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Estimation of the Scale Parameter of Quasi Lindley Distribution in the Presence of Outliers

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Abstract: In this paper, we estimate the scale parameter of quasi Lindley distribution in the presence of a single outlier based on order statistics by using the maximum likelihood and Bayes estimation methods. We derive the exact expressions for the single and product moments of order statistics from quasi Lindley distribution in the presence of multiple outliers and use these moments to study the robustness of the best linear unbiased estimator and some other linear estimators of the scale parameter. We compute some numerical results and use real data to show the effect of outliers on the estimators.

Keywords: Outliers, Slippage model, Order statistics, Estimator, Moments

1 Introduction

Lindley distribution is one of the important lifetime distributions, which has received a large share of researchers' attention during the past years, for example: [1], [2] and [3]. [4] suggested the two-parameter Lindley distribution or Quasi Lindley distribution (*QLD*), and they studied some properties of it. The probability density distribution (pdf) of *QLD* with shape parameter α and scale parameter θ is

$$f(x;\alpha,\theta)=\frac{1}{\theta(1+\alpha)}(\alpha+\frac{x}{\theta})e^{-\frac{x}{\theta}},x>0,\ \theta>0,\ \alpha>0. \tag{1}$$

The cumulative distribution function (cdf) of *QLD* is

$$F(x;\alpha,\theta)=1-[1+\frac{x}{\theta(1+\alpha)}]e^{-\frac{x}{\theta}},x>0,\ \theta>0,\ \alpha>0. \tag{2}$$

The one parameter Lindley distribution $(\frac{1}{\theta})$ is a special case from *QLD* when $\alpha=\frac{1}{\theta}$. Also, gamma distribution $(2,\frac{1}{\theta})$ is a special case from *QLD* When $\alpha=0$. For more details see [4].
Over the past few years, researchers have published a number of studies of this distribution, for example: [5], [6], [7], [8], [9] and [10].

Outlier is the observation that is not consistent with the rest of the observations. There are several models for outliers, one of them and most commonly studied is a contamination model called the slippage model. Under this model, we consider the observations $X_1,\ X_2,\ldots,\ X_{n-p}$ are independent random variables $(r.v.'s)$ with pdf $f_1(x)$ and cdf $F_1(x)$ while the other observations (outliers) $X_{n-p+1},\ \ldots,\ X_n$ are independent (and independent $X_1,\ X_2,\ldots,\ X_{n-p}$) with pdf $f_2(x)$ and cdf $F_2(x)$, where f_2 and F_2 are differs of f_1 and F_1 respectively, in the values of location and/or scale parameters.
Let $X_{1:n},\ X_{2:n},\ldots,\ X_{n:n}$ denote the order statistics obtained by arranging the $X_i^{'}s$ in increasing order of magnitude. The

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marginal pdf of the r^{th} order statistic under the multiple outliers model is

$$f_{r:n}[p](x) = \sum_{s=max(0,r-p-1)}^{min(n-p-1,r-1)} C_1 f_1(x)\{F_1(x)\}^s\{F_2(x)\}^{r-s-1} \times \{1-F_1(x)\}^{n-p-s-1}\{1-F_2(x)\}^{p-r+s+1} + \sum_{s=max(0,r-p)}^{min(n-p,r-1)} C_2 f_2(x)\{F_1(x)\}^s\{F_2(x)\}^{r-s-1} \times \{1-F_1(x)\}^{n-p-s}\{1-F_2(x)\}^{p-r+s}, \qquad -\infty < x < \infty, \tag{3}$$

where

$$C_1 = \frac{(n-p)!p!}{s!(r-s-1)!(n-p-s-1)!(p-r+s+1)!}$$

and

$$C_2 = \frac{(n-p)!p!}{s!(r-s-1)!(n-p-s)!(p-r+s)!}.$$

The joint pdf of the r^{th} and s^{th} order statistics $(1 \leqslant r < s \leqslant n)$ is

$$\begin{aligned} f_{r,s:n}[p](x,y) = & \sum_{j=0}^{s-r-1}\sum_{i=max(0,s-p-j-2)}^{min(n-p-j-2,r-1)} A_1 f_1(x) f_1(y) F_1(x)^i F_2(x)^{r-1-i}\{F_1(y)-F_1(x)\}^j \\ & \times \{F_2(y)-F_2(x)\}^{s-r-1-j}\{1-F_1(y)\}^{n-p-i-j-2}\{1-F_2(y)\}^{p-s+i+j+2} \\ & + \sum_{j=0}^{s-r-1}\sum_{i=max(0,s-p-j-1)}^{min(n-p-j-1,r-1)} A_2 f_1(x) f_2(y) F_1(x)^i F_2(x)^{r-1-i}\{F_1(y)-F_1(x)\}^j \\ & \times \{F_2(y)-F_2(x)\}^{s-r-1-j}\{1-F_1(y)\}^{n-p-i-j-1}\{1-F_2(y)\}^{p-s+i+j+1} \\ & + \sum_{j=0}^{s-r-1}\sum_{i=max(0,s-p-j-1)}^{min(n-p-j-1,r-1)} A_2 f_2(x) f_1(y) F_1(x)^i F_2(x)^{r-1-i}\{F_1(y)-F_1(x)\}^j \\ & \times \{F_2(y)-F_2(x)\}^{s-r-1-j}\{1-F_1(y)\}^{n-p-i-j-1}\{1-F_2(y)\}^{p-s+i+j+1} \\ & + \sum_{j=0}^{s-r-1}\sum_{i=max(0,s-p-j)}^{min(n-p-j,r-1)} A_3 f_2(x) f_2(y) F_1(x)^i F_2(x)^{r-1-i}\{F_1(y)-F_1(x)\}^j \\ & \times \{F_2(y)-F_2(x)\}^{s-r-1-j}\{1-F_1(y)\}^{n-p-i-j}\{1-F_2(y)\}^{p-s+i+j}, \\ & -\infty < x < y < \infty, \end{aligned} \tag{4}$$

where

$$A_1 = \frac{(n-p)!p!}{i!(r-i-1)!j!(s-r-1-j)!(n-p-i-j-2)!(p-s+i+j+2)!},$$

$$A_2 = \frac{(n-p)!p!}{i!(r-i-1)!j!(s-r-1-j)!(n-p-i-j-1)!(p-s+i+j+1)!},$$

$$and \ A_3 = \frac{(n-p)!p!}{i!(r-i-1)!j!(s-r-1-j)!(n-p-i-j)!(p-s+i+j)!},$$

see [11]
The joint pdf of all n order statistics when $p=1$ is

$$f_{1,\,2,\,\dots,\,n:n}[\mathbf{1}](x_1,\,x_2,\,\dots,\,x_n) = (n-1)!\prod_{i=1}^{n} f_1(x_i)\sum_{i=1}^{n}\frac{f_2(x_i)}{f_1(x_i)}, \quad -\infty < x_1 < x_2 < \dots < x_n < \infty, \tag{5}$$

see [12].
In this paper, we study the effect of outliers on some estimators. We first find the maximum likelihood and Bayes estimators of the scale parameter of (QLD) based on order statistics in the presence of a single outlier, we derive the single and product moments of order statistics from QLD and study the robustness of some linear estimators of the scale parameter in the presence of multiple outliers.

