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Estimation on Lindely Weibull Distribution based on Progressive First Failure Censored Data

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Abstract: In this paper, we discuss the problem of estimating the parameters of the Lindely Weibull distribution based on progressive first-failure censoring scheme. The maximum likelihood and Bayes methods of estimation are used for this purpose. Markov chain Monte Carlo (MCMC) method is used to measure estimates of Bayes using Gibbs sampling procedure and the corresponding credible intervals are also to be constructed. To illustrate the proposed methods, we provide a numerical example. Finally, the Bayes estimates of the parameters are compared with their corresponding maximum likelihood estimates via Monte Carlo simulation study.

Keywords: Lindely Weibull distribution, progressive first-failure censoring scheme, maximum likelihood estimation, bayes estimation

1 Introduction

Censoring is extremely common in life tests. There are several types of censored tests. One of the most common censored test is type II censoring. It is noted that one can use type II censoring for saving time and money. However, when the lifetimes of products are very high, the experimental time of a type II censoring life test can be still too long. A generalization of type II censoring is progressive type II censoring, which is useful when the loss of live test units at points other than the termination point is inevitable. The estimation of parameters from different lifetime distribution based on progressive type II censored samples is studied by several authors including [1],[2], [3], [4],[5] and [6]. [7] described a life test during which the experimenter might decide to group the test units into several sets, each as an assembly of test units, then run all the test units simultaneously until the first failure occurs in each group. Such a censoring scheme is named first-failure censoring. If an experimenter wishes to remove some sets of test units before observing the first failures in these sets this life test plan is called a progressive first-failure censoring scheme which has been introduced by [8].

Recently, [9] studied the problem of Bayesian estimation and optimal censoring of inverted generalized linear exponential distribution under progressive first failure censored samples, [10] discussed estimation and prediction for progressive first failure censored inverted exponentiated Rayleigh distribution, [11] discussed the problem of estimating the parameters of the generalized linear exponential distribution based on progressive first- failure censoring scheme and [12] studied inference for inverse power lomax distribution with progressive first-failure censoring.

2 A Progressive First-Failure Censoring Scheme

First-failure censoring is combined with progressive censoring and can be defined as: Suppose that n independent groups with k items within each group are placed on a life test, R_1 groups and the group in which the first failure is observed are randomly removed from the test as soon as the first failure (say $X_{1:m:n:k}$)has occurred, R_2 groups and the group in which the second first failure is observed are randomly removed from the test when the second failure (say $X_{2:m:n:k}$) has occurred, and finally $R_{m(m\leq n)}$ groups and the group in which the m-th first failure is observed are randomly removed

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from the test as soon as the m-th failure has occurred (say $X_{m:m:n:k}$). The $X_{1:m:n:k} < X_{2:m:n:k} < \ldots < X_{m:m:n:k}$ are called progressively first-failure censored order statistics. $R = (R_1, R_2, \ldots, R_m)$ is the progressive censoring scheme. It is clear that m is number of the first-failure observed $(1 < m \leq n)$ and $n = m + \sum_{i=1}^{m} R_i$. If the failure times of the $n \times k$ components originally in the test are from a continuous population with probability density function $f(x)$ and distribution function $F(x)$, the joint probability density function for $X_{1:m:n:k}, X_{2:m:n:k}, \ldots, X_{m:m:n:k}$ is given by

$$f_{1,2,\ldots,m}(x_{1:m:n:k}, x_{2:m:n:k}, \ldots, x_{m:m:n:k}) = Ak^m \prod_{i=1}^{m} f(x_{i:m:n:k})(1 - F(x_{i:m:n:k}))^{k(R_i+1)-1}, \tag{1}$$

$$0 < x_{1:m:n:k} < x_{2:m:n:k} < \ldots < x_{m:m:n:k} < \infty,$$

where

$$A = n(n - R_1 - 1)(n - R_1 - R_2 - 2)\ldots(n - R_1 - R_2 - \ldots - R_{m-1} - m + 1).$$

Special cases
From (1) it is clear that the progressive first-failure censored scheme containing the following censoring scheme as special cases:

1. The first failure censored scheme when R=(0,0,...,0).
2. The progressive type-II censored order statistics if k=1.
3. Usually type-II censored order statistics when k=1 and R=(0,0,...,n-m).
4. The complete sample case when k=1 and R=(0,0,...,0).

Also, it should be noted that $X_{1:m:n:k}, X_{2:m:n:k}, \ldots, X_{m:m:n:k}$ can be viewed as a progressive type-II censored sample from a population with distribution function $1 - (1 - F(x))^k$. For this reason, results for progressive type-II censored order statistics can be extended to progressively first failure censored order statistics easily. Also, the progressive first-failure censored scheme has advantages in terms of reducing the time of test, in which more items are used, but only m of $n \times k$ items are failures.

3 The Lindely Weibull Distribution

We suggest a new generalization of the Weibull (W) distribution named the Lindley Weibull (LIW) model. The W distribution has been widely used in reliability analysis and in applications of several different fields. Although its common use, a negative point of the distribution is the limited shape of its hazard rate function (hrf) that can only be monotonically increasing or decreasing or constant.

Generally, practical problems require a wider range of possibilities in the medium risk, for example, when the lifetime data present a bathtub shaped hazard function like human mortality and machine life cycles. Researchers in the last years developed various extensions and modified forms of the W distribution to obtain more flexible distributions. Some extensions of the W distribution are available in the literature such as the (exponentiated W) in [13], (additive W) in [14], (beta-W) in [15], (extended W) in [16] and [17] proposed a new class of distributions called the Lindley generator (Li-G) with one extra parameter. For an arbitrary baseline cumulative distribution function (cdf) G(x, ξ), the Li-G family with one extra positive shape parameter θ has cdf and probability density function (pdf) given by

$$F(x;\theta,\xi) = 1 - [1 - G(x;\xi)]^{\theta}[1 - \frac{\theta}{\theta+1} log\overline{G}(x;\xi)]$$

and

$$f(x;\theta,\xi) = \frac{\theta^2}{\theta+1} g(x;\xi)[1 - G(x;\xi)]^{\theta-1}[1 - log\overline{G}(x;\xi)],$$

where

$$g(x;\xi) = \frac{dG(x;\xi)}{dx}, \quad \overline{G}(x;\xi) = 1 - G(x;\xi).$$

The cdf and pdf of the W distribution are given by

$$G(x;\alpha,\beta) = 1 - exp[-(\alpha x)^{\beta}]$$

and

$$g(x;\alpha,\beta) = \beta\alpha^{\beta}x^{\beta-1}exp[-(\alpha x)^{\beta}].$$

Based on the Li-G family, we construct the Lindely Weibull distribution. The basic motivations for the LIW distribution in practice are: (i) to make the kurtosis more flexible as compared to the baseline model, (ii) to produce skewness for symmetrical distributions, (iii) to construct heavy-tailed distributions that are not longer-tailed for modeling real data, (iv) to generate distributions with symmetric, left-skewed, right-skewed and reversed-J shaped, (v) to provide consistently better fits than other generated models under the same underlying distribution. The LiW distribution has the following probability density function (pdf)

$$f(x) = f(x;\theta,\alpha,\beta) = \frac{\beta\theta^2}{\theta+1}[\alpha^{\beta}x^{\beta-1} + \alpha^{2\beta}x^{2\beta-1}]exp[-\theta(\alpha x)^{\beta}], \tag{2}$$

cumulative distribution function (cdf)

$$F(x) = F(x;\theta,\alpha,\beta) = 1 - exp[-\theta(\alpha x)^{\beta}][1 + \frac{\theta}{\theta+1}(\alpha x)^{\beta}], \tag{3}$$

