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q-Beta Polynomials and their Applications

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Abstract: The aim of this paper is to construct generating functions for q-beta polynomials. By using these generating functions, we define the q -beta polynomials and also derive some fundamental properties of these polynomials. We give some functional equations and partial differential equations (PDEs) related to these generating functions. By using these equations, we find some identities related to these polynomials, binomial coefficients, the gamma function and the beta function. We obtain a relation between the q-beta polynomials and the q-Bernstein basis functions. We give relations between the q-Beta polynomials, the Bernoulli polynomials, the Euler polynomials and the Stirling numbers. We also give a probability density function associated with the beta polynomials. By applying the Mellin transform, the Fourier transform and the Laplace transform to the generating functions, we obtain not only interpolation function, but also some series representations for the q-Beta polynomials. Furthermore, by using the p-adic q-Volkenborn integral, we give relations between the q-beta polynomials, the q-Euler numbers and the Carlitz's q-Bernoulli numbers.

Keywords: q-Bernstein basis functions; Generating functions; q-Beta polynomials; p-adic q-Volkenborn integral; Beta distribution; Beta function; Fourier transforms; Laplace transforms; Mellin transforms; Bernoulli polynomials; Euler polynomials; Stirling numbers

1 Introduction

Polynomials have many algebraic operations. They have been used several branches of Mathematics, Physics and Engineering. Because of closure under addition, multiplication, differentiation, integration, and composition, they have been utilized in computational models of scientific and engineering problems [4, 5, 12].

Recently, generating functions have been played an important role in the investigation of many fundamental properties of the polynomials and sequences. These functions can be use to find many identities and formulas for the polynomials and sequences. There are various applications of these functions in many areas of Mathematics and Mathematical Physics, especially including Statistics, Probability, Analytic Number Theory [13, 19].

We summarize our paper results as follows:

In Section 2, we define the q-beta polynomials. We give some properties of these polynomials. In Section 3, we construct generating functions for the q-beta polynomials. We investigate some properties of these functions. We give a relation between the Bernstein basis functions and the q-beta polynomials. In Section 4, we give integral representations for the (q)-beta polynomials. Using these integral representations, we derive some interesting identities. In Section 5 and 6, we define some functional equations and PDEs of the generating functions. By using these functions, we obtain not only differentiating and recurrence relations, but also some identities for the q-beta polynomials. In section 7, we give relations between the q-Beta polynomials, the Bernoulli polynomials, the Euler polynomials and the Stirling numbers. In Section 8, we define probability density function which is related to the beta distribution and the q-beta polynomials. We give mean and variance of this probability density function. In Section 9, by applying the Mellin transform, the Fourier transform and the Laplace transform to the generating functions for the q-Beta polynomials, we define interpolation function and some series representations for the q-Beta polynomials. In Section 10, by using the p -adic Volkenborn integrals on $\mathbb{Z}_p$, we derive some identities, which are related to the q-beta polynomials, the q-Euler numbers and the Carlitz's q-Bernoulli numbers.

2 The q-beta polynomials $\mathfrak{B}_k^n(x;q)$

In this section, we define the q-beta polynomials. We investigate and derive some properties of these polynomials.

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Definition 2.1. Let $x \in [-1,0]$. Let n and k be nonnegative integers. Then we define

$$\mathfrak{B}_k^n(x;q) = [x]^k [x+1]^{n-k}, \quad (1)$$

where $k = 0,1,2,\ldots,n$ and

$$[x] = \frac{1-q^x}{1-q}$$

with $q \in \mathbb{C}$ $(|q| < 1)$.

By using the following well-known identity

$$[a+b] = [a] + q^a [b], [7]$$

we modify (1) as follows:

$$\mathfrak{B}_k^n(x;q) = \sum_{j=0}^{n-k} \binom{n-k}{j} q^{n-j} [x]^{n-j},$$

$$\mathfrak{B}_k^n(x;q) = \sum_{j=0}^{n-k}\sum_{d=0}^{n-j} (-1)^{n-j-d} \binom{n-k}{j}\binom{n-j}{d} \frac{q^{x(n-j-d)+n-j}}{(1-q)^{n-j}}, \quad (2)$$

and

$$\mathfrak{B}_k^n(x;q) = \sum_{m=0}^{\infty}\sum_{j=0}^{n-k}\sum_{d=0}^{n-j}(-1)^{n-j-d} \binom{m+n-j-1}{n-j-1}\binom{n-k}{j} \binom{n-j}{d} q^{x(n-j-d)+n-j+m}.$$

Remark 2.1. For mathematical convenience, we usually set

$$\mathfrak{B}_k^n(x;q) = 0$$

if $k < 0$ or $k > n$ [16].