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2 Estimation in the Presence of Single Outlier

Let $X_1,\ X_2,...,\ X_{n-1}$ be independent observations from $QLD(\alpha,\theta)$ with pdf (1) and cdf (2), and let X_n be an observation from $QLD(\beta,\gamma)$ with pdf

$$f_2(x;\beta,\gamma)=\frac{1}{\gamma(1+\beta)}(\beta+\frac{x}{\gamma})e^{-\frac{x}{\gamma}},x>0,\ \gamma>0,\ \beta>0, \tag{6}$$

and cdf

$$F_2(x;\beta,\gamma)=1-[1+\frac{x}{\gamma(1+\beta)}]e^{-\frac{x}{\gamma}},x>0,\ \gamma>0,\ \beta>0. \tag{7}$$

Suppose that $\alpha=\beta$ and $\gamma=\theta/g$, where $0<g<1$, (this model is known as a single-scale outlier model). The joint pdf of all n order statistics $X_{1:n},\ X_{2:n},\ ...,X_{n:n}$ from QLD in the presence of a single outlier is given by replacing (1) and (6) in (5) as

$$f_{(1,\ 2,...,\ n:n)}[\mathbf{1}](x_1,\ x_2,\ ...,x_n)\propto(\frac{1}{\theta})^n\Pi_{i=1}^n t_i(\theta)e^{-\frac{\sum_{i=1}^n x_i}{\theta}}B_1(\theta),$$
$$x_i>0,\ \theta>0,\ \alpha>0, \tag{8}$$

where

$$t_i(\theta)=\alpha+\frac{x_i}{\theta},and\ B_1(\theta)=\sum_{i=1}^n ge^{-(g-1)\frac{x_i}{\theta}}\frac{t_i(\theta/g)}{t_i(\theta)}. \tag{9}$$

2.1 Maximum likelihood estimator

The maximum likelihood estimator (MLE) of the scale parameter θ of QLD in the presence of a single outlier can be obtained by solving the following nonlinear equation

$$\sum_{i=1}^n x_i-n\theta-\sum_{i=1}^n\frac{x_i}{t_i(\theta)}+\theta^2\frac{\frac{\partial B_1(\theta)}{\partial\theta}}{B_1(\theta)}=0, \tag{10}$$

where

$$\frac{\partial B_1(\theta)}{\partial\theta}=\sum_{i=1}^n g(1-g)e^{-\frac{(g-1)x_i}{\theta}}\grave{t}_i(\theta)T_i(\theta),$$
$$\grave{t}_i(\theta)=-\frac{x_i}{\theta^2},\ and\ T_i(\theta)=\frac{t_i(\theta/g)t_i(\theta)-\alpha}{(t_i(\theta))^2}. \tag{11}$$

2.2 Bayes estimator

To obtain Bayes estimator of the scale parameter θ of QLD -when the parameter α is known- in the presence of a single outlier, we consider the inverse gamma conjugate prior density for the parameter θ as follows

$$\pi(\theta)=\frac{a^b}{\Gamma(b)}(\frac{1}{\theta})^{b+1}e^{-\frac{a}{\theta}},\qquad\theta>0,\ a>0,\ b>0. \tag{12}$$

It follows from (8) and (12) that the posterior distribution of θ is given by

$$\pi^*(\theta|x_1,\ x_2,\ ...,x_n,\alpha)\propto(\frac{1}{\theta})^{n+b+1}\Pi_{i=1}^n t_i(\theta)e^{-\frac{\sum_{i=1}^n x_i+a}{\theta}}B_1(\theta),$$
$$x_i>0,\ \theta>0,\ \alpha>0,\ a>0,\ b>0. \tag{13}$$

Under the squared error loss function (SEL), Bayes estimator of the scale parameter θ of QLD in the presence of a single outlier is

$$\hat{\theta}_B\simeq\tilde{\theta}+\frac{1}{2}(\frac{n+b+1}{\tilde{\theta}^2}-2\frac{\sum_{i=1}^n x_i+a}{\tilde{\theta}^3}+\sum_{i=1}^n\chi(t_i(\tilde{\theta}))+\chi(B_1(\tilde{\theta})))^{-4}$$
$$\times(-\frac{n+b+1}{\tilde{\theta}^3}+6\frac{\sum_{i=1}^n x_i+a}{\tilde{\theta}^4}+\sum_{i=1}^n\varphi(t_i(\tilde{\theta}))+\varphi(B_1(\tilde{\theta}))), \tag{14}$$

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where

$\chi(w) = \frac{w\frac{\partial^2 w}{\partial\theta^2} - (\frac{\partial w}{\partial\theta})^2}{w^2}$, $\varphi(w) = \frac{w^2\frac{\partial^3 w}{\partial\theta^3} - 3w\frac{\partial w}{\partial\theta}\frac{\partial^2 w}{\partial\theta^2} + 2(\frac{\partial w}{\partial\theta})^2}{w^3}$,

$B_1(\theta)$ and $t_i(\theta)$ are as in (9),

$\frac{\partial B_1(\theta)}{\partial\theta}$, $\grave{t}_i(\theta)$ and $T_i(\theta)$ are as in (11),

$$
\begin{aligned}
\frac{\partial^2 B_1(\theta)}{\partial\theta^2} &= \sum_{i=1}^{n} g(1-g)e^{-\frac{(g-1)x_i}{\theta}}(((1-g)(\grave{t}_i(\theta))^2(\theta) + t_i^{(2)}(\theta))T_i(\theta) + \grave{t}_i(\theta)\grave{T}_i(\theta)),\\
\frac{\partial^3 B_1(\theta)}{\partial\theta^3} &= \sum_{i=1}^{n} g(1-g)e^{-\frac{(g-1)x_i}{\theta}}\\
&\times ((((1-g)^2(\grave{t}_i(\theta))^3 + 2(1-g)\grave{t}_i(\theta))t_i^{(2)}(\theta) + t_i^{(2)}(\theta) + t_i^{(3)}(\theta))T_i(\theta)\\
&+ ((\grave{t}_i(\theta))^2 + 2t_i^{(2)}(\theta) + \grave{t}_i(\theta))\grave{T}_i(\theta) + \grave{t}_i(\theta)T_i^{(2)}(\theta)),\\
\grave{T}_i(\theta) &= \grave{t}_i(\theta)(\frac{gt_i(\theta) + t_i(\theta/g)}{(t_i(\theta))^2} - 2\frac{T_i(\theta)}{t_i(\theta)}),\\
T_i^{(2)}(\theta) &= \frac{2g(\grave{t}_i(\theta))^2 t_i(\theta) + (gt_i(\theta) + t_i(\theta/g))(t_i(\theta)t_i^{(2)}(\theta) - 2\grave{t}_i(\theta))}{(t_i(\theta))^3}\\
&- 2\frac{(t_i^{(2)}(\theta)t_i(\theta) - 2(\grave{t}_i(\theta))^2)T_i(\theta) + t_i(\theta)\grave{t}_i(\theta)\grave{T}_i(\theta)}{(t_i(\theta))^2},\\
t_i^{(2)}(\theta) &= 2\frac{x_i}{\theta^3},\ t_i^{(3)}(\theta) = -6\frac{x_i}{\theta^4},
\end{aligned}
$$

and $\tilde{\theta}$ is the posterior mode.

Proof. As we know, Bayes estimator of any function U in the parameter θ under the SEL is the posterior mean in the form

$$
\begin{aligned}
E[U(\theta) \mid x_1,\, x_2,\, ...,x_n)] &= \int_0^\infty U(\theta)\pi^*(\theta)d\theta\\
&= \frac{\int_0^\infty U(\theta)L(\theta \mid x_1,\, x_2,\, ...,x_n)\pi(\theta)d\theta}{\int_0^\infty L(\theta \mid x_1,\, x_2,\, ...,x_n)\pi(\theta)d\theta}.
\end{aligned}
\tag{15}
$$

By using Lindley approximation, see [13].

$$
\begin{aligned}
E(U(\theta) \mid x_1,\, x_2,\, ...,x_n) \simeq \{\{U(\theta) &+ \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} U_{ij}\varepsilon_{ij}\\
&+ \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m} h_{ijk}U_l\varepsilon_{ij}\varepsilon_{kl}]\}|_{\tilde{\theta}},
\end{aligned}
\tag{16}
$$

we first find the logarithm of the posterior distribution (13) denoted by $S(\theta)$ as

$$
S(\theta) = -(n+b+1)Log(\theta) + \frac{\sum_{i=1}^{n} x_i + a}{\theta} + \sum_{i=1}^{n} Log(t_i(\theta)) + Log(B_1(\theta)). \tag{17}
$$

By differentiating $S(\theta)$ with respect to θ, we get

$$
\frac{\partial S(\theta)}{\partial\theta} = \frac{-(n+b+1)}{\theta} - \frac{\sum_{i=1}^{n} x_i + a}{\theta^2} + \frac{1}{\theta^2}\sum_{i=1}^{n}\frac{x_i(\theta)}{t_i(\theta)} + \frac{\grave{B_1}(\theta)}{B_1(\theta)} \tag{18}
$$

The posterior mode $(\tilde{\theta})$ can be obtained by equating (18) to zero and solving it in θ.
Second, we differentiate (18) with respect to θ to get ε_{11}, as follow

$$
\varepsilon_{11} = \frac{n+b+1}{\theta^2} - 2\frac{\sum_{i=1}^{n} x_i + a}{\theta^3} + \sum_{i=1}^{n}\chi(t_i(\theta)) + \chi(B_1(\theta)) \tag{19}
$$

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The second derivative of (18) with respect to θ gives $h_{111}(\theta)$ as

$$h_{111}(\theta) = -\frac{n+b+1}{\theta^3} + 6\frac{\sum_{i=1}^{n} x_i + a}{\theta^4} + \sum_{i=1}^{n} \varphi(t_i(\theta)) + \varphi(B_1(\theta)). \quad (20)$$

By choosing $U(\theta) = \theta$; then, substituting of (19) and (20) in (16), we get (14).

