AIC)

The AIC is a technique developed by Hirotuge Akaike [34] (Japanese Statistician) based on a sample to estimate the likelihood of the model to estimate future values. The smaller the AIC value, the better the model fits. The good model is the one that has minimum AIC among all the other models.

10.2 Bayesian Information Criterion (BIC)

The BIC is a technique developed by Schwarz [35] that's used for model selection criteria. This technique measures the trade-off between the model fit and the complexity of the model. A lower BIC value indicates a better fit.

10.3 Akaike Information Criterion Corrected (AICC)

The AICC introduced by Hurvich and Tsai [36] found that AICC which is the modified version of AIC is not asymptotically efficient. Hence they suggested a biased corrected version of AIC known as Akaike Information Corrected Criterion statistics.

Table 1: MLEs and statistic measures for data set

Distribution	ML Estimates	AIC	BIC	AICC
Alpha Power Two Parameter Pranav	$\hat{\alpha}$=1.1325701 $\hat{\beta}$=3.1409934 $\hat{\theta}$=1.1724847	909.637	917.5989	909.8746
Two Parameter Pranav	$\hat{\theta}$= 0.0915615 $\hat{\alpha}$=83.664476	922.9262	928.2341	923.0439
Two Parameter Lindley	$\hat{\theta}$=0.0443123 $\hat{\alpha}$=0.0000044	950.2435	955.8185	950.3612
Power Lindley	$\hat{\theta}$=0.0068519 $\hat{\beta}$=1.4992002	921.5439	926.8518	921.7815

Two Parameter Akash	$\hat{\theta}$=0.0699417 $\hat{\alpha}$=10.720725	931.4603	937.0353	931.5779
Pranav	$\hat{\theta}$=0.0948905	931.4013	934.0553	931.4401
Akash	$\hat{\theta}$= 0.0710344	931.3277	933.9817	931.3665
Ishita	$\hat{\theta}$=0.0711589	932.4164	935.0704	932.4552
Sujatha	$\hat{\theta}$=0.0702382	931.278	933.9319	931.3168
Shanker	$\hat{\theta}$=0.0473755	947.3519	950.0059	947.3907
Exponential	$\hat{\theta}$=0.0237182	997.7186	1000.373	997.7574

On the basis of model selection techniques, the proposed AP2PP distribution has the lesser AIC, BIC and AICC values as compared to other distributions. Therefore, we can conclude that the Alpha power two-parameter Pranav (AP2PP) distribution leads to a better fit compared to two-parameter Pranav, two parameter Akash, Power Lindley, two parameter Lindley, Pranav, Akash, Sujatha, Ishita, Lindley, Shanker, and Exponential distributions.

Descriptive statistics

Table 2: Descriptive Statistics of Hypertensive data

Data	N	Mean	Standard Deviation	Standard error
Hypertensive data	105	42.162	17.612	1.719