Remark 2.2. Observe that

$$\lim_{q \to 1} \mathfrak{B}_k^n(x;q) = \mathfrak{B}_k^n(x;1) = x^k (1+x)^{n-k} \quad (3)$$

[2, 6, 15, 16].

2.1 Some properties of the q-beta polynomials

By substituting $x = 0$ and $x = -1$ into (1), we have

$$\mathfrak{B}_k^n(0;q) = \mathfrak{B}_k^n(-1;q) = 0.$$

By (1), we obtain

$$\prod_{j=1}^{v} \mathfrak{B}_{k_j}^{n_j}(x;q) = \mathfrak{B}_{k_1+k_2+\cdots+k_v}^{n_1+n_2+\cdots+n_v}(x;q).$$

Using (1), the q-beta polynomials are easy obtain. We now compute some of them as follows:

$$\mathfrak{B}_0^0(x;q) = 1, \mathfrak{B}_0^1(x;q) = 1 + q[x], \mathfrak{B}_1^1(x;q) = [x],$$

$$\mathfrak{B}_0^2(x;q) = q^2 [x]^2 + 2q[x] + 1, \mathfrak{B}_1^2(x;q) = q[x]^2 + [x],$$

$$\mathfrak{B}_2^2(x;q) = [x]^2,$$

$$\mathfrak{B}_0^3(x;1) = \sum_{j=0}^{3} \binom{3}{j} (q[x])^{3-j}, \mathfrak{B}_1^3(x;q) = q^2[x]^3 + 2q[x]^2 + [x],$$

$$\mathfrak{B}_2^3(x;q) = q[x]^3 + [x]^2, \mathfrak{B}_3^3(x;q) = [x]^3,$$

$$\mathfrak{B}_{n-1}^n(x;q) = [x]^{n-1} + q[x]^n, \mathfrak{B}_n^n(x;q) = [x]^n,$$

$$\mathfrak{B}_0^n(x;q) = \sum_{j=0}^{n} \binom{n}{j} (q[x])^{n-j}.$$

3 Generating function for the q-beta polynomials

We now construct generating functions for the q-beta polynomials. We also derive *functional equations* and *PDEs* related to these functions. By using these equations, we give some fundamental properties of these polynomials.

In [16], we constructed generatiog functions for the functions $\mathfrak{M}_{k,n}(x)$. We now define q-version of these functions by means of the following generating functions:

$$\mathfrak{h}_{k,q}(t,x) = \left(\frac{[x]}{[1+x]}\right)^k e^{t[1+x]} = \sum_{n=0}^{\infty} \mathfrak{M}_{k,n}(x;q) \frac{t^n}{n!}, \quad (4)$$

where $k \in \mathbb{N}_0 = \{0,1,2,\cdots\}$.

By using (4), we have

$$\mathfrak{M}_{k,n}(x;q) = [x]^k [1+x]^{n-k},$$

where $n,k \in \mathbb{N}_0$ ([16]).

From (4), if $n \geq k$, we defined

$$\mathfrak{M}_{k,n}(x;q) = \mathfrak{B}_k^n(x;q)$$

and if $n < k$, we set

$$\mathfrak{M}_{k,n}(x;q) = \mathfrak{b}_{k,n}(x;q),$$

where

$$\mathfrak{b}_{k,n}(x) = \frac{[x]^k}{[1+x]^{k-n}}$$

and $n \in \{0,1,2,\cdots,k-1\}$.

Thus, from , we have

$$\mathfrak{M}_{k,n}(x;q) = \mathfrak{B}_k^n(x;q) + \mathfrak{b}_{k,n}(x;q).$$

Generating functions for the q-beta polynomials $\mathfrak{B}_k^n(x;q)$ can be defined as follows:

Definition 3.1.

$$F_{q,k}(t,x) = \mathfrak{h}_k(t,x) - \sum_{n=0}^{k-1} \mathfrak{b}_{k,n}(x)\frac{t^n}{n!} = \sum_{n=k}^{\infty} \mathfrak{B}_k^n(x;q)\frac{t^n}{n!}, \tag{5}$$

where $\mathfrak{B}_k^0(x;q) = \ldots = \mathfrak{B}_k^{k-1}(x;q) = 0$.
Note that there is one generating function for each value of k.
We now give these generating functions explicitly by the following theorem.

Theorem 3.1. Let k be nonnegative integer. Then we have

$$F_{q,k}(t,x) = \left(\frac{[x]}{[1+x]}\right)^k e^{t[1+x]}, \tag{6}$$

where $t, q \in \mathbb{C}$ ($|q| < 1$) and $x \neq -1$.