4 Maximum Likelihood Estimation

In this section we deduce the maximum likelihood estimates of the unknown parameters α, θ and β of the LIW(α,θ,β) with pdf and cdf given in (2) and (3), respectively. Thus, from (1) the likelihood function for progressive first-failure censored scheme take the following form

$$\begin{aligned} L(\alpha,\theta,\beta;\underline{x}) = Ak^m \prod_{i=1}^{m} \frac{\beta\theta^2}{\theta+1}\left(\alpha^{\beta}x_i^{\beta-1} + \alpha^{2\beta}x_i^{2\beta-1}\right)(exp[-\theta(\alpha x_i)^{\beta}])^{k(R_i+1)} \\ \times\left(1 + \frac{\theta}{\theta+1}(\alpha x_i)^{\beta}\right)^{k(R_i+1)-1}. \end{aligned} \tag{4}$$

where

$$\underline{x} = (x_1, ..., x_m)$$

and

$$A = n(n-R_1-1)(n-R_1-R_2-2)...(n-R_1-R_2-...-R_{m-1}-m+1).$$

The logarithm of (4) can be written as

$$\begin{aligned} log\, L(\alpha,\theta,\beta) = \log A + m\log k + m\log\beta + 2m\log\theta - m\log(\theta+1) \\ + \sum_{i=1}^{m}\log\left(\frac{x_i^{\beta-1}}{\alpha^{-\beta}} + \frac{x_i^{2\beta-1}}{\alpha^{-2\beta}}\right) - \theta\sum_{i=1}^{m}(k(R_i+1))\frac{x_i^{\beta}}{\alpha^{-\beta}} \\ + \sum_{i=1}^{m}(k(R_i+1)-1) + log\left(1 + \frac{\theta x_i^{\beta}}{(\theta+1)\alpha^{-\beta}}\right). \end{aligned} \tag{5}$$

Taking the derivatives with respect to α, θ and β of (5) and putting them equal to zero we get

$$\begin{aligned} \frac{\partial l}{\partial\alpha} = \sum_{i=1}^{m}\frac{\left(\beta\alpha^{\beta-1}x_i^{\beta-1} + 2\beta x_i^{2\beta-1}\alpha^{2\beta-1}\right)}{\left(\alpha^{\beta}x_i^{\beta-1} + x_i^{2\beta-1}\alpha^{2\beta}\right)} - \beta\theta\sum_{i=1}^{m}(k(R_i+1))x_i^{\beta}\alpha^{\beta-1} \\ + \sum_{i=1}^{m}\frac{\alpha^{\beta-1}\beta\theta(-1+k(R_i+1))x_i^{\beta}}{(1+\theta)\left(1+\frac{\alpha^{\beta}\theta x_i^{\beta}}{1+\theta}\right)} = 0, \end{aligned} \tag{6}$$

$$\frac{\partial l}{\partial \theta}=\frac{2m}{\theta}-\frac{m}{\theta+1}-\sum_{i=1}^{m}(k(R_i+1))x_i^{\beta}\alpha^{\beta}+\sum_{i=1}^{m}\frac{(-1+k(R_i+1))(\frac{-\alpha^{\beta}\theta x_i^{\beta}}{(1+\theta)^2}+\frac{\alpha^{\beta}x_i^{\beta}}{1+\theta})}{1+\frac{\alpha^{\beta}\theta x_i^{\beta}}{1+\theta}}=0, \tag{7}$$

$$\begin{aligned}\frac{\partial l}{\partial \beta}=&\frac{m}{\beta}+\sum_{i=1}^{m}\frac{\left(\alpha^{\beta}ln(\alpha)x_i^{\beta-1}+\alpha^{\beta}ln(x_i)x_i^{\beta-1}+2\alpha^{2\beta}ln(\alpha)x_i^{2\beta-1}+2\alpha^{2\beta}ln(x_i)x_i^{2\beta-1}\right)}{\left(\alpha^{\beta}x_i^{\beta-1}+\alpha^{2\beta}x_i^{2\beta-1}\right)}\\&-\theta\sum_{i=1}^{m}\left(k\alpha^{\beta}ln(\alpha)(1+R_i)x_i^{\beta}+k\alpha^{\beta}ln(x_i)(1+R_i)x_i^{\beta}\right)\\&+\sum_{i=1}^{m}\frac{(-1+k(1+R_i))(\frac{\alpha^{\beta}\theta ln(\alpha)x_i^{\beta}}{1+\theta}+\frac{\alpha^{\beta}\theta ln(x_i)x_i^{\beta}}{1+\theta})}{1+\frac{\alpha^{\beta}\theta x_i^{\beta}}{1+\theta}}=0.\end{aligned} \tag{8}$$

Since (6), (7) and (8) cannot be solved analytically, some numerical methods such as Newtons method must be employed to get the maximum likelihood estimates of α, θ and β .

4.1 Approximate confidence interval

In this section we obtained the approximate confidence interval for Lindely Weibull distribution model. Let s = (α,θ,β), the fisher information matrix of the parameters s is given by

$$I(s)=E\begin{bmatrix}\frac{-\partial^2 l}{\partial\alpha^2} & \frac{-\partial^2 l}{\partial\alpha\partial\theta} & \frac{-\partial^2 l}{\partial\alpha\partial\beta}\\ \frac{-\partial^2 l}{\partial\theta\partial\alpha} & \frac{-\partial^2 l}{\partial\theta^2} & \frac{-\partial^2 l}{\partial\theta\partial\beta}\\ \frac{-\partial^2 l}{\partial\beta\partial\alpha} & \frac{-\partial^2 l}{\partial\beta\partial\theta} & \frac{-\partial^2 l}{\partial\beta^2}\end{bmatrix}.$$

The expectation of the above expressions are not easy to derive; hence, we make use of the observed fisher information matrix. Let $\hat{s}$ = $(\hat{\alpha},\hat{\theta},\hat{\beta})$ be the MLEs of the parameters s = (α,θ,β). The observed fisher information matrix is given by

$$I(\hat{s})=\begin{bmatrix}\frac{-\partial^2 l}{\partial\alpha^2} & \frac{-\partial^2 l}{\partial\alpha\partial\theta} & \frac{-\partial^2 l}{\partial\alpha\partial\beta}\\ \frac{-\partial^2 l}{\partial\theta\partial\alpha} & \frac{-\partial^2 l}{\partial\theta^2} & \frac{-\partial^2 l}{\partial\theta\partial\beta}\\ \frac{-\partial^2 l}{\partial\beta\partial\alpha} & \frac{-\partial^2 l}{\partial\beta\partial\theta} & \frac{-\partial^2 l}{\partial\beta^2}\end{bmatrix} \qquad at \quad s=\hat{s}.$$

Thus, the observed variance?covariance matrix of MLEs $(\hat{\alpha},\hat{\theta},\hat{\beta})$ is the inverse of the observed fisher information matrix, given by

$$I^{-1}(\hat{s})=\begin{bmatrix}Var(\hat{\alpha}) & Cov(\hat{\alpha},\hat{\theta}) & Cov(\hat{\alpha},\hat{\beta})\\ Cov(\hat{\theta},\hat{\alpha}) & Var(\hat{\theta}) & Cov(\hat{\theta},\hat{\beta})\\ Cov(\hat{\beta},\hat{\alpha}) & Cov(\hat{\beta},\hat{\theta}) & Var(\hat{\beta})\end{bmatrix}.$$

The asymptotic normality of the MLE will be used to compute the approximate confidence intervals for parameters α, θ and β. Therefore, (1-γ)100 % confidence intervals for parameters α, θ and β become

$$\hat{\alpha}\pm Z_{\frac{\gamma}{2}}\sqrt{Var(\hat{\alpha})},\hat{\theta}\pm Z_{\frac{\gamma}{2}}\sqrt{Var(\hat{\theta})} \quad and\ \hat{\beta}\pm Z_{\frac{\gamma}{2}}\sqrt{Var(\hat{\beta})},$$

where $Z_{\frac{\gamma}{2}}$ is the percentile of the distribution standard normal with right-tail probability $\frac{\gamma}{2}$.