2.3 Numerical results

In this subsection, we calculate the MLE using (10) and Bayes estimator in (14) and their corresponding mean squared errors (MSE) when $\alpha = 2$ by using Monte Carlo simulation study for $n = 5,\ 6,\ 7,\ 10,\ 20,\ 30$, $p = 0,\ 1$, and $g = 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5$. We use the following algorithm:

1. For given values of a, b $(a = 1,\ b = 1)$, we generate a random value θ from inverse gamma distribution (12).
2. For the chosen value of α and the generated value of θ in step (1), we generate random sample of size $n-1$ from $QLD(\alpha,\theta)$ and generate random sample of size 1 from $QLD(\alpha,\theta/g)$.
   Union and arrangement the values of the tow samples give an order statistics random sample of size n with a single outlier.
3. We compute the MLE $(\hat{\theta}_{ML})$ using (10) and compute Bayes estimator $(\hat{\theta}_B)$ in (14).
4. We compute the SEL $(\hat{\theta}-\theta)^2$ where $\hat{\theta}$ stands for an estimator (maximum likelihood or Bayes).
5. We repeat The above steps (2-4) 1000 times, then compute the mean of estimators and the MSE.

Table 1 show the numerical results.
From Table1, we can observe the following:

1. The MSE of Bayes estimator are less than MSE of maximum likelihood estimator when there is no outlier.
2. If there is an outlier, the MSE of Bayes estimator are less than MSE of maximum likelihood estimator in the following cases:$(n = 6,\ 7, g = 0.5)$, $(n = 10, g = 0.4,\ 0.5)$ and $(n = 20,\ 30, g = 0.2,\ 0.3,\ 0.4,\ 0.5)$, that means the presence of outlier affects strongly on Bayes estimator in the small samples, this effect decreases when the size of sample increases.
3. At $g = 0.1$ the maximum likelihood estimator is better than Bayes estimator for all values of n.

Table 1: The MLE and Bayes estimator and their corresponding MSE of the scale parameter θ of $QLD(\alpha = 2, \theta = 1.3775)$ in the presence of a single outlier.

g^* means any value of g.

p	g	n	$\hat{\theta}_{ML}$	$MSE(\hat{\theta}_{ML})$	$\hat{\theta}_B$	$MSE(\hat{\theta}_B)$	n	$\hat{\theta}_{ML}$	$MSE(\hat{\theta}_{ML})$	$\hat{\theta}_B$	$MSE(\hat{\theta}_B)$
0	g^*	5	1.343	.3199	1.159	.2750	10	1.364	.1746	1.243	.1515
	0.1		1.350	.3593	22.86	31812		1.354	.1723	3.037	148.4
	0.2		1.359	.3615	2.376	26.34		1.383	.1910	1.446	1.700
	0.3		1.381	.3536	1.551	10.18		1.412	.1857	1.306	.1896
	0.4		1.396	.3847	1.311	1.671		1.396	.1846	1.269	.1596
	0.5		1.358	.3461	1.188	.3903		1.367	.1777	1.242	.1568
0	g^*	6	1.366	.2569	1.193	.2167	20	1.365	.0785	1.297	.0732
1	0.1		1.351	.2991	9.993	1785		1.384	.0909	1.420	.2568
	0.2		1.378	.2942	1.849	13.70		1.378	.0802	1.315	.0776
	0.3		1.345	.3064	1.356	1.351		1.368	.0844	1.297	.0790
	0.4		1.385	.3310	1.269	.4910		1.367	.0889	1.296	.0819
	0.5		1.392	.2910	1.221	.2708		1.358	.0838	1.288	.0787
0	g^*	7	1.348	.2038	1.193	.1778	30	1.377	.0492	1.329	.0464
1	0.1		1.365	.2400	5.874	333.0		1.381	.0593	1.349	.0613
	0.2		1.375	.2616	1.717	4.839		1.386	.0576	1.338	.0537
	0.3		1.341	.2479	1.284	1.178		1.379	.0555	1.329	.0518
	0.4		1.379	.2571	1.240	.2636		1.385	.0542	1.335	.0500
	0.5		1.382	.2362	1.222	.2081		1.370	.0569	1.321	.0538

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3 Estimation in the Presence of Multiple Outliers

Let $X_1,\ X_2,...,\ X_{n-1}$ be independent observations from $QLD(\alpha,\theta)$ with pdf (1) and cdf (2), and let $X_{n-p+1},...,\ X_n$ be independent observations from $QLD(\beta,\gamma)$ with pdf (6) and cdf (7), and independent of $(X_1,\ X_2,...,\ X_{n-1})$.
In this section we derive the explicit formulas for the single and product moments of order statistics from QLD under the slippage multiple-outlier model.

3.1 Single moments

By using (3), writing $[F_1(x)]^s = [1-[1-F_1(x)]]^s$ and $[F_2(x)]^{r-s-1} = [1-[1-F_2(x)]]^{r-s-1}$ then expanding, we have

$$\begin{aligned}\mu_{r:n}^{(k)}[p] =& \sum_{s=max(0,r-p-1)}^{min(n-p-1,r-1)} C_1 \sum_{a=0}^{s} \binom{s}{a} \sum_{b=0}^{r-s-1} \binom{r-s-1}{b} (-1)^{a+b} \\ & \int_0^\infty x^k f_1(x) \times \{1-F_1(x)\}^{n-p-s+a-1} \{1-F_2(x)\}^{p-r+s+b+1} dx \\ & + \sum_{s=max(0,r-p)}^{min(n-p,r-1)} C_2 \sum_{a=0}^{s} \binom{s}{a} \sum_{b=0}^{r-s-1} \binom{r-s-1}{b} (-1)^{a+b} \\ & \int_0^\infty x^k f_2(x) \times \{1-F_1(x)\}^{n-p-s+a} \{1-F_2(x)\}^{p-r+s+b} dx\end{aligned}$$

By using (1), (2) , (6) and (7), then expanding binomial in x and making some algebraic simplifications, we get the single moment of the r^{th} order statistic $\mu_{r:n}^{(k)}[p]$, in the presence of multiple outliers as the following:

$$\mu_{r:n}^{(k)}[p] = \Sigma_1 \phi_{c,d} + \Sigma_2 \phi^*_{c,d}, \qquad 1 \leqslant r \leqslant n,\ k = 0,1,..., \tag{21}$$

where

$$\begin{aligned}\Sigma_1 =& \sum_{s=max(0,r-p-1)}^{min(n-p-1,r-1)} C_1 \sum_{a=0}^{s} \binom{s}{a} \sum_{b=0}^{r-s-1} \binom{r-s-1}{b} \sum_{c=0}^{n-p-s+a-1} \binom{n-p-s+a-1}{c} \\ & \times \sum_{d=0}^{p-r+s+b+1} \binom{p-r+s+b+1}{d} (-1)^{a+b}, \\ \Sigma_2 =& \sum_{s=max(0,r-p)}^{min(n-p,r-1)} C_2 \sum_{a=0}^{s} \binom{s}{a} \sum_{b=0}^{r-s-1} \binom{r-s-1}{b} \sum_{c=0}^{n-p-s+a} \binom{n-p-s+a}{c} \\ & \times \sum_{d=0}^{p-r+s+b} \binom{p-r+s+b}{d} (-1)^{a+b}, \\ \phi_{c,d} =& \frac{1}{(\theta(\alpha+1))^{c+1}} \frac{1}{(\gamma(\beta+1))^{d}} (\alpha + \frac{k+c+d+2}{\theta z}) \frac{\Gamma(k+c+d+1)}{z^{k+c+d+1}}, \\ \phi^*_{c,d} =& \frac{1}{(\theta(\alpha+1))^{c}} \frac{1}{(\gamma(\beta+1))^{d+1}} (\beta + \frac{k+c+d+2}{\gamma z}) \frac{\Gamma(k+c+d+1)}{z^{k+c+d+1}},\end{aligned}$$