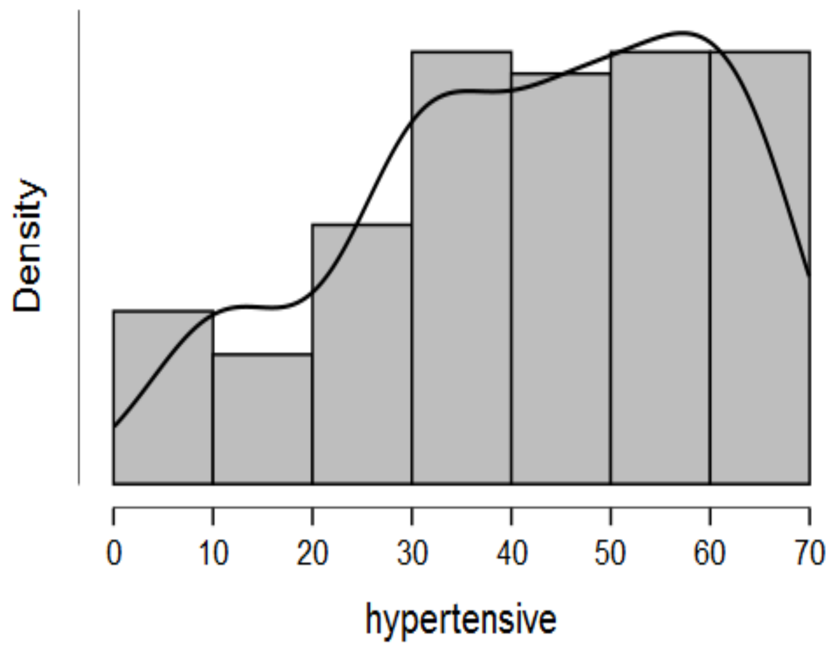


Fig.5: Probability density plot of hypertensive data

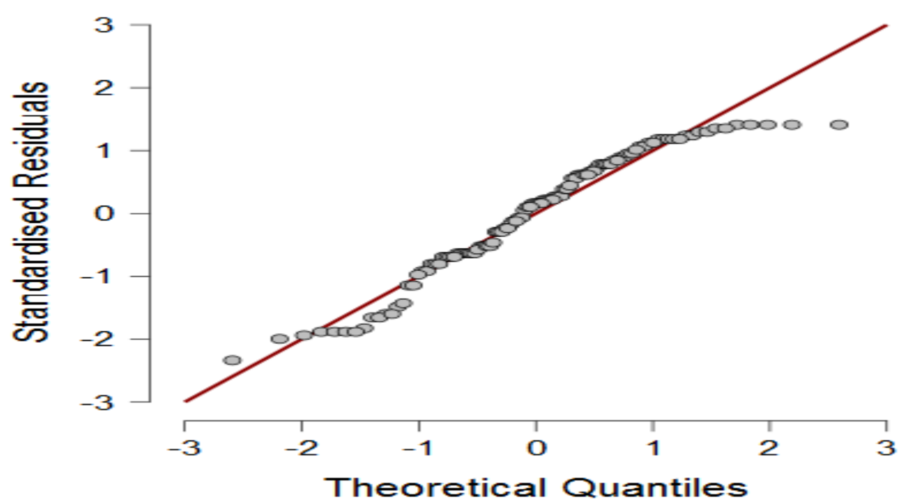
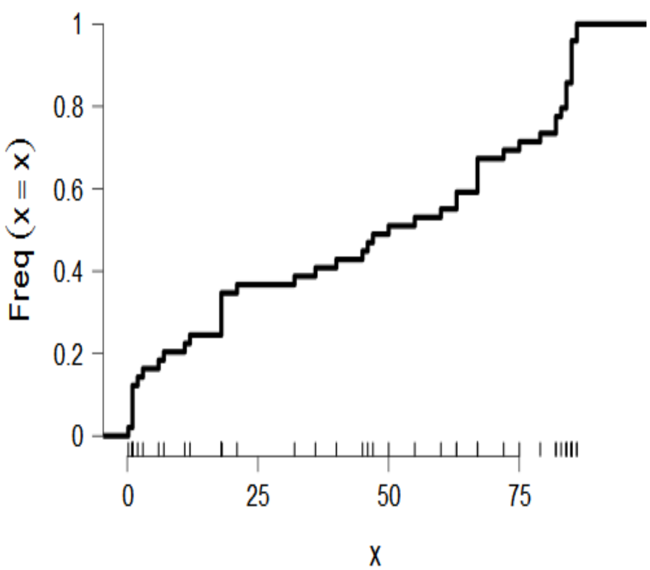