5 Bayesian Estimation of the Parameters

Assume that the prior densities for the parameters α, θ and β are the Gamma distribution

$$g_1(\alpha) = \frac{b^a}{\Gamma(a)}\alpha^{a-1}e^{-b\alpha}, \qquad a,b>0, \qquad \alpha>0, \tag{9}$$

$$g_2(\theta) = \frac{d^c}{\Gamma(c)}\theta^{c-1}e^{-d\theta}, \qquad c,d>0, \qquad \theta>0, \tag{10}$$

$$g_3(\beta) = \frac{f^h}{\Gamma(h)}\beta^{h-1}e^{-f\beta}, \qquad h,f>0, \qquad \beta>0. \tag{11}$$

Hence, the joint prior distribution for α, θ and β is

$$g(\alpha,\theta,\beta) = \frac{b^a d^c f^h}{\Gamma(a)\Gamma(c)\Gamma(h)}\alpha^{a-1}\theta^{c-1}\beta^{h-1}e^{-(b\alpha+d\theta+f\beta)}. \tag{12}$$

From (4) and (12) we get the joint posterior as

$$\begin{aligned} q(\alpha,\theta,\beta|\underline{x}) = {} & KAk^m \prod_{i=1}^{m} \frac{\alpha^{a-1}\beta^h\theta^{c+1}}{\theta+1}\left(\alpha^\beta x_i^{\beta-1} + \alpha^{2\beta}x_i^{2\beta-1}\right)(exp[-\theta(\alpha x_i)^\beta]^{k(R_i+1)}) \\ & \times(1+\frac{\theta}{\theta+1}(\alpha x_i)^\beta)^{k(R_i+1)-1}exp(-(b\alpha+d\theta+f\beta)). \end{aligned} \tag{13}$$

Where K is the normalizing constant given from

$$K^{-1} = \int_0^\infty\int_0^\infty\int_0^\infty q(\alpha,\theta,\beta|\underline{x})d\alpha d\theta d\beta.$$

Under squared error loss function, the Bayes estimator of a function u(α,θ,β) is the posterior mean of the function and is given by a ratio of three integrals as follows

$$\hat{u}_\beta(\alpha,\theta,\beta) = E(u(\alpha,\theta,\beta|\underline{x})) = \int_\alpha\int_\theta\int_\beta u(\alpha,\theta,\beta)q(\alpha,\theta,\beta|\underline{x})d\alpha d\theta d\beta. \tag{14}$$

Under Linex loss function, the Bayes estimator of u(α,θ,β) is given by

$$\begin{aligned} \hat{u}_\beta(\alpha,\theta,\beta) &= -\frac{1}{\xi}ln(E(e^{-\xi u(\alpha,\theta,\beta)}|\underline{x})) \\ &= -\frac{1}{\xi}ln(\int_\alpha\int_\theta\int_\beta e^{-\xi u(\alpha,\theta,\beta)}q(\alpha,\theta,\beta|\underline{x})d\alpha d\theta d\beta). \end{aligned} \tag{15}$$

It is clear from equations (14) and (15), that both of the integrals can not be obtained in a simple closed form and hence numerical methods of integration must be used. Therefore, we use the Monte Carlo integration (MCI) sampling procedure to compute Bayes estimate under two different types of loss functions.

5.1 Bayes estimation using Monte Carlo integration

The Bayes estimators of the parameters α, θ and β can not be obtained in simple closed form. So, we can use MCI procedure to get the Bayes estimators of the parameters.

We can obtain Bayes estimation using MCI by generating α_i, θ_i and β_i, $i = 1,2,......,m.$ from the prior distribution given by (9), (10) and (11), respectively. Then, we have the Bayes estimators under squared error and Linex loss functions as the following.

5.1.1 The Bayes estimators under the squared error loss function

We can write the Bayes estimates of α, θ and β under the squared error loss function as

$$\hat{\alpha}_{BS} = \frac{\sum_{i=1}^{m} \alpha_i L(\alpha_i, \theta_i, \beta_i; \underline{x})}{\sum_{i=1}^{m} L(\alpha_i, \theta_i, \beta_i; \underline{x})}, \tag{16}$$

$$\hat{\theta}_{BS} = \frac{\sum_{i=1}^{m} \theta_i L(\alpha_i, \theta_i, \beta_i; \underline{x})}{\sum_{i=1}^{m} L(\alpha_i, \theta_i, \beta_i; \underline{x})}, \tag{17}$$

and

$$\hat{\beta}_{BS} = \frac{\sum_{i=1}^{m} \beta_i L(\alpha_i, \theta_i, \beta_i; \underline{x})}{\sum_{i=1}^{m} L(\alpha_i, \theta_i, \beta_i; \underline{x})}. \tag{18}$$

5.1.2 The Bayes estimators under Linex loss function

The posterior expectation with respect to the posterior density of α, θ and β are given, respectively, by

$$E_\alpha(e^{-\xi\alpha}|\underline{x}) = \frac{\sum_{i=1}^{m} e^{-\xi\alpha_i} L(\alpha_i, \theta_i, \beta_i; \underline{x})}{\sum_{i=1}^{m} L(\alpha_i, \theta_i, \beta_i; \underline{x})}, \tag{19}$$

$$E_\theta(e^{-\xi\theta}|\underline{x}) = \frac{\sum_{i=1}^{m} e^{-\xi\theta_i} L(\alpha_i, \theta_i, \beta_i; \underline{x})}{\sum_{i=1}^{m} L(\alpha_i, \theta_i, \beta_i; \underline{x})}, \tag{20}$$

and

$$E_\beta(e^{-\xi\beta}|\underline{x}) = \frac{\sum_{i=1}^{m} e^{-\xi\beta_i} L(\alpha_i, \theta_i, \beta_i; \underline{x})}{\sum_{i=1}^{m} L(\alpha_i, \theta_i, \beta_i; \underline{x})}. \tag{21}$$

Hence, the Bayes estimates of α, θ and β under Linex loss function are given, respectively, by

$$\hat{\alpha}_{BL} = -\frac{1}{\xi} ln[\frac{\sum_{i=1}^{m} e^{-\xi\alpha_i} L(\alpha_i, \theta_i, \beta_i; \underline{x})}{\sum_{i=1}^{m} L(\alpha_i, \theta_i, \beta_i; \underline{x})}], \tag{22}$$

$$\hat{\theta}_{BL} = -\frac{1}{\xi} ln[\frac{\sum_{i=1}^{m} e^{-\xi\theta_i} L(\alpha_i, \theta_i, \beta_i; \underline{x})}{\sum_{i=1}^{m} L(\alpha_i, \theta_i, \beta_i; \underline{x})}], \tag{23}$$

and

$$\hat{\beta}_{BL} = -\frac{1}{\xi} ln[\frac{\sum_{i=1}^{m} e^{-\xi\beta_i} L(\alpha_i, \theta_i, \beta_i; \underline{x})}{\sum_{i=1}^{m} L(\alpha_i, \theta_i, \beta_i; \underline{x})}]. \tag{24}$$

5.1.3 The Bayes estimators under General Entropy Loss Function

The Bayes estimates of α, θ and β under general entropy loss function are given, respectively, by

$$\hat{\alpha}_{BG} = [\frac{\sum_{i=1}^{m} \alpha_i^{-\varphi} L(\alpha_i, \theta_i, \beta_i; \underline{x})}{\sum_{i=1}^{m} L(\alpha_i, \theta_i, \beta_i; \underline{x})}]^{\frac{-1}{\varphi}}, \tag{25}$$

$$\hat{\theta}_{BG} = [\frac{\sum_{i=1}^{m} \theta_i^{-\varphi} L(\alpha_i, \theta_i, \beta_i; \underline{x})}{\sum_{i=1}^{m} L(\alpha_i, \theta_i, \beta_i; \underline{x})}]^{\frac{-1}{\varphi}}, \tag{26}$$

and

$$\hat{\beta}_{BG} = [\frac{\sum_{i=1}^{m} \beta_i^{-\varphi} L(\alpha_i, \theta_i, \beta_i; \underline{x})}{\sum_{i=1}^{m} L(\alpha_i, \theta_i, \beta_i; \underline{x})}]^{\frac{-1}{\varphi}}. \tag{27}$$

5.2 HPD credible interval

Now, we construct the HPD credible intervals of α using the generated importance samples. Let $\alpha_{(1)} < \alpha_{(2)} < ... < \alpha_{(M)}$ denote the ordered values $\alpha_1, \alpha_2, ..., \alpha_M$. Then using the algorithm proposed by[18], the (1-γ)100% where $0 < \gamma < 1$, HPD credible interval for α is given by $(\alpha_{(j)}, \alpha_{(j+[(1-\gamma)M])})$, where j is chosen such that
$\alpha_{(j+[(1-\gamma)M])} - \alpha_{(j)} = \min_{1 \le i \le \gamma M}(\alpha_{(i+[(1-\gamma)M])} - \alpha_{(i)}), \qquad j = 1, 2, ...M.$

6 Simulation study

In this section we report some numerical experiments performed to evaluate the behavior of the proposed methods, we simulated 1000 progressively first- failure censored samples from a LW (α, θ, β) distribution. We used different sample of sizes (n), different effective sample of sizes (m), different k (k= 2, 5) and different of sampling schemes. First, we used the non informative gamma priors, we call it prior 0: $\lambda_1 = \lambda_2 = \lambda_3 = \mu_1 = \mu_2 = \mu_3 = 0$.