and

$$z = \frac{n-p-s+a}{\theta} + \frac{p-r+s+b+1}{\gamma}.$$

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3.2 Product moments

By using (4) then expanding $\{F_1(x)\}^i = \{1-(1-F_1(x))\}^i$,$\{F_2(x)\}^{r-1-i} = \{1-(1-F_2(x))\}^{r-1-i}$,$\{F_1(y)-F_1(x)\}^j = \{(1-F_1(x))-(1-F_1(y))\}^j$ and $\{F_2(y)-F_2(x)\}^{s-r-1-j} = \{(1-F_2(x))-(1-F_2(y))\}^{s-r-1-j}$ binomially, we get

$$\begin{aligned}
\mu_{r,s:n}^{(k,l)}[p] = & \sum_{j=0}^{s-r-1}\sum_{i=max(0,s-p-j-2)}^{min(n-p-j-2,r-1)} A_1 \sum_{a=0}^{i}\binom{i}{a}\sum_{b=0}^{r-i-1}\binom{r-i-1}{b}\sum_{c=0}^{j}\binom{j}{c} \\
& \times \sum_{d=0}^{s-r-j-1}\binom{s-r-j-1}{d}(-1)^{a+b+c+d}\int_0^\infty x^k y^m f_1(x) f_1(y)\{1-F_1(x)\}^{a+j-c} \\
& \times \{1-F_2(x)\}^{b+s-r-1-j-d}\{1-F_1(y)\}^{n+c-p-i-j-2}\{1-F_2(y)\}^{p-s+i+j+d+2}dxdy \\
& + \sum_{j=0}^{s-r-1}\sum_{i=max(0,s-p-j-1)}^{min(n-p-j-1,r-1)} A_2 \sum_{a=0}^{i}\binom{i}{a}\sum_{b=0}^{r-i-1}\binom{r-i-1}{b}\sum_{c=0}^{j}\binom{j}{c} \\
& \times \sum_{d=0}^{s-r-j-1}\binom{s-r-j-1}{d}(-1)^{a+b+c+d}\int_0^\infty x^k y^m f_1(x) f_2(y)\{1-F_1(x)\}^{a+j-c} \\
& \times \{1-F_2(x)\}^{b+s-r-1-j-d}\{1-F_1(y)\}^{n+c-p-i-j-1}\{1-F_2(y)\}^{p-s+i+j+d+1}dxdy \\
& + \sum_{j=0}^{s-r-1}\sum_{i=max(0,s-p-j-1)}^{min(n-p-j-1,r-1)} A_2 \sum_{a=0}^{i}\binom{i}{a}\sum_{b=0}^{r-i-1}\binom{r-i-1}{b}\sum_{c=0}^{j}\binom{j}{c} \\
& \times \sum_{d=0}^{s-r-j-1}\binom{s-r-j-1}{d}(-1)^{a+b+c+d}\int_0^\infty x^k y^m f_2(x) f_2(y)\{1-F_1(x)\}^{a+j-c} \\
& \times \{1-F_2(x)\}^{b+s-r-1-j-d}\{1-F_1(y)\}^{n+c-p-i-j-1}\{1-F_2(y)\}^{p-s+i+j+d+1}dxdy \\
& + \sum_{j=0}^{s-r-1}\sum_{i=max(0,s-p-j)}^{min(n-p-j,r-1)} A_3 \sum_{a=0}^{i}\binom{i}{a}\sum_{b=0}^{r-i-1}\binom{r-i-1}{b}\sum_{c=0}^{j}\binom{j}{c} \\
& \times \sum_{d=0}^{s-r-j-1}\binom{s-r-j-1}{d}(-1)^{a+b+c+d}\int_0^\infty x^k y^m f_2(x) f2(y)\{1-F_1(x)\}^{a+j-c} \\
& \times \{1-F_2(x)\}^{b+s-r-1-j-d}\{1-F_1(y)\}^{n+c-p-i-j}\{1-F_2(y)\}^{p-s+i+j+d}dxdy.
\end{aligned}$$

By using (1), (2) , (6) and (7), expanding binomial in x and in y then integral for x then for y using the following relation:

$$\int_0^u x^n e^{-\mu x}dx = \frac{n!}{\mu^{n+1}} - e^{-\mu u}\sum_{k=0}^{n}\frac{n!}{k!}\frac{u^k}{\mu^{n-k+1}},$$

$u>0$, Real $\mu>0$, $n=0,1,2...$ [14], we get the product moments of the r^{th} and s^{th} order statistics $\mu_{r,s:n}^{(k,m)}[p]$ under the slippage multiple-outlier as the following:

$$\begin{aligned}
\mu_{r,s:n}^{(k,m)}[p] = & \Sigma_1 \frac{1}{(\theta(\alpha+1))^{q+u+2}}\frac{1}{(\gamma(\beta+1))^{t+v}}\psi 1 + \Sigma_3 \frac{1}{(\theta(\alpha+1))^{q+u}}\frac{1}{(\gamma(\beta+1))^{t+v+2}}\psi 4 \\
& + \Sigma_2 \frac{1}{(\theta(\alpha+1))^{q+u+1}}\frac{1}{(\gamma(\beta+1))^{t+v+1}}(\psi 2+\psi 3), \quad 1\leqslant r<s\leqslant n, \quad k,m=0,1,...,
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
\Sigma_1 = & \sum_{j=0}^{s-r-1}\sum_{i=max(0,s-p-j-2)}^{min(n-p-j-2,r-1)} A_1 \sum_{a=0}^{i}\binom{i}{a}\sum_{b=0}^{r-1-i}\binom{r-1-i}{b}\sum_{c=0}^{j}\binom{j}{c} \\
& \times \sum_{d=0}^{s-r-1-j}\binom{s-r-1-j}{d}\sum_{q=0}^{a+j-c}\binom{a+j-c}{q}\sum_{t=0}^{b+s-r-j-d-1}\binom{b+s-r-j-d-1}{t} \\
& \times \sum_{u=0}^{n-p-i-j+c-2}\binom{n-p-i-j+c-2}{u}\sum_{v=0}^{p-s+i+j+d+2}\binom{p-s+i+j+d+2}{v}(-1)^{a+b+c+d}
\end{aligned}$$

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$$\Sigma_2 = \sum_{j=0}^{s-r-1} \sum_{i=max(0,s-p-j-1)}^{min(n-p-j-1,r-1)} A_2 \sum_{a=0}^{i} \binom{i}{a} \sum_{b=0}^{r-1-i} \binom{r-1-i}{b} \sum_{c=0}^{j} \binom{j}{c}$$
$$\times \sum_{d=0}^{s-r-1-j} \binom{s-r-1-j}{d} \sum_{q=0}^{a+j-c} \binom{a+j-c}{q} \sum_{t=0}^{b+s-r-j-d-1} \binom{b+s-r-j-d-1}{t}$$
$$\times \sum_{u=0}^{n-p-i-j+c-1} \binom{n-p-i-j+c-1}{u} \sum_{v=0}^{p-s+i+j+d+1} \binom{p-s+i+j+d+1}{v} (-1)^{a+b+c+d}$$

$$\Sigma_3 = \sum_{j=0}^{s-r-1} \sum_{i=max(0,s-p-j)}^{min(n-p-j,r-1)} A_3 \sum_{a=0}^{i} \binom{i}{a} \sum_{b=0}^{r-1-i} \binom{r-1-i}{b} \sum_{c=0}^{j} \binom{j}{c}$$
$$\times \sum_{d=0}^{s-r-1-j} \binom{s-r-1-j}{d} \sum_{q=0}^{a+j-c} \binom{a+j-c}{q} \sum_{t=0}^{b+s-r-j-d-1} \binom{b+s-r-j-d-1}{t}$$
$$\times \sum_{u=0}^{n-p-i-j+c} \binom{n-p-i-j+c}{u} \sum_{v=0}^{p-s+i+j+d} \binom{p-s+i+j+d}{v} (-1)^{a+b+c+d}$$