Fig. 6: Q-Q Normal plot of hypertensive data



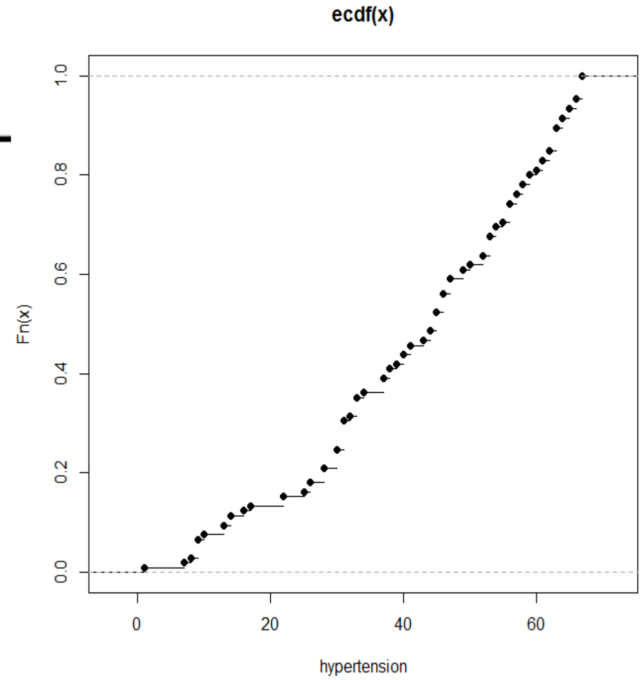


Fig. 7: Cumulative distribution plot of Hypertensive data **Fig.8:** Empirical Cumulative distribution function of data set

11 Distribution Fitting

In this section, we have fitted the hypertensive data set in normal, lognormal, Weibull and exponential distributions.

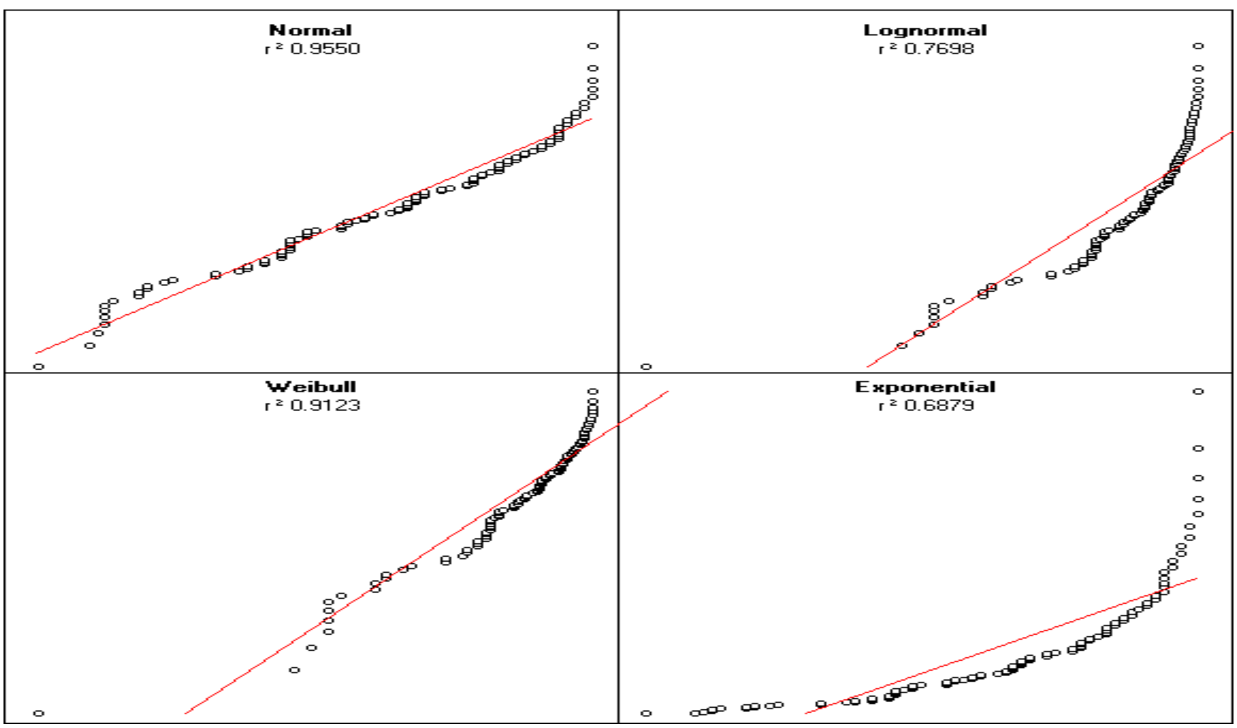


Fig.9: Hypertensive data fitted to Weibull, exponential, lognormal and Normal distribution