For computing Bayes estimators, other than prior 0, we also used informative prior, including prior 1, $\lambda_1 = 0.1$, $\lambda_2 = 0.2$, $\lambda_3 = 0.3$, $\mu_1 = 0.4$, $\mu_2 = 0.5$ and $\mu_3 = 0.6$. In two cases, we used the squared error loss function to compute the Bayes estimates, mean squared errors (MSEs) and 95% credible intervals. For comparison purposes, we also compute the MLEs and the 95%confidence intervals based on the observed Fisher information matrix. From Tables (1-6) we deduce that:

1. It can be seen that the mean squared errors decrease as n increase. Moreover, as m increases, the MSE of estimates decrease.
2. The mean squared errors decrease as the value of the group size k increases.
3. The MSE of Bayesian estimators (SEL, LINEX and Entropy) is always similar in most cases.
4. It can be observed that MLEs ara better than the Bayes estimates.
5. Length of confidence and HPD credible intervals decreases as n increase. HPD credible intervals are better than confidence intervals in respect of average length.

7 Real data analysis

Here we discuss a real-life situation to illustrate different procedures studied in this paper. The considered data are given in[19], and it explains survival period of 45 patients treated with chemotherapy. The data are given below as 1, 63, 105, 129, 182, 216, 250, 262, 301, 301, 342, 354, 356, 358, 380, 383, 383, 388, 394, 408, 460, 489, 499, 523, 524, 535, 562, 569, 675, 676, 748, 778, 786, 797, 955, 968, 1000, 1245, 1271, 1420, 1551, 1694, 2363, 2754, 2950. Next, we generate a first-failure cenosred sample after randomly grouping this data set into n = 15 groups with k = 3 items within each group and report it in table 7. Finally, the following first-failure censored sample is obtained: 2754, 460, 489, 499, 523, 524, 535, 562, 569, 675, 676, 748, 1, 63, 250. Now, we generate progressive first-failure censored sample using three different censoring schemes with m = 8. The different censoring schemes are presented in table 8. In all the three cases we calculate the ML and Bayes estimates of the parameters. In Bayes estimation we use non-informative priors as we have no prior information about the parameters. For importance sampling procedure, we take M = 1000. Also, we obtain 95% confidence and HPD credible intervals for the parameters.

8 Conclusions

In this article, we address the problem of estimating the unknown parameters of Lindley Weibull (LIW) distribution using progressive first failure censoring. We first derive the MLE and asymptotic confidence intervals and then, Bayes estimators using non-informative and informative gamma priors. Asymptotic confidence intervals are constructed using observed Fisher information matrix. Because the Bayes estimate cannot be obtained in closed forms, the MCMC technique is used to compute the Bayes estimate and associated HPD credible interval. The performance of the point and interval estimates is examined by a Monte Carlo simulation study. Simulation results suggest that the ML estimation is better than the Bayes estimation.

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Table 1: Average values of the different estimators and the corresponding MSEs when $\alpha = 0.02$, $\theta = 0.2$ and $\beta = 0.05$ with prior 0.

(k,n,m)	Scheme	MLE			Bayes SEL		
		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$
(2,20,10)	$(10,0^9)$	0.0334 (0.00081)	0.6004 (0.21272)	0.3559 (0.11431)	0.03 (0.00079)	0.6 (0.20935)	0.4 (0.11428)
	$(0^3,2,3,3,2,0^3)$	0.0873 (0.01587)	0.5347 (0.17742)	0.2886 (0.06517)	0.09 (0.01584)	0.5 (0.17878)	0.3 (0.06517)
	$(0^9,10)$	0.0447 (0.0024)	0.496 (0.10505)	0.2634 (0.04959)	0.04 (0.00212)	0.5 (0.1059)	0.3 (0.0496)
(2,40,10)	$(30,0^9)$	0.0744 (0.00692)	0.5009 (0.12392)	0.3057 (0.06735)	0.07 (0.00703)	0.5 (0.12439)	0.3 (0.06736)
	$(0^3,10^3,0^4)$	0.111 (0.03533)	0.5316 (0.13843)	0.2776 (0.05341)	0.11 (0.03535)	0.5 (0.13893)	0.3 (0.0534)
	$(0^9,30)$	0.0227 (0.0009)	0.5467 (0.14262)	0.2055 (0.02612)	0.02 (0.00039)	0.5 (0.13897)	0.2 (0.02613)
(2,40,20)	$(20,0^{19})$	0.0512 (0.00161)	0.4493 (0.07293)	0.3268 (0.08322)	0.05 (0.00161)	0.4 (0.07319)	0.3 (0.08322)
	$(0^8,5^4,0^8)$	0.0286 (0.00036)	0.5174 (0.12179)	0.3055 (0.06859)	0.03 (0.00036)	0.5 (0.1215)	0.3 (0.06859)
	$(0^{19},20)$	0.0724 (0.0077)	0.4461 (0.08262)	0.241 (0.03901)	0.07 (0.00839)	0.4 (0.0824)	0.2 (0.03901)
(5,20,10)	$(10,0^9)$	0.0099 (0.00014)	0.1975 (0.0123)	0.5712 (0.30034)	0.01 (0.00014)	0.2 (0.01242)	0.6 (0.30031)
	$(0^3,2,3,3,2,0^3)$	0.0316 (0.00046)	0.1738 (0.00689)	0.3828 (0.13518)	0.03 (0.00046)	0.2 (0.00688)	0.4 (0.13517)
	$(0^9,10)$	0.0918 (0.03643)	0.1335 (0.01038)	0.3939 (0.13447)	0.09 (0.03738)	0.1 (0.01038)	0.4 (0.13445)
(5,40,10)	$(30,0^9)$	0.0355 (0.00094)	0.1715 (0.00267)	0.4439 (0.16261)	0.04 (0.00094)	0.2 (0.00267)	0.4 (0.16257)
	$(0^3,10^3,0^4)$	0.0511 (0.0036)	0.1656 (0.00256)	0.3844 (0.12127)	0.05 (0.00359)	0.2 (0.00257)	0.4 (0.12129)
	$(0^9,30)$	0.0811 (0.01025)	0.1769 (0.00271)	0.232 (0.03633)	0.08 (0.01023)	0.2 (0.0027)	0.2 (0.03629)
(5,40,20)	$(20,0^{19})$	0.017 (0.00008)	0.171 (0.00239)	0.4978 (0.21608)	0.02 (0.00008)	0.2 (0.00239)	0.5 (0.21608)
	$(0^8,5^4,0^8)$	0.0286 (0.00018)	0.1222 (0.00732)	0.4471 (0.16341)	0.03 (0.00017)	0.1 (0.00732)	0.4 (0.16343)
	$(0^{19},20)$	0.0281 (0.00061)	0.1684 (0.00302)	0.3097 (0.07158)	0.03 (0.00063)	0.2 (0.00302)	0.3 (0.07158)