$$\psi 1 = \alpha^2 h(\kappa_1,\kappa_2) + \frac{\alpha}{\theta} h(\kappa_1+1,\kappa_2) + \frac{\alpha}{\theta} h(\kappa_1,\kappa_2+1) + \frac{1}{\theta^2} h(\kappa_1+1,\kappa_2+1),$$
$$\psi 2 = \alpha\beta h(\kappa_1,\kappa_2) + \frac{\alpha}{\gamma} h(\kappa_1+1,\kappa_2) + \frac{\beta}{\theta} h(\kappa_1,\kappa_2+1) + \frac{1}{\theta\gamma} h(\kappa_1+1,\kappa_2+1),$$
$$\psi 3 = \alpha\beta h^*(\kappa_1,\kappa_2) + \frac{\beta}{\theta} h^*(\kappa_1+1,\kappa_2) + \frac{\alpha}{\gamma} h^*(\kappa_1,\kappa_2+1) + \frac{1}{\theta\gamma} h^*(\kappa_1+1,\kappa_2+1),$$
$$\psi 4 = \beta^2 h^*(\kappa_1,\kappa_2) + \frac{\beta}{\gamma} h^*(\kappa_1+1,\kappa_2) + \frac{\beta}{\gamma} h^*(\kappa_1,\kappa_2+1) + \frac{1}{\gamma^2} h^*(\kappa_1+1,\kappa_2+1),$$
$$h(\kappa_1,\kappa_2) = \frac{(\kappa_1-1)!\Gamma(\kappa_2)}{z_{21}^{\kappa_1} z_{22}^{\kappa_2}} - \frac{(\kappa_1-1)!}{z_{21}^{\kappa_1}} \sum_{l=0}^{\kappa_1-1} \frac{z_{21}^{l}\Gamma(\kappa_2+l)}{l!(z_{21}+z_{22})^{\kappa_2+l}},$$
$$h^*(\kappa_1,\kappa_2) = \frac{(\kappa_1-1)!\Gamma(\kappa_2)}{z_{23}^{\kappa_1} z_{24}^{\kappa_2}} - \frac{(\kappa_1-1)!}{z_{23}^{\kappa_1}} \sum_{l=0}^{\kappa_1-1} \frac{z_{23}^{l}\Gamma(\kappa_2+l)}{l!(z_{23}+z_{24})^{\kappa_2+l}},$$
$$\kappa_1 = k+q+t+1, \quad \kappa_2 = m+u+v+1.$$
$$z_{21} = \frac{a+j-c+1}{\theta} + \frac{b+s-r-j-d-1}{\gamma},$$
$$z_{22} = \frac{n+c-p-i-j-1}{\theta} + \frac{p-s+i+j+d+2}{\gamma},$$
$$z_{23} = \frac{a+j-c}{\theta} + \frac{b+s-r-j-d}{\gamma},$$
$$and\ z_{24} = \frac{n+c-p-i-j}{\theta} + \frac{p-s+i+j+d+1}{\gamma}.$$

3.3 Robust linear estimators

Suppose $X_1,\ X_2, ...,\ X_{n-1}$ are independent observations from $QLD(\alpha,\theta)$, and $X_{n-p+1}, ...,\ X_n$ are p independent outliers arise from $QLD(\beta,\gamma)$ (and independent of $X_1,\ X_2, ...,\ X_{n-1}$.) Assume $\alpha=\beta$ and $\gamma=\theta/g$, where $0<g<1$,.

Consider the following linear estimators of the mean of DLD $\frac{\alpha+2}{\alpha+1}\theta$:

1.Best linear estimator

$$\theta^* = \sum_{i=1}^{n} a_i X_{i:n} = \grave{\mathbf{A}}\mathbf{X},$$

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where $\grave{\mathbf{A}} = (a_1,\ a_2,\ ..., a_n)$ is the vector of the coefficients of the BLUE.
The bias and variance of the BLUE are

$$\begin{aligned} Bias(\theta^*) &= E(\theta^*) - \theta = \frac{\grave{\eta}\omega^{-1}}{\grave{\eta}\omega^{-1}\eta}\mu[p] - \theta \\ &= \sum_{i=1}^{n} a_i \mu_{i:n}[p] - \theta. \end{aligned} \tag{23}$$

$$\begin{aligned} Var(\theta^*) &= \frac{\grave{\eta}\omega^{-1}\Sigma[p]\omega^{-1}\eta}{(\grave{\eta}\omega^{-1}\eta)^2} \\ &= \grave{\mathbf{A}}\Sigma[p]\mathbf{A}. \end{aligned} \tag{24}$$

where $\grave{\mu[p]} = (\mu_{1:n}[p],\ \mu_{2:n}[p],\ ...,\ \mu_{n:n}[p])$, and $\Sigma[p] = [Cov(X_{r:n}, X_{s:n})[p], 1 \le r \le s \le n]$.

2. Complete sample mean $\bar{X}$
3. One sided trimmed estimator

$$T_{m,n} = \frac{1}{n-m}\sum_{i=1}^{n-m} X_{i:n},$$

4. One sided Winsorized estimator

$$W_{m,n} = \frac{1}{n}\sum_{i=1}^{n-m-1} X_{i:n} + (m+1)X_{n-m:n}.$$

5. Cikkagouder-Kunchur estimator

$$CK_n = \frac{1}{n}\sum_{i=1}^{n}(1 - \frac{2i}{n(n+1)})X_{i:n},$$

3.4 Numerical results

By using the means, variances, and covariances of order statistics which evaluate from (21) and (22) when $\alpha = 2$ and $\theta = 1$, we compute the MSE and bias of these estimators for $n = 5,\ 6,\ 7,\ p = 0,\ 1,\ 2,$ and $g = 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5$ We summarize the numerical results in Tables (2), (3),(4) and (5) .
From Table 2, Table 3, Table 4, and Table 5 , we can observe:

1. For $p = 0$, $\frac{\alpha+1}{\alpha+2}\bar{X}$ and BLUE are unbiased estimators but BLUE has the minimum MSE among all estimators.
2. The bias increases as g decreases and/or/ p increases for most estimators.
3. The MSE increases as g decreases and/or p increases for most estimators.
4. At $g = .5$ and $g = .4$ & $p = 1$, for $n = 5$, $g = .4$ & $p = 1$, for $n = 6$ and $g = .5$ & $p = 1$, for $n = 7$ $\frac{\alpha+1}{\alpha+2}CK_n$ has the lower MSE than $\frac{\alpha+1}{\alpha+2}T_{1:n}$.
5. $\frac{\alpha+1}{\alpha+2}T_{1:n}$ performs better than $\frac{\alpha+1}{\alpha+2}CK_n$ for $g < .5$ and $p = 1$ whereas $\frac{\alpha+1}{\alpha+2}T_{2:n}$ has the best performance for the small value of g and $p = 2$ (the small value of g means the probability of the presence of outliers is big) .
6. The effect of outliers on L-Estimators decreases as n increases.

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Table 2: Bias and MSE of the L-Estimators in the presence of outliers $n = 5$.

p		$g=g^*$									
		Bias	MSE								
0	$\frac{\alpha+1}{\alpha+2}\bar{X}$	.0000	.1750								
	$\frac{\alpha+1}{\alpha+2}T_{1,5}$	-.3012	.2032								
	$\frac{\alpha+1}{\alpha+2}T_{2,5}$	-.4987	.3291								
	$\frac{\alpha+1}{\alpha+2}T_{3,5}$	-.6534	.4855								
	$\frac{\alpha+1}{\alpha+2}W_{1,5}$	.0897	.2704								
	$\frac{\alpha+1}{\alpha+2}W_{2,5}$	-.1666	.2356								
	$\frac{\alpha+1}{\alpha+2}W_{3,5}$	-.4338	.3350								
	BLUE	.0000	.1744								
	$\frac{\alpha+1}{\alpha+2}CK_5$	-.2638	.1655								
		$g=0.1$		$g=0.2$		$g=0.3$		$g=0.4$		$g=0.5$	
		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
1	$\frac{\alpha+1}{\alpha+2}\bar{X}$	1.800	6.880	.8000	1.655	.4660	.7452	.2500	.3796	.2000	.3200
	$\frac{\alpha+1}{\alpha+2}T_{1,5}$	-.0540	.2040	-.1003	.1947	-.1401	.1897	-.1875	.1876	-.2032	.1880
	$\frac{\alpha+1}{\alpha+2}T_{2,5}$	-.3633	.2581	-.3841	.2667	-.4032	.2752	-.4278	.2872	-.4365	.2917
	$\frac{\alpha+1}{\alpha+2}T_{3,5}$	-.5723	.4151	-.5835	.4239	-.5941	.4325	-.6083	.4444	-.6135	.4489
	$\frac{\alpha+1}{\alpha+2}W_{1,5}$	.5087	.7753	.4264	.6463	.3569	.5470	.2755	.4438	.2726	.3659
	$\frac{\alpha+1}{\alpha+2}W_{2,5}$	.0718	.3481	.0338	.3219	-.0006	.3004	-.0447	.2766	.0218	.2351
	$\frac{\alpha+1}{\alpha+2}W_{3,5}$	-.2976	.3128	-.3167	.3132	-.3348	.3143	-.3588	.3170	-.2273	.2325
	BLUE	1.921	7.893	.8443	1.849	.4869	.8086	.2580	.3964	.2056	.3302
	$\frac{\alpha+1}{\alpha+2}CK_5$	.9574	2.567	.2871	.5635	.0612	.2659	-.0867	.1730	-.1214	.1628
2	$\frac{\alpha+1}{\alpha+2}\bar{X}$	3.600	20.06	1.600	4.415	.9320	1.749	.5000	.7093	.4000	.5450
	$\frac{\alpha+1}{\alpha+2}T_{1,5}$	1.044	2.793	.3772	.6950	.1387	.3498	-.0341	.2234	-.0788	.2061
	$\frac{\alpha+1}{\alpha+2}T_{2,5}$	-.1111	.2533	-.1956	.2401	-.2618	.2405	-.3351	.2529	-.3583	.2596
	$\frac{\alpha+1}{\alpha+2}T_{3,5}$	-.4396	.3403	-.4773	.3575	-.5100	.3752	-.5499	.4003	-.5633	.4097
	$\frac{\alpha+1}{\alpha+2}W_{1,5}$	2.651	13.91	1.294	3.510	.8388	1.646	.5286	.8435	.4442	.5880
	$\frac{\alpha+1}{\alpha+2}W_{2,5}$	.5355	1.021	.3729	.7264	.2489	.5444	.1156	.3934	.1438	.3186
	$\frac{\alpha+1}{\alpha+2}W_{3,5}$	-.0713	.3962	-.1371	.3588	-.1936	.3354	-.2615	.3186	-.1421	.2440
	BLUE	3.781	22.21	1.666	4.809	.9633	1.873	.5121	.7404	.4084	.5637
	$\frac{\alpha+1}{\alpha+2}CK_5$	2.230	8.346	.8570	1.637	.3953	.5979	.0938	.2480	.0233	.2034