12 Conclusion

In this study, we have proposed and studied the Alpha power two-parameter Pranav distribution. The reliability and statistical properties of the distribution are provided, such as survival function, hazard function, moment generating function, moments, order statistics, entropies, Bonferroni and Lorenz curves. The study reveal that the alpha power two-parameter Pranav distribution depicts a decreasing and upside-down bathtub hazard rates. The method of maximum likelihood estimation approach is used to estimate the parameters for the proposed distribution. The flexibility of the AP2PP model has been explained using the data set. It has been observed that the AP2PP distribution provides a better fit as compared to 2PPD, 2PLD, Power Lindley, 2PAD, Pranav distribution, Akash distribution, Sujatha distribution, Ishita, Lindley and Exponential distributions in considered dataset. The quantile-quantile plot is also obtained to show the normality of the data.

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References

[1] Alzaatreh, C. Lee and F. Famoye, A new method for generating families of continuous distributions, *Metron*, **71**, 63–79 (2013).

[2] M. Eugene, C. Lee and F. Famoye, Beta-normal distribution and its applications, *Communications in Statistics-Theory and Methods*, **31**, 497–512 (2002).

[3] K. Zografos and N. Balakrishnan, On families of beta- and generalized gamma-generated distributions and associated inference, *Statistical Methodology*, **6**, 344-362 (2007).

[4] G.M. Cordeiro and M. A. Castro, A new family of generalized distributions*, Journal of Statistical Computational and Simulation*, **81**, 883–898 (2013).

[5] M. Bourguignon, R.B. Silva and G.M. Cordeiro, the Weibull-G family of probability distribution, *Journal of Data Science*, **12**, 53-68 (2014).

[6] C. Kus, A new lifetime distribution*, Computational Statistics and Data Analysis*, **51**, 4497–4509 (2007).

[7] S. Dey and D. Kumar*,* Alpha power transformed Lindley distribution, properties and associated inference with application to earthquake data, *Annals of Data Science*, **6**(4), 625-650 (2019).

[8] B.O. Peng, Zhengqiu Xu and Min Wang, The exponentiated Lindley geometric distribution with applications, *Entropy*, **21**(5), 510 (2019).

[9] G. M. Cordeiro, A. Z Afify, H. M Yousof, S. Cakmakyapan and G. Ozel, The Lindley Weibull distribution: Properties and applications, *An Acad. Bras Cienc*, **90**(3), 2579-2598 (2018).

[10] S. Ibrahim, The properties of Topp Leone exponentiated weibull distribution with application to survival data, *Research Journal of Mathematical and Statistical Sciences*, **9**(1), 9-15 (2021).

[11] N. Rasheed, Topp-leone dagum distribution: properties and its applications**,** *Research journal of Mathematical and Statistical Sciences*, **8**(1), 16-30 (2020).

[12] N. Rasheed, Log component Rayleigh distribution with applications to the reliability data sets, *Research Journal of Mathematical and Statistical Sciences*, **9**(1), 1-8 (2021).

[13] M.A. Aldahlan, Alpha power transformed log-logistic distribution with application to breaking stress data, *Advances in Mathematical Physics*, (2020).

[14] M. Nasaar, S. Dey and D. Kumar, Alpha power transformed inverse Lindley distribution: A distribution with an upside-down bathtub-shaped hazard function, *Journal of Computational and Applied Mathematics*, **348**, 130-145 (2019).

[15] I. Elbatal, A. Zubair, A.M. Abdullah and M. Elgarhy, A new alpha power transformed family of distributions, properties and application to Weibull model, *Journal of Nonlinear Sciences and Applications*, **12**, 1-20 (2019).