(k,n,m)	Scheme	Linex c_1			Linex c_2		
		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$
(2,20,10)	$(10,0^9)$	0.03 (0.00079)	0.6 (0.20934)	0.4 (0.11428)	0.03 (0.00079)	0.6 (0.20937)	0.4 (0.11428)
	$(0^3,2,3,3,2,0^3)$	0.09 (0.01583)	0.5 (0.17877)	0.3 (0.06517)	0.09 (0.01584)	0.5 (0.17879)	0.3 (0.06517)
	$(0^9,10)$	0.04 (0.00212)	0.5 (0.1059)	0.3 (0.0496)	0.04 (0.00212)	0.5 (0.1059)	0.3 (0.0496)
(2,40,10)	$(30,0^9)$	0.07 (0.00703)	0.5 (0.12439)	0.3 (0.06736)	0.07 (0.00703)	0.5 (0.12439)	0.3 (0.06736)
	$(0^3,10^3,0^4)$	0.11 (0.03491)	0.5 (0.13893)	0.3 (0.0534)	0.11 (0.03579)	0.5 (0.13894)	0.3 (0.0534)
	$(0^9,30)$	0.02 (0.00038)	0.5 (0.13885)	0.2 (0.02613)	0.02 (0.00039)	0.5 (0.13908)	0.2 (0.02613)

Continuation of Table 1

(k,n,m)	Scheme	Linex c_1			Linex c_2		
		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$
(2,40,20)	$(20,0^{19})$	0.05 (0.00161)	0.4 (0.07319)	0.3 (0.08322)	0.05 (0.00161)	0.4 (0.07319)	0.3 (0.08322)
	$(0^8,5^4,0^8)$	0.03 (0.00036)	0.5 (0.12149)	0.3 (0.06859)	0.03 (0.00036)	0.5 (0.1215)	0.3 (0.06859)
	$(0^{19},20)$	0.07 (0.00838)	0.4 (0.0824)	0.2 (0.03901)	0.07 (0.00841)	0.4 (0.0824)	0.2 (0.03901)
(5,20,10)	$(10,0^9)$	0.01 (0.00014)	0.2 (0.01242)	0.6 (0.30031)	0.01 (0.00014)	0.2 (0.01242)	0.6 (0.30031)
	$(0^3,2,3,3,2,0^3)$	0.03 (0.00046)	0.2 (0.00688)	0.4 (0.13517)	0.03 (0.00046)	0.2 (0.00688)	0.4 (0.13517)
	$(0^9,10)$	0.09 (0.03737)	0.1 (0.01038)	0.4 (0.13445)	0.09 (0.03739)	0.1 (0.01038)	0.4 (0.13445)
(5,40,10)	$(30,0^9)$	0.04 (0.00094)	0.2 (0.00267)	0.4 (0.16257)	0.04 (0.00094)	0.2 (0.00267)	0.4 (0.16257)
	$(0^3,10^3,0^4)$	0.05 (0.00359)	0.2 (0.00257)	0.4 (0.12129)	0.05 (0.00359)	0.2 (0.00257)	0.4 (0.12129)
	$(0^9,30)$	0.08 (0.01019)	0.2 (0.0027)	0.2 (0.03629)	0.08 (0.01026)	0.2 (0.0027)	0.2 (0.03629)
(5,40,20)	$(20,0^{19})$	0.02 (0.00008)	0.2 (0.00239)	0.5 (0.21608)	0.02 (0.00008)	0.2 (0.00239)	0.5 (0.21608)
	$(0^8,5^4,0^8)$	0.03 (0.00017)	0.1 (0.00732)	0.4 (0.16343)	0.03 (0.00017)	0.1 (0.00732)	0.4 (0.16343)
	$(0^{19},20)$	0.03 (0.00063)	0.2 (0.00302)	0.3 (0.07158)	0.03 (0.00063)	0.2 (0.00302)	0.3 (0.07158)

Continuation of Table 1

(k,n,m)	Scheme	Entropy		
		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$
(2,20,10)	$(10,0^9)$	0.03 (0.00079)	0.6 (0.20932)	0.4 (0.11428)
	$(0^3,2,3,3,2,0^3)$	0.09 (0.01581)	0.5 (0.17876)	0.3 (0.06517)
	$(0^9,10)$	0.04 (0.0021)	0.5 (0.10589)	0.3 (0.0496)
(2,40,10)	$(30,0^9)$	0.07 (0.00703)	0.5 (0.12438)	0.3 (0.06736)
	$(0^3,10^3,0^4)$	0.11 (0.03419)	0.5 (0.13893)	0.3 (0.0534)
	$(0^9,30)$	0.02 (0.0003)	0.5 (0.13876)	0.2 (0.02613)
(2,40,20)	$(20,0^{19})$	0.05 (0.00161)	0.4 (0.07319)	0.3 (0.08322)
	$(0^8,5^4,0^8)$	0.03 (0.00036)	0.5 (0.12149)	0.3 (0.06859)
	$(0^{19},20)$	0.07 (0.00824)	0.4 (0.0824)	0.2 (0.03901)
(5,20,10)	$(10,0^9)$	0.01 (0.00014)	0.2 (0.01242)	0.6 (0.30031)
	$(0^3,2,3,3,2,0^3)$	0.03 (0.00046)	0.2 (0.00688)	0.4 (0.13517)
	$(0^9,10)$	0.09 (0.03736)	0.1 (0.01038)	0.4 (0.13445)

Continuation of Table 1

(k,n,m)	Scheme	Entropy		
		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$
(5,40,10)	$(30,0^9)$	0.04 (0.00094)	0.2 (0.00267)	0.4 (0.16257)
	$(0^3,10^3,0^4)$	0.05 (0.00359)	0.2 (0.00257)	0.4 (0.12129)
	$(0^9,30)$	0.08 (0.00996)	0.2 (0.0027)	0.2 (0.03629)
(5,40,20)	$(20,0^{19})$	0.02 (0.00008)	0.2 (0.00239)	0.5 (0.21608)
	$(0^8,5^4,0^8)$	0.03 (0.00017)	0.1 (0.00732)	0.4 (0.16343)
	$(0^{19},20)$	0.03 (0.00063)	0.2 (0.00302)	0.3 (0.07158)

Table 2: Average values of the different estimators and the corresponding MSEs when $\alpha = 0.02$, $\theta = 0.2$ and $\beta = 0.05$ with prior 1.

(k,n,m)	Scheme	MLE			Bayes SEL		
		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$
(2,20,10)	$(10,0^9)$	0.0374 (0.00115)	0.5626 (0.16134)	0.3237 (0.08397)	0.04 (0.0011)	0.6 (0.1616)	0.3 (0.08396)
	$(0^3,2,3,3,2,0^3)$	0.0397 (0.00118)	0.6333 (0.20288)	0.2661 (0.0486)	0.04 (0.00116)	0.6 (0.20302)	0.3 (0.0486)
	$(0^9,10)$	0.1091 (0.03082)	0.5436 (0.14484)	0.2651 (0.05399)	0.11 (0.03696)	0.5 (0.1476)	0.3 (0.05399)
(2,40,10)	$(30,0^9)$	0.088 (0.00916)	0.4472 (0.08514)	0.3134 (0.07309)	0.08 (0.00496)	0.4 (0.08643)	0.3 (0.07308)
	$(0^3,10^3,0^4)$	0.1551 (0.04766)	0.4827 (0.10312)	0.2641 (0.04654)	0.16 (0.04783)	0.5 (0.10709)	0.3 (0.04652)
	$(0^9,30)$	0.024 (0.00028)	0.6012 (0.17294)	0.2713 (0.05208)	0.02 (0.00032)	0.6 (0.17114)	0.3 (0.05208)
(2,40,20)	$(20,0^{19})$	0.0274 (0.00029)	0.4362 (0.06663)	0.4011 (0.12868)	0.03 (0.00029)	0.4 (0.06716)	0.4 (0.12869)
	$(0^8,5^4,0^8)$	0.0799 (0.00666)	0.4281 (0.06185)	0.3013 (0.06485)	0.07 (0.00516)	0.4 (0.06122)	0.3 (0.06486)
	$(0^{19},20)$	0.0223 (0.00025)	0.5429 (0.1241)	0.1969 (0.02191)	0.02 (0.00026)	0.5 (0.1255)	0.2 (0.02191)
(5,20,10)	$(10,0^9)$	0.045 (0.00235)	0.1195 (0.0086)	0.5759 (0.3061)	0.04 (0.0023)	0.1 (0.0086)	0.6 (0.30605)
	$(0^3,2,3,3,2,0^3)$	0.039 (0.00111)	0.1061 (0.01157)	0.5098 (0.22135)	0.04 (0.00109)	0.1 (0.01157)	0.5 (0.22137)
	$(0^9,10)$	0.1031 (0.02812)	0.1275 (0.01397)	0.4116 (0.17447)	0.1 (0.02622)	0.1 (0.01395)	0.4 (0.17454)
(5,40,10)	$(30,0^9)$	0.0157 (0.00008)	0.1428 (0.00491)	0.5899 (0.30264)	0.02 (0.00008)	0.1 (0.0049)	0.6 (0.30255)
	$(0^3,10^3,0^4)$	0.0445 (0.00212)	0.1402 (0.00812)	0.4695 (0.19471)	0.04 (0.00217)	0.1 (0.00812)	0.5 (0.19478)
	$(0^9,30)$	0.1596 (0.0467)	0.1499 (0.00525)	0.2517 (0.04364)	0.17 (0.0565)	0.1 (0.00524)	0.3 (0.04362)
(5,40,20)	$(20,0^{19})$	0.015 (0.0001)	0.1575 (0.00359)	0.548 (0.26612)	0.02 (0.0001)	0.2 (0.00359)	0.5 (0.26609)
	$(0^8,5^4,0^8)$	0.0331 (0.00177)	0.1495 (0.00445)	0.4845 (0.20727)	0.03 (0.00179)	0.1 (0.00445)	0.5 (0.20723)
	$(0^{19},20)$	0.0434 (0.00318)	0.1781 (0.00212)	0.28 (0.05827)	0.04 (0.00323)	0.2 (0.00214)	0.3 (0.05828)