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Table 3: Bias and MSE of the L-Estimators in the presence of outliers $n = 6$.

p		$g=g^*$									
		Bias	MSE								
0	$\frac{\alpha+1}{\alpha+2}\bar{X}$	.0000	.1458								
	$\frac{\alpha+1}{\alpha+2}T_{1,6}$	-.2711	.1712								
	$\frac{\alpha+1}{\alpha+2}T_{2,6}$	-.4513	.2760								
	$\frac{\alpha+1}{\alpha+2}T_{3,6}$	-.5934	.4065								
	$\frac{\alpha+1}{\alpha+2}W_{1,6}$	.1319	.2434								
	$\frac{\alpha+1}{\alpha+2}W_{2,6}$	-0.0790	.1956								
	$\frac{\alpha+1}{\alpha+2}W_{3,6}$	-0.2979	.2378								
	BLUE	.0000	.1453								
	$\frac{\alpha+1}{\alpha+2}CK_6$	-0.2237	.1387								
		$g=0.1$		$g=0.2$		$g=0.3$		$g=0.4$		$g=0.5$	
		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
1	$\frac{\alpha+1}{\alpha+2}\bar{X}$	1.500	4.802	.6666	1.173	.3883	.5418	.2083	.2879	.1666	.2465
	$\frac{\alpha+1}{\alpha+2}T_{1,6}$	-.0496	.1650	-.0919	.1592	-.1280	.1564	-.1706	.1562	-.1847	.1569
	$\frac{\alpha+1}{\alpha+2}T_{2,6}$	-.3232	.2115	.3434	.2197	-.3618	.2277	-.3853	.2388	-.3936	.2429
	$\frac{\alpha+1}{\alpha+2}T_{3,6}$	-.5111	.3383	-.5227	.3471	-.5337	.3557	-.5483	.3674	-.5536	.3718
	$\frac{\alpha+1}{\alpha+2}W_{1,6}$	.5104	.6758	.4343	.5651	.3704	.4800	.2965	.3918	-.0455	.1660
	$\frac{\alpha+1}{\alpha+2}W_{2,6}$	.1527	.3202	.1145	.2922	.0800	.2690	.0367	.2432	-.2336	.1865
	$\frac{\alpha+1}{\alpha+2}W_{3,6}$	-.1487	.2401	-.1705	.2368	-.1909	.2345	-.2177	.2327	-.4204	.2785
	BLUE	1.608	5.558	.7063	1.318	.4070	.5890	.2155	.3003	.1716	.2540
	$\frac{\alpha+1}{\alpha+2}CK_6$	.8636	2.071	.2656	.4646	.0644	.2240	-0.0671	.1477	.2656	.4646
2	$\frac{\alpha+1}{\alpha+2}\bar{X}$	3.000	13.95	1.333	3.090	.7766	1.239	.4166	.5169	.3333	.4027
	$\frac{\alpha+1}{\alpha+2}T_{1,6}$	.8344	1.822	.2970	.4781	.1022	.2568	-.0417	.1763	-.0796	.1657
	$\frac{\alpha+1}{\alpha+2}T_{2,6}$	-.1007	.1959	-.1770	.1905	-.2367	.1945	-.3028	.2080	-.3238	.2144
	$\frac{\alpha+1}{\alpha+2}T_{3,6}$	-.3842	.2670	-.4213	.2847	-.4534	.3022	-.4924	.3265	-.5056	.3355
	$\frac{\alpha+1}{\alpha+2}W_{1,6}$	2.277	9.981	1.150	2.632	.7710	1.292	.5097	.6983	.0833	.2266
	$\frac{\alpha+1}{\alpha+2}W_{2,6}$	.5770	.9204	.4231	.6558	.3066	.4915	.1823	.3533	-.1421	.1877
	$\frac{\alpha+1}{\alpha+2}W_{3,6}$	.0867	.3644	.0150	.3137	-.0457	.2796	-.1180	.2507	-.3565	.2530
	BLUE	3.172	15.674	1.396	3.404	.8065	1.337	.4281	.5416	.3413	.4176
	$\frac{\alpha+1}{\alpha+2}CK_6$	1.981	6.577	.7662	1.305	.3578	.4861	.0915	.2083	.7662	1.305

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Table 4: Bias and MSE of the L-Estimators in the presence of outliers $n = 7$.

p		$g = g^*$ Bias	$g = g^*$ MSE								
0	$\frac{\alpha+1}{\alpha+2}\bar{X}$	.0000	.1250								
	$\frac{\alpha+1}{\alpha+2}T_{1,7}$	-.2474	.1478								
	$\frac{\alpha+1}{\alpha+2}T_{2,7}$	-.4137	.2370								
	$\frac{\alpha+1}{\alpha+2}T_{3,7}$	-.5455	.3487								
	$\frac{\alpha+1}{\alpha+2}W_{1,7}$	.1617	.2242								
	$\frac{\alpha+1}{\alpha+2}W_{2,7}$	-.0173	.1721								
	$\frac{\alpha+1}{\alpha+2}W_{3,7}$	-.2025	.1843								
	BLUE	.0000	.1245								
	$\frac{\alpha+1}{\alpha+2}CK_7$	-.1941	.1194								
		$g = 0.1$		$g = 0.2$		$g = 0.3$		$g = 0.4$		$g = 0.5$	
		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
1	$\frac{\alpha+1}{\alpha+2}\bar{X}$	1.285	3.545	.5714	.8801	.3328	.4159	.1785	.2294	.1428	.1989
	$\frac{\alpha+1}{\alpha+2}T_{1,7}$	-.0460	.1387	-.0851	.1348	-.1182	.1333	-.1572	.1340	-.1699	.1348
	$\frac{\alpha+1}{\alpha+2}T_{2,7}$	-.2926	.1791	-.3120	.1867	-.3297	.1941	-.3521	.2042	-.3599	.2079
	$\frac{\alpha+1}{\alpha+2}T_{3,7}$	-.4640	.2849	-.4758	.2934	.4868	.3016	-.5014	.3126	-.5067	.3167
	$\frac{\alpha+1}{\alpha+2}W_{1,7}$	.5067	.6017	.4359	.5049	.3770	.4307	.3093	.3538	.2876	.3312
	$\frac{\alpha+1}{\alpha+2}W_{2,7}$	.2043	.3007	.1667	.2722	.1331	.2487	.0913	.2222	.0770	.2140
	$\frac{\alpha+1}{\alpha+2}W_{3,7}$	-.0498	.2051	-.0729	.1990	-.0943	.1941	-.1220	.1890	-.1319	.1876
	BLUE	1.383	4.131	.6071	.9922	.3451	.4450	.1850	.2389	.1473	.2047
	$\frac{\alpha+1}{\alpha+2}CK_7$	.7823	1.695	.6905	1.062	.0637	.1913	-.0543	.1284	-0.0818	.1210
2	$\frac{\alpha+1}{\alpha+2}\bar{X}$	2.571	10.27	1.142	2.288	.6657	.9284	.3571	.3976	.2857	.3137
	$\frac{\alpha+1}{\alpha+2}T_{1,7}$	.6942	1.289	.2439	.3553	.0787	.2014	-.0454	.1460	-.0784	.1390
	$\frac{\alpha+1}{\alpha+2}T_{2,7}$	-.0925	.1601	-.1624	.1585	-.2171	.1638	-.2775	.1769	-.2968	.1828
	$\frac{\alpha+1}{\alpha+2}T_{3,7}$	-.3437	.2201	-.3798	.2369	-.4109	.2533	-.4485	.2758	-.4612	.2841
	$\frac{\alpha+1}{\alpha+2}W_{1,7}$	2.008	7.569	1.045	2.080	.7200	1.063	.4939	.6001	.4366	.5117
	$\frac{\alpha+1}{\alpha+2}W_{2,7}$	.5926	.8339	.4479	.5973	.3391	.4497	.2237	.3240	.1880	.2921
	$\frac{\alpha+1}{\alpha+2}W_{3,7}$	.1821	.3501	.1089	.2924	.0473	.2526	-.0251	.2169	-.0491	.2078
	BLUE	2.704	11.110	1.201	2.541	.6846	.9868	.3679	.4175	.2932	.3257
	$\frac{\alpha+1}{\alpha+2}CK_7$	1.778	5.297	.2445	.3880	.3251	.4025	.0869	.1779	.0314	.1487