[16] U. Ceren and C. Salen, Alpha power inverted exponential distribution, *Gazi University Journal of Science*, **31**(3), 954-965 (2018).

[17] A. Ramadan, Zein Eldin, Muhammad Ahsan ul Haq, Sharqa Hashmi and Mahmoud El-Sehety, Alpha power transformed inverse Lomax distribution with different methods of estimation, *Journal of Probability and Statistics,* (2020). https://doi.org/10.1155/2020/1860813

[18] K. M. Sakthivel and A. Arul chezhian, Alpha power transformed Pareto distribution and its properties with applications, *Journal of Mathematik*, **1**, 52-57 (2020).

[19] I. Elbatal, M. Elgarhy and B. M. Golam Kibria, Alpha power transformed weibull-g family of distributions: theory and applications, *Journal of Statistical Theory and Applications*, **20**(2), 340-354 (2021).

[20] S. Nasira and P. N. Mwitha, Alpha power transformed Frechet distribution, *Applied Mathematics and Information Sciences*, **13**(1), 129-141 (2019).

[21] M. Maryam and R. Kannan, On the alpha power transformed Aradhana distribution: its properties and application, *Indian Journal of Science and Technology*, **14**(30), 2483-2493 (2021).

[22] M. Maryam and R. Kannan, A new generalization of garima distribution with application to real life data, *Applied Mathematics and Information Sciences*, **15**(5), 577-592 (2021).

[23] K. K. Shukla, A Pranav distribution with properties and its applications, *Biometrics & Biostatistics International Journal*, **7**(3), 244-254 (2018).

[24] C. C. Odom, E.C. Nduka and M. A. Ijomah, A modification of Pranav distribution using Quadratic Rank Transmutation map approach, *International Journal of Scientific Research in Mathematical and Statistical Sciences*, **6**(2), 193-202 (2019).

[25] E. Umeh and A. Ibenegbu, A two-parameter Pranav distribution with properties and its applications, *Journal of Biostatistics and Epidemiology*, **5**(1), 74-90 (2019).

[26] B. Abebe and K.K. Shukla, A discrete Pranav distribution and its applications, *Biometrics & Biostatistics International Journal*, **8**(1), 33-37 (2019).

[27] A. Mahdavi and D. Kundu, A new method for generating distributions with an application to exponential distribution, *Communications in Statistics- Theory and Methods*, **46**(13), 6543–6557 (2017).

[28] A. Renyi, *On Measures of Entropy and Information*, *Proceedings of fourth Berkeley Symposium Mathematics Statistics and Probability*, vol 1. University of California Press, Berkeley, 547–561(1961).

[29] C. Tsallis, Possible Generalization of Boltzmann-Gibbs Statistics, *Journal of Statistical Physics*, **52**, 479-487 (1988).

[30] C. E. Bonferroni, Elementi di Statistica Generale, *Libreria Seeber, Firenze*, (1930).

[31] M.O Lorenz, Methods of measuring the concentration of wealth, *American Statistical Association*, **9**, 209–219 (1905).

[32] R Core Team, R. version 3.6.2, *A Language and environment for statistical computing. R Foundation for Statistical Computing*, Vienna, Austria, (2019). https://www.R-project.org/

[33] G. Mancia, M. Bombelli, G. Brambilla, R. Facchetti, R. Sega, E. Toso and G. Grassi, Long-term prognostic value of white coat hypertension, an insight from diagnostic use of both ambulatory and home blood pressure measurements, *Hypertension*, **62**,168–174 (2013).

[34] H. Akaike, A bayesian extension of the minimum aic procedure of autoregressive model fitting, *Biometrika*, **66**(2), 227-242 (1979).

[35] E. G. Schwarz, Estimating the dimension of the model, *Annals of Statistics*, Institute of Mathematical Statistics, **6**(2), 461-464 (1978).

[36] M. Hurvich and C.L. Tsai, Regression and time series model selection in small samples, *Biometrika,* **76**, 297-307 (1989).