Continuation of Table 2

(k,n,m)	Scheme	Linex c_1			Linex c_2		
		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$
(2,20,10)	$(10,0^9)$	0.04 (0.0011)	0.6 (0.1616)	0.3 (0.08396)	0.04 (0.0011)	0.6 (0.16161)	0.3 (0.08396)
	$(0^3,2,3,3,2,0^3)$	0.04 (0.00116)	0.6 (0.20301)	0.3 (0.0486)	0.04 (0.00116)	0.6 (0.20302)	0.3 (0.0486)
	$(0^9,10)$	0.11 (0.03685)	0.5 (0.14758)	0.3 (0.05399)	0.11 (0.03707)	0.5 (0.14762)	0.3 (0.05399)
(2,40,10)	$(30,0^9)$	0.08 (0.00495)	0.4 (0.08642)	0.3 (0.07308)	0.08 (0.00496)	0.4 (0.08643)	0.3 (0.07308)
	$(0^3,10^3,0^4)$	0.15 (0.04726)	0.5 (0.10703)	0.3 (0.04652)	0.16 (0.04839)	0.5 (0.10715)	0.3 (0.04652)
	$(0^9,30)$	0.02 (0.00032)	0.6 (0.17112)	0.3 (0.05208)	0.02 (0.00032)	0.6 (0.17116)	0.3 (0.05208)
(2,40,20)	$(20,0^{19})$	0.03 (0.00029)	0.4 (0.06716)	0.4 (0.12869)	0.03 (0.00029)	0.4 (0.06716)	0.4 (0.12869)
	$(0^8,5^4,0^8)$	0.07 (0.00515)	0.4 (0.06121)	0.3 (0.06486)	0.07 (0.00517)	0.4 (0.06122)	0.3 (0.06486)
	$(0^{19},20)$	0.02 (0.00026)	0.5 (0.1255)	0.2 (0.02191)	0.02 (0.00026)	0.5 (0.12551)	0.2 (0.02191)
(5,20,10)	$(10,0^9)$	0.04 (0.0023)	0.1 (0.0086)	0.6 (0.30605)	0.04 (0.0023)	0.1 (0.0086)	0.6 (0.30605)
	$(0^3,2,3,3,2,0^3)$	0.04 (0.00109)	0.1 (0.01157)	0.5 (0.22137)	0.04 (0.00109)	0.1 (0.01157)	0.5 (0.22137)
	$(0^9,10)$	0.1 (0.02621)	0.1 (0.01395)	0.4 (0.17454)	0.1 (0.02623)	0.1 (0.01395)	0.4 (0.17454)
(5,40,10)	$(30,0^9)$	0.02 (0.00008)	0.1 (0.0049)	0.6 (0.30255)	0.02 (0.00008)	0.1 (0.0049)	0.6 (0.30255)
	$(0^3,10^3,0^4)$	0.04 (0.00217)	0.1 (0.00812)	0.5 (0.19478)	0.04 (0.00217)	0.1 (0.00812)	0.5 (0.19478)
	$(0^9,30)$	0.17 (0.05574)	0.1 (0.00524)	0.3 (0.04362)	0.17 (0.05729)	0.1 (0.00524)	0.3 (0.04362)
(5,40,20)	$(20,0^{19})$	0.02 (0.0001)	0.2 (0.00359)	0.5 (0.26609)	0.02 (0.0001)	0.2 (0.00359)	0.5 (0.26609)
	$(0^8,5^4,0^8)$	0.03 (0.00179)	0.1 (0.00445)	0.5 (0.20723)	0.03 (0.00179)	0.1 (0.00445)	0.5 (0.20723)
	$(0^{19},20)$	0.04 (0.00323)	0.2 (0.00214)	0.3 (0.05828)	0.04 (0.00323)	0.2 (0.00214)	0.3 (0.05828)

Continuation of Table 2

(k,n,m)	Scheme	Entropy		
		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$
(2,20,10)	$(10,0^9)$	0.04 (0.0011)	0.6 (0.16159)	0.3 (0.08396)
	$(0^3,2,3,3,2,0^3)$	0.04 (0.00116)	0.6 (0.203)	0.3 (0.0486)
	$(0^9,10)$	0.11 (0.03661)	0.5 (0.14755)	0.3 (0.05399)
(2,40,10)	$(30,0^9)$	0.08 (0.00489)	0.4 (0.08642)	0.3 (0.07308)
	$(0^3,10^3,0^4)$	0.14 (0.0453)	0.5 (0.10689)	0.3 (0.04652)
	$(0^9,30)$	0.02 (0.00032)	0.6 (0.17109)	0.3 (0.05208)

Continuation of Table 2

(k,n,m)	Scheme	Entropy		
		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$
(2,40,20)	$(20,0^{19})$	0.03 (0.00029)	0.4 (0.06716)	0.4 (0.12869)
	$(0^8,5^4,0^8)$	0.07 (0.00504)	0.4 (0.06121)	0.3 (0.06486)
	$(0^{19},20)$	0.02 (0.00026)	0.5 (0.12549)	0.2 (0.02191)
(5,20,10)	$(10,0^9)$	0.04 (0.0023)	0.1 (0.0086)	0.6 (0.30605)
	$(0^3,2,3,3,2,0^3)$	0.04 (0.00109)	0.1 (0.01157)	0.5 (0.22137)
	$(0^9,10)$	0.1 (0.02618)	0.1 (0.01395)	0.4 (0.17454)
(5,40,10)	$(30,0^9)$	0.02 (0.00008)	0.1 (0.0049)	0.6 (0.30255)
	$(0^3,10^3,0^4)$	0.04 (0.00217)	0.1 (0.00812)	0.5 (0.19478)
	$(0^9,30)$	0.17 (0.05399)	0.1 (0.00524)	0.3 (0.04362)
(5,40,20)	$(20,0^{19})$	0.02 (0.0001)	0.2 (0.00359)	0.5 (0.26609)
	$(0^8,5^4,0^8)$	0.03 (0.00179)	0.1 (0.00445)	0.5 (0.20723)
	$(0^{19},20)$	0.04 (0.00322)	0.2 (0.00214)	0.3 (0.05828)

Table 3: 95% confidence intervals and lenghts when $\alpha = 0.02$, $\theta = 0.2$ and $\beta = 0.05$ with prior 0.