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Table 5: Bias and MSE of the L-Estimators in the presence of outliers $n = 10$.

p		$g=g^*$ Bias	MSE								
0	$\frac{\alpha+1}{\alpha+2}\bar{X}$	.0000	.0875								
	$\frac{\alpha+1}{\alpha+2}T_{1,10}$	-.1983	.1044								
	$\frac{\alpha+1}{\alpha+2}T_{2,10}$	-.3350	.1647								
	$\frac{\alpha+1}{\alpha+2}T_{3,10}$	-.4450	.2412								
	$\frac{\alpha+1}{\alpha+2}W_{1,10}$	.2147	.1901								
	$\frac{\alpha+1}{\alpha+2}W_{2,10}$	.0918	.1401								
	$\frac{\alpha+1}{\alpha+2}W_{3,10}$	-.0344	.1198								
	BLUE	.0000	.0871								
	$\frac{\alpha+1}{\alpha+2}CK_{10}$	-.9009	.8768								
		$g=0.1$		$g=0.2$		$g=0.3$		$g=0.4$		$g=0.5$	
		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
1	$\frac{\alpha+1}{\alpha+2}\bar{X}$	.9000	1.763	.4000	.4575	.2330	.2300	.1250	.1386	.1000	.1237
	$\frac{\alpha+1}{\alpha+2}T_{1,10}$	-.0383	.0940	-.0705	.1281	-.0975	.0926	-.1287	.0941	-.1387	.0949
	$\frac{\alpha+1}{\alpha+2}T_{2,10}$	-.2314	.1221	-.2489	.0661	-.2645	.1338	-.2840	.1414	-.2907	.1442
	$\frac{\alpha+1}{\alpha+2}T_{3,10}$	-.3695	.1912	-.3811	.1982	-.3917	.2049	-.4055	.2138	-.4103	.2170
	$\frac{\alpha+1}{\alpha+2}W_{1,10}$	.4881	.4628	.4294	.3925	.3814	.3388	.3273	.2835	.3103	.2673
	$\frac{\alpha+1}{\alpha+2}W_{2,10}$	.2824	.2630	.2480	.2361	.2179	.2138	.1813	.1889	.1690	.1811
	$\frac{\alpha+1}{\alpha+2}W_{3,10}$	.1102	.1691	.1510	.1585	.0653	.1496	.0383	.1395	.0290	.1363
	BLUE	.9752	2.080	.4274	.5179	.2459	.2495	.1299	.1436	.1034	.1267
	$\frac{\alpha+1}{\alpha+2}CK_{10}$	-.8913	1.442	-.8924	1.004	-.8935	.9244	-.8951	.8914	-.8957	.8861
2	$\frac{\alpha+1}{\alpha+2}\bar{X}$	1.800	5.060	.8000	1.147	.4660	.4812	.2500	.2210	.2000	.1800
	$\frac{\alpha+1}{\alpha+2}T_{1,10}$	.4609	.6050	.1565	.1893	.0416	.1208	-.0477	.0969	-.0720	.0943
	$\frac{\alpha+1}{\alpha+2}T_{2,10}$	-.0757	.1041	-.1326	.1061	-.1768	.1116	-.2256	.1222	-.2412	.1267
	$\frac{\alpha+1}{\alpha+2}T_{3,10}$	-.2668	.1437	-.2995	.1570	-.3273	.1698	-.3606	.1871	-.3717	.1934
	$\frac{\alpha+1}{\alpha+2}W_{1,10}$	1.520	4.069	.8505	1.248	.6219	.7011	.4595	.4350	.4176	.3810
	$\frac{\alpha+1}{\alpha+2}W_{2,10}$	.5890	.6501	.4680	.4750	.3784	.3648	.2842	.2688	.2554	.2436
	$\frac{\alpha+1}{\alpha+2}W_{3,10}$	.3149	.3237	.2453	.2625	.1878	.2190	.1214	.1773	.0998	.1659
	BLUE	1.931	5.856	.8482	1.292	.4888	.5265	.2588	.2324	.2061	.1867
	$\frac{\alpha+1}{\alpha+2}CK_{10}$	-.8796	2.017	-.8823	1.132	-.8850	.9710	-.8886	.9051	-.8900	.8946

4 Real Data

The following data is real data represents the waiting times (in minutes) before the service of one hundred (100) bank customers.

Table 6: The real data .

0.8	0.8	1.3	1.5	1.8	1.9	1.9	2.1	2.6	2.7
2.9	3.1	3.2	3.3	3.5	3.6	4.0	4.1	4.2	4.2
4.3	4.3	4.4	4.4	4.6	4.7	4.7	4.8	4.9	4.9
5.0	5.3	5.5	5.7	5.7	6.1	6.2	6.2	6.2	6.3
6.7	6.9	7.1	7.1	7.1	7.1	7.4	7.6	7.7	8.0
8.2	8.6	8.6	8.6	8.8	8.8	8.9	8.97	9.5	9.6
9.7	9.8	10.7	10.9	11.0	11.0	11.1	11.2	11.2	11.5
11.9	12.4	12.5	12.9	13.0	13.1	13.3	13.6	13.7	13.9
14.1	15.4	15.4	17.3	17.3	18.1	18.2	18.4	18.9	19.0
19.9	20.6	21.3	21.4	21.9	23.0	27.0	31.6	33.1	38.5

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[1] used this data set and obtained the ML estimates of the rate parameter $\hat{\theta} = 0.187$ with $SEL(\hat{\theta}) = 0.013$ for Lindley distribution and $\hat{\theta} = 0.101$ with $SEL(\hat{\theta}) = 0.010$ for the exponential distribution.
[7] used the same data. They found the ML estimates of the rate parameter θ as follows

Model	$\hat{\theta}$	$\hat{\alpha}$	$1/\hat{\theta}$
Lindley	0.186571		5.359886
Exponential	0.101245		9.87703
QLD	.0.196209	0.066138	5.09660

Here, we find the ML and Bayes estimates of the scale parameter θ using (10), and (14), we consider $(a = b = 1)$. The results are presented in Table 7.

Table 7: The ML and Bayes estimates from the real data when there is no outlier.

Estimate	$\alpha = 0$	$\alpha = .02$	$\alpha = .03$	$\alpha = .04$	$\alpha = .06$	$\alpha = .1$	$\alpha = .2$
$\hat{\theta}_{ML}$	4.9385	4.9831	5.0048	5.0262	5.0678	5.1477	5.3324
$SEL(\hat{\theta}_{ML})$	0.0037	0.0002	0.0000	0.0006	0.0046	0.0218	0.1104
$\hat{\theta}_B$	4.8962	4.9398	4.9609	4.9817	5.0224	5.1003	5.2805
$SEL(\hat{\theta}_B)$	0.0107	0.0036	0.0015	0.0003	0.0005	0.0100	0.0786

We observe from Table 7 that $\hat{\theta}_{ML}$ is better than $\hat{\theta}_B$ in the following cases: $(\alpha = 0)$, ($\alpha = .02$ and $\alpha = .03$. But $\hat{\theta}_B$ is better in the following cases: $\alpha = .04$, $\alpha = .06$, $\alpha = .1$ and $\alpha = .2$.