(k,n,m)	Scheme	confidence interval			lenght		
		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$
(2,20,10)	$(10,0^9)$	[-0.2616,0.3172]	[-1.1912,2.3722]	[0.1486,0.6137]	0.57877	3.56343	0.46513
	$(0^3,2,3,3,2,0^3)$	[-2.0095,2.2065]	[-1.6179,2.5731]	[0.0598,0.4897]	4.21601	4.19093	0.42989
	$(0^9,10)$	[-0.7482,0.8238]	[-1.5777,2.5295]	[0.0227,0.5339]	1.57201	4.1072	0.51120
(2,40,10)	$(30,0^9)$	[-1.2054,1.3524]	[-1.6184,2.5279]	[0.1047,0.5786]	2.55779	4.14627	0.47394
	$(0^3,10^3,0^4)$	[-1.5998,1.8063]	[-1.0604,2.0039]	[0.1013,0.4249]	3.40609	3.06436	0.32360
	$(0^9,30)$	[-0.3569,0.3908]	[-1.4089,2.405]	[0.0518,0.4173]	0.74768	3.81387	0.36554
(2,40,20)	$(20,0^{19})$	[-0.3466,0.4512]	[-0.3106,1.1358]	[0.1644,0.4217]	0.79785	1.4464	0.25734
	$(0^8,5^4,0^8)$	[-0.3459,0.4144]	[-0.6127,1.6002]	[0.1432,0.3874]	0.76030	2.21284	0.24420
	$(0^{19},20)$	[-2.6812,2.8479]	[-1.9707,2.825]	[0.0009,0.4555]	5.52907	4.79576	0.45457
(5,20,10)	$(10,0^9)$	[-0.049,0.0678]	[-0.3059,0.6836]	[0.2108,0.9124]	0.11676	0.989505	0.70157
	$(0^3,2,3,3,2,0^3)$	[-0.3021,0.3646]	[-0.2976,0.6068]	[0.1239,0.6325]	0.66670	0.904434	0.50858
	$(0^9,10)$	[-1.3693,1.5372]	[-0.2762,0.4035]	[0.0364,0.8233]	2.90648	0.67969	0.78691
(5,40,10)	$(30,0^9)$	[-0.3014,0.3728]	[-0.302,0.6764]	[0.1419,0.5975]	0.67425	0.97845	0.45555
	$(0^3,10^3,0^4)$	[-0.391,0.4893]	[-0.2026,0.5069]	[0.1596,0.5675]	0.88028	0.709521	0.40791
	$(0^9,30)$	[-2.883,3.0448]	[-0.682,1.0547]	[-0.0026,0.3611]	5.92785	1.73669	0.36369
(5,40,20)	$(20,0^{19})$	[-0.0708,0.1033]	[-0.1438,0.4737]	[0.246,0.7036]	0.17416	0.617463	0.45757
	$(0^8,5^4,0^8)$	[-0.202,0.2664]	[-0.1095,0.3685]	[0.1726,0.5314]	0.46843	0.477932	0.35882
	$(0^{19},20)$	[-0.4006,0.4673]	[-0.3197,0.6618]	[0.1122,0.5032]	0.86795	0.981472	0.39100

Table 4: 95%credible intervals and lenghts when $\alpha = 0.02$, $\theta = 0.2$ and $\beta = 0.05$ with prior 0.

(k,n,m)	Scheme	credible interval			lenght		
		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$
(2,20,10)	$(10,0^9)$	[0.03,0.03]	[0.6,0.6]	[0.4,0.4]	0.000242177	0.0037315	0.000104824
	$(0^3,2,3,3,2,0^3)$	[0.08,0.09]	[0.5,0.5]	[0.3,0.3]	0.0109802	0.0149261	0.000103165
	$(0^9,10)$	[0.04,0.04]	[0.5,0.5]	[0.3,0.3]	0.000865361	0.00714831	0.000135225
(2,40,10)	$(30,0^9)$	[0.07,0.07]	[0.5,0.5]	[0.3,0.3]	0.00444408	0.00471704	0.0000987826
	$(0^3,10^3,0^4)$	[0.1,0.11]	[0.5,0.5]	[0.3,0.3]	0.00621952	0.00357858	0.0000431964
	$(0^9,30)$	[0.02,0.02]	[0.5,0.5]	[0.2,0.2]	0.000226412	0.00996565	0.0000853009
(2,40,20)	$(20,0^{19})$	[0.05,0.05]	[0.4,0.4]	[0.3,0.3]	0.000272694	0.000843553	0.0000252731
	$(0^8,5^4,0^8)$	[0.03,0.03]	[0.5,0.5]	[0.3,0.3]	0.000172681	0.00232227	0.0000250953
	$(0^{19},20)$	[0.05,0.08]	[0.4,0.4]	[0.2,0.2]	0.0234134	0.00671674	0.0000704685
(5,20,10)	$(10,0^9)$	[0.01,0.01]	[0.2,0.2]	[0.6,0.6]	5.29173×10^{-6}	0.000320189	0.000394885
	$(0^3,2,3,3,2,0^3)$	[0.03,0.03]	[0.2,0.2]	[0.4,0.4]	0.000249188	0.00044147	0.000182729
	$(0^9,10)$	[0.09,0.09]	[0.1,0.1]	[0.4,0.4]	0.00944347	0.000256039	0.000186573
(5,40,10)	$(30,0^9)$	[0.04,0.04]	[0.2,0.2]	[0.4,0.4]	0.000188038	0.000413565	0.0000914301
	$(0^3,10^3,0^4)$	[0.05,0.05]	[0.2,0.2]	[0.4,0.4]	0.000187531	0.000195387	0.0000999241
	$(0^9,30)$	[0.08,0.09]	[0.2,0.2]	[0.2,0.2]	0.0150912	0.00102271	0.00011986
(5,40,20)	$(20,0^{19})$	[0.02,0.02]	[0.2,0.2]	[0.5,0.5]	9.20555×10^{-6}	0.000148458	0.0000544297
	$(0^8,5^4,0^8)$	[0.03,0.03]	[0.1,0.1]	[0.4,0.4]	0.000107113	0.000088064	0.0000324409
	$(0^{19},20)$	[0.03,0.03]	[0.2,0.2]	[0.3,0.3]	0.000242278	0.000652	0.000066341

Table5:95% confidence intervals and lenghts when $\alpha = 0.02$, $\theta = 0.2$ and $\beta = 0.05$ with prior 1.

(k,n,m)	Scheme	confidence interval			lenght		
		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$
(2,20,10)	$(10,0^9)$	[-0.5848,0.6597]	[-1.493,2.7289]	[0.0975,0.4299]	1.24451	4.22196	0.33233
	$(0^3,2,3,3,2,0^3)$	[-0.6833,0.7613]	[-1.8851,3.1252]	[0.0906,0.4755]	1.44457	5.01023	0.38491
	$(0^9,10)$	[-2.4763,2.6408]	[-2.9534,3.9873]	[-0.0223,0.5924]	5.11711	6.94073	0.61464
(2,40,10)	$(30,0^9)$	[-0.986,1.1514]	[-0.8606,1.6567]	[0.1245,0.5121]	2.13739	2.51732	0.38756
	$(0^3,10^3,0^4)$	[-10.2268,10.5815]	[-3.8544,4.6345]	[-0.2103,0.7203]	20.8084	8.48895	0.93056
	$(0^9,30)$	[-0.1717,0.2051]	[-1.1199,2.3302]	[0.101,0.5754]	0.37672	3.45004	0.47437
(2,40,20)	$(20,0^{19})$	[-0.1567,0.2138]	[-0.3685,1.1092]	[0.2279,0.6062]	0.37054	1.47775	0.37825
	$(0^8,5^4,0^8)$	[-0.796,0.9372]	[-0.6341,1.4993]	[0.1411,0.393]	1.73323	2.1334	0.25187
	$(0^{19},20)$	[-0.3254,0.3589]	[-0.9878,2.0297]	[0.0751,0.3005]	0.68432	3.01757	0.22542
(5,20,10)	$(10,0^9)$	[-0.209,0.2962]	[-0.1049,0.2447]	[0.2489,0.9961]	0.50516	0.34953	0.74719
	$(0^3,2,3,3,2,0^3)$	[-0.2573,0.3319]	[-0.1705,0.3488]	[0.1688,0.8904]	0.58926	0.51926	0.72152
	$(0^9,10)$	[-1.5949,1.8112]	[-0.2334,0.395]	[0.0087,0.732]	3.40611	0.62835	0.72325
(5,40,10)	$(30,0^9)$	[-0.0909,0.1358]	[-0.1498,0.3741]	[0.2513,0.9141]	0.22665	0.52391	0.66281
	$(0^3,10^3,0^4)$	[-0.2627,0.3427]	[-0.1884,0.434]	[0.213,0.7307]	0.60534	0.62242	0.51774
	$(0^9,30)$	[-6.908,7.3107]	[-0.516,0.7089]	[-0.0934,0.6118]	14.2187	1.22484	0.70516
(5,40,20)	$(20,0^{19})$	[-0.1049,0.1453]	[-0.1972,0.5134]	[0.2546,0.7346]	0.25023	0.71053	0.47996
	$(0^8,5^4,0^8)$	[-0.1145,0.176]	[-0.0722,0.2894]	[0.272,0.6747]	0.29051	0.36154	0.40264
	$(0^{19},20)$	[-0.4115,0.5011]	[-0.1378,0.4451]	[0.1229,0.3982]	0.91263	0.58290	0.27531

Table 6: 95%credible intervals and lenghts when $\alpha = 0.02$, $\theta = 0.2$ and $\beta = 0.05$ with prior 1.

(k,n,m)	Scheme	credible interval			lenght		
		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$
(2,20,10)	$(10,0^9)$	[0.04,0.04]	[0.6,0.6]	[0.3,0.3]	0.00114425	0.00800339	0.000048277
	$(0^3,2,3,3,2,0^3)$	[0.04,0.04]	[0.6,0.6]	[0.3,0.3]	0.0014012	0.013843	0.000102158
	$(0^9,10)$	[0.09,0.11]	[0.5,0.5]	[0.3,0.3]	0.0216325	0.0188735	0.000112905
(2,40,10)	$(30,0^9)$	[0.08,0.08]	[0.4,0.4]	[0.3,0.3]	0.000965609	0.00327106	0.000105455
	$(0^3,10^3,0^4)$	[0.14,0.34]	[0.4,0.5]	[0.3,0.3]	0.196035	0.0496483	0.0004256
	$(0^9,30)$	[0.02,0.02]	[0.6,0.6]	[0.3,0.3]	0.000101896	0.00705761	0.0000774068

Continuation of Table 6

(k,n,m)	Scheme	credible interval			lenght		
		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$
(2,40,20)	$(20,0^{19})$	[0.03,0.03]	[0.4,0.4]	[0.4,0.4]	0.000055836	0.00155493	0.0000883702
	$(0^8,5^4,0^8)$	[0.07,0.07]	[0.4,0.4]	[0.3,0.3]	0.00217669	0.00160358	0.0000280296
	$(0^{19},20)$	[0.02,0.02]	[0.5,0.5]	[0.2,0.2]	0.000228227	0.0024037	0.0000126326
(5,20,10)	$(10,0^9)$	[0.04,0.04]	[0.1,0.1]	[0.6,0.6]	0.000115748	0.0000595539	0.00021393
	$(0^3,2,3,3,2,0^3)$	[0.04,0.04]	[0.1,0.1]	[0.5,0.5]	0.000133515	0.000122821	0.00026768
	$(0^9,10)$	[0.1,0.11]	[0.1,0.1]	[0.4,0.4]	0.00779305	0.000121839	0.000334387
(5,40,10)	$(30,0^9)$	[0.02,0.02]	[0.1,0.1]	[0.6,0.6]	0.0000308529	0.000148534	0.000298476
	$(0^3,10^3,0^4)$	[0.04,0.04]	[0.1,0.1]	[0.5,0.5]	0.000234433	0.000373319	0.000111128
	$(0^9,30)$	[0.15,0.3]	[0.1,0.1]	[0.3,0.3]	0.149934	0.000738215	0.000254691
(5,40,20)	$(20,0^{19})$	[0.02,0.02]	[0.2,0.2]	[0.5,0.5]	0.0000335207	0.000162418	0.000121668
	$(0^8,5^4,0^8)$	[0.03,0.03]	[0.1,0.1]	[0.5,0.5]	0.000036449	0.0000575496	0.000116061
	$(0^{19},20)$	[0.04,0.04]	[0.2,0.2]	[0.3,0.3]	0.000443825	0.000143974	0.0000510502

Table 7: Random grouping to the real data set of survival period of patients treated with chemotherapy.

item	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	63	105	129	182	216	250	262	301	301	342	354	356	358	380
2	383	383	388	394	408	460	489	499	523	524	535	562	569	675	676
3	748	778	786	797	955	968	1000	1245	1271	1420	1551	1694	2363	2754	2950

Table 8: Different censoring schemes.

(k,n,m)	censoring scheme
(2,15,8)	R_1=(7,0*7)
	R_2=(0*2,3,1,1,2,0*2)
	R_3=(0*7,7)

Table 9: The MLE and Bayes estimates of the parameters for the real data set

(k,n,m)	Scheme	MLE			SEL		
		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$
(3,15,8)	$(7,0^7)$	0.796 (0.06084)	0.0172 (0.00542)	0.5185 (0.02195)	0.7393 (0.05232)	0.0172 (0.00542)	0.5183 (0.02193)
	$(0^2,3,1,1,2,0^2)$	0.0079 (0.00001)	0.2348 (0.00002)	0.6302 (0.03366)	0.0079 (0.00001)	0.234 (0.00003)	0.6301 (0.03365)
	$(0^7,7)$	0.0088 (0.00001)	0.1933 (0.00032)	0.6545 (0.03654)	0.0089 (0.00001)	0.1933 (0.00032)	0.6542 (0.03651)

Continuation of Table 9

(k,n,m)	Scheme	Linex			Entropy		
		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$
(3,15,8)	$(7,0^7)$	0.7383 (0.05218)	0.0172 (0.00542)	0.5183 (0.02193)	0.7374 (0.05204)	0.0172 (0.00542)	0.5183 (0.02193)
	$(0^2,3,1,1,2,0^2)$	0.0079 (0.00001)	0.234 (0.00003)	0.6301 (0.03365)	0.0079 (0.00001)	0.234 (0.00003)	0.6301 (0.03365)
	$(0^7,7)$	0.0089 (0.00001)	0.1933 (0.00032)	0.6542 (0.03651)	0.0089 (0.00001)	0.1933 (0.00032)	0.6542 (0.03651)

Table 10: ACI intervals and the corresponding length of the parameters for the real data set

(k,n,m)	Scheme	ACI		
		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$
		interval	interval	interval
(3,15,8)	$(7,0^7)$	(-4.6876,6.2796)	(-0.0452,0.0796)	(0.2385,0.7984)
	$(0^2,3,1,1,2,0^2)$	(-0.0495,0.0653)	(-0.5634,1.0329)	(0.1902,1.0701)
	$(0^7,7)$	(-0.0395,0.0572)	(-0.3209,0.7074)	(0.2332,1.0757)

Continuation of Table 10

(k,n,m)	Scheme	ACI		
		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$
		length	length	length
(3,15,8)	$(7,0^7)$	10.9672	0.124737	0.559951
	$(0^2,3,1,1,2,0^2)$	0.114765	1.59627	0.879971
	$(0^7,7)$	0.096608	1.02829	0.842535

Table 11: HPD credible intervals and the corresponding length of the parameters for the real data set

(k,n,m)	Scheme	HPD		
		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$
		interval	interval	interval
(3,15,8)	$(7,0^7)$	(0.6894,0.804)	(0.0172,0.0172)	(0.5183,0.5184)
	$(0^2,3,1,1,2,0^2)$	(0.0079,0.0079)	(0.2337,0.2344)	(0.6299,0.6303)
	$(0^7,7)$	(0.0088,0.0089)	(0.193,0.1935)	(0.6539,0.6544)

Continuation of Table 11

(k,n,m)	Scheme	HPD		
		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$
		length	length	length
(3,15,8)	$(7,0^7)$	0.114518	5.16664×10^{-6}	0.000109072
	$(0^2,3,1,1,2,0^2)$	5.28691×10^{-6}	0.000703505	0.000325039
	$(0^7,7)$	2.83076×10^{-6}	0.000477714	0.000491189

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