The box plot detected that the last four data are outliers.

We calculate the maximum likelihood and Bayes estimates by considering that only the last data is an outlier $(p = 1)$. The results are presented in Table 8.

Table 8: The ML and Bayes estimates from the real data when there is a single outlier.

g	Estimate	$\alpha = 0$	$\alpha = .02$	$\alpha = .03$	$\alpha = .04$	$\alpha = .06$	$\alpha = .1$	$\alpha = .2$
0.1	$\hat{\theta}_{ML}$	4.8042	4.8495	4.8716	4.8933	4.9357	5.0171	5.2053
	$SEL(\hat{\theta}_{ML})$	0.0383	0.0226	0.0164	0.0113	0.0041	0.0002	0.0421
	$\hat{\theta}_B$	4.7626	4.8068	4.8283	4.8494	4.8907	4.9701	5.1537
	$SEL(\hat{\theta}_B)$	0.0563	0.0373	0.0294	0.0226	0.0119	0.0008	0.0236
0.2	$\hat{\theta}_{ML}$	4.8286	4.8733	4.8950	4.9164	4.9582	5.0386	5.2247
	$SEL(\hat{\theta}_{ML})$	0.0293	0.0160	0.0110	0.0069	0.0017	0.0014	0.0505
	$\hat{\theta}_B$	4.7867	4.8303	4.8515	4.8723	4.9131	4.9915	5.1730
	$SEL(\hat{\theta}_B)$	0.0454	0.0287	0.0220	0.0162	0.0075	0.0000	0.0299
0.3	$\hat{\theta}_{ML}$	4.8518	4.8964	4.9181	4.9394	4.9811	5.0611	5.2464
	$SEL(\hat{\theta}_{ML})$	0.0219	0.0107	0.0067	0.0036	0.0003	0.0037	0.0607
	$\hat{\theta}_B$	4.8098	4.8533	4.8744	4.8952	4.9358	5.0139	5.1947
	$SEL(\hat{\theta}_B)$	0.0361	0.0215	0.0157	0.0109	0.0041	0.0001	0.0379

From Table 8, we observe that $\hat{\theta}_{ML}$ is better than $\hat{\theta}_B$ in the following cases $(\alpha = 0)$ $\alpha = .02$, $\alpha = .03$, $\alpha = .04$, and $\alpha = .06$ but $\hat{\theta}_B$ is better in the following cases: $\alpha = .1$, and $\alpha = .2$. We also observe that SEL of two estimators decreases when α and/or g increases just for $\alpha = 0$, $\alpha = .02$, $\alpha = .03$, $\alpha = .04$, and $\alpha = .06$.
On the other hand, we evaluate $\frac{\alpha+1}{\alpha+2}\bar{X}$, $\frac{\alpha+1}{\alpha+2}T_{m,100}$, $\frac{\alpha+1}{\alpha+2}W_{m,100}$, and $\frac{\alpha+1}{\alpha+2}CK_{100}$ estimates of the scale parameter θ for $m = 1,\ 2,\ 3,$ and 4 and for diffrent values of α. We present the results in Table 9.

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Table 9: The estimates of linear estimators from the real data.

Estimate	$\alpha=0$	$\alpha=.02$	$\alpha=.03$	$\alpha=.04$	$\alpha=.06$	$\alpha=.1$	$\alpha=.2$
$\frac{\alpha+1}{\alpha+2}\bar{X}$	4.9384	4.9873	5.0114	5.0352	5.0822	5.1736	5.3874
$SEL(\frac{\alpha+1}{\alpha+2}\bar{X})$	0.0037	0.0001	0.0001	0.0012	0.0067	0.0301	0.1500
$\frac{\alpha+1}{\alpha+2}T_{1,100}$	4.7939	4.8413	4.8647	4.8879	4.9335	5.0221	5.2297
$SEL(\frac{\alpha+1}{\alpha+2}T_{1,100})$	0.0424	0.0251	0.0182	0.0125	0.0044	0.0004	0.0527
$\frac{\alpha+1}{\alpha+2}T_{2,100}$	4.6739	4.7202	4.7430	4.7656	4.8101	4.8965	5.0988
$SEL(\frac{\alpha+1}{\alpha+2}T_{2,100})$	0.1062	,0.0782	0.0660	0.0549	0.0360	0.0107	0.0097
$\frac{\alpha+1}{\alpha+2}T_{3,100}$	4.5592	4.6044	4.6266	4.6486	4.6920	4.7763	4.9737
$SEL(\frac{\alpha+1}{\alpha+2}T_{3,100})$	0.1942	0.1564	0.1393	0.1234	0.0948	0.0500	0.0000
$\frac{\alpha+1}{\alpha+2}T_{4,100}$	4.4661	4.5103	4.5321	4.5537	4.5962	4.6788	4.8721
$SEL(\frac{\alpha+1}{\alpha+2}T_{4,100})$	0.2850	0.2397	0.2188	0.1991	0.1630	0.1031	0.0163
$\frac{\alpha+1}{\alpha+2}W_{1,100}$	4.9115	4.9601	4.9840	5.0078	5.0545	5.1453	5.358
$SEL(\frac{\alpha+1}{\alpha+2}W_{1,100})$	0.0078	0.0015	0.0002	0.0000	0.0029	0.0211	0.1281
$\frac{\alpha+1}{\alpha+2}W_{2,100}$	4.8965	4.9449	4.9688	4.9925	5.0391	5.1296	5.3416
$SEL(\frac{\alpha+1}{\alpha+2}W_{2,100})$	0.0107	0.0030	0.0009	0.0000	0.0015	0.0168	0.1167
$\frac{\alpha+1}{\alpha+2}W_{3,100}$	4.8275	4.8753	4.8988	4.9221	4.9681	5.0573	5.2663
$SEL(\frac{\alpha+1}{\alpha+2}W_{3,100})$	0.0297	0.0155	0.0102	0.0060	0.0010	0.0032	0.0709
$\frac{\alpha+1}{\alpha+2}W_{4,100}$	4.7475	4.7945	4.8176	4.8405	4.8857	4.9735	5.1790
$SEL(\frac{\alpha+1}{\alpha+2}W_{4,100})$	0.0637	0.0422	0.0332	0.0254	0.0130	0.0006	0.0320
$\frac{\alpha+1}{\alpha+2}CK_{100}$	4.8704	4.9186	4.9424	4.9659	5.0123	5.1023	5.3132
$SEL(\frac{\alpha+1}{\alpha+2}CK_{100})$	0.0167	0.0066	0.0033	0.0011	0.0001	0.0104	0.0981

From Table 9, we note the following: (i) estimate of $\frac{\alpha+1}{\alpha+2}\bar{X}$ at $\alpha = .03$ has the lowest SEL among other values of α.
(ii) SEL of every estimator $\frac{\alpha+1}{\alpha+2}T_{m,100}$ decreases as α increases.
(iii) For the trimmed estimators, $\frac{\alpha+1}{\alpha+2}T_{3,100}$ gives the best estimate among other estimators for $\alpha = .2$ while $\frac{\alpha+1}{\alpha+2}T_{1,100}$ gives the best estimate among other trimmed estimators for the other values of α.
(iv) For the Winsorized estimators, $\frac{\alpha+1}{\alpha+2}W_{1,100}$ has the smallest SEL for $\alpha = 0$, $\alpha = .02$, $\alpha = .03$, $\alpha = .04$ and $\alpha = .06$, but $\frac{\alpha+1}{\alpha+2}W_{4,100}$ has the smallest SEL for $\alpha = .1$, $\alpha = .2$
(v) $\frac{\alpha+1}{\alpha+2}CK_n$ has lowest SEL at $\alpha = 0.06$ among other values of α and among other estimators.

(vi) $\frac{\alpha+1}{\alpha+2}\bar{X}$ at $\alpha = 0$, $\alpha = .02$ and $\alpha = .03$, $\frac{\alpha+1}{\alpha+2}W_{1,100}$ at $\alpha = .04$, and $\frac{\alpha+1}{\alpha+2}T_{1,100}$ at $\alpha = .1$ have the lowest SEL among other estimators.

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Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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