Proof. Substituting (1) into the right-hand side of (5), we obtain

$$F_{q,k}(t,x) = \sum_{n=0}^{\infty}\left([x]^k [x+1]^{n-k}\right)\frac{t^n}{n!}.$$

Therefore

$$F_{q,k}(t,x) = \frac{[x]^k}{[x+1]^k}\sum_{n=0}^{\infty}[x+1]^n \frac{t^n}{n!}.$$

The right-hand side of the above equation is a Taylor series for $e^{[1+x]t}$. Thus we arrive at the desired result.□

Observe that

$$F_{q,k}(t,x) = e^{t[1+x]}\sum_{m=0}^{\infty}(-1)^m \binom{m+k-1}{k-1} q^m [x]^{k+m},$$

where $|q[x]| < 1$.

$$F_{q,k}(t,0) = 0,$$

$$F_{q,k}(t,-1) = \infty,$$

and

$$F_{q,1}(t,1) = \frac{e^{(q+1)t}}{q+1}.$$

From the above equation, we get

$$\mathfrak{B}_1^n(1;q) = \sum_{j=0}^{n-1}\binom{n-1}{j} q^j.$$

3.1 Identity for the q-Bernstein basis functions and the q-beta polynomials

Here we give relation between the q-Bernstein basis functions and the q-beta polynomials. Firstly, we give definition of the q-Bernstein basis functions as follows:

$$\mathfrak{b}_k^n(x;q) = \binom{n}{k}[x]^k q^{(n-k)x}[1-x]^{n-k}$$

or

$$\mathfrak{b}_k^n(x;q) = \binom{n}{k}[x]^k (1-[x])^{n-k} \tag{7}$$

[17].

We set

$$\mathfrak{b}_k^n(x^v;q) = \binom{n}{k}[x]^{vk}(1-[x]^v)^{n-k} \tag{8}$$

We note that

$$\lim_{q\to 1}\mathfrak{b}_k^n(x^v;q) = B_k^n(x^v) = \binom{n}{k} x^{vk}(1-x^v)^{n-k},$$

where $B_k^n(x)$ denotes the Bernstein basis functions [1, 4, 5, 10, 12, 14, 15, 17].

Theorem 3.2. The following identity holds true.

$$\mathfrak{b}_k^n(x;q)\mathfrak{B}_k^n(x;q) = \mathfrak{b}_k^n(x^2;q).$$

Proof. Multiplying both sides of Equations (1) and (7), we get

$$\mathfrak{B}_k^n(x;q)\mathfrak{b}_k^n(x;q) = \binom{n}{k}[x]^{2k}\left(1-[x]^2\right)^{n-k}.$$

Using (8), we arrive at the desired result.□

4 Integral Representations

In this section, integral representations for the q-beta polynomials are given.

Theorem 4.1. Let $0 < q < 1$. Then we have

$$\int_0^1 \mathfrak{B}_k^n(x;q)dx = \sum_{j=0}^{n-k}\sum_{d=0}^{n-j}(-1)^{n-j-d+1}\binom{n-k}{j}\binom{n-j}{d} \times \frac{[n-j-d]\,q^{n-j}}{(1-q)^{n-j-1}(n-j-d)\ln(q)}.$$

Proof. From (2), for $0 < q < 1$, we get

$$\int_0^1 \mathfrak{B}_k^n(x;q)dx = \sum_{j=0}^{n-k}\sum_{d=0}^{n-j}(-1)^{n-j-d+1}\binom{n-k}{j}\binom{n-j}{d} \times \frac{1}{(1-q)^{n-j}}\int_0^1 q^{x(n-j-d)}dx.$$

Thus, we arrive at the desired result.□

We also easily see that

$$\int_{-1}^{0}\mathfrak{B}_k^n(x;q)dx=\sum_{j=0}^{n-k}\sum_{d=0}^{n-j}(-1)^{n-j-d+1}\binom{n-k}{j}\binom{n-j}{d}\times\frac{[j+d-n]\,q^{n-j}}{(1-q)^{n-j-1}(j+d-n)\ln(q)}$$

where $0<q<1$.

Theorem 4.2. Then we have

$$\int_{-1}^{0}\mathfrak{B}_k^n(x;1)dx=\sum_{j=0}^{n-k}(-1)^{n-j}\binom{n-k}{j}\frac{1}{n-j+1} \tag{9}$$

and

$$\int_{-1}^{0}\mathfrak{B}_k^n(x;1)dx=\frac{(-1)^k}{(n+1)\binom{n}{k}}. \tag{10}$$

Proof of (10). Integrating equation (3) with respect to x from -1 to 0, we get

$$\int_{-1}^{0}\mathfrak{B}_k^n(x;1)dx=\int_{-1}^{0}x^k(1+x)^{n-k}dx.$$

From the above equation, we get

$$\int_{-1}^{0}\mathfrak{B}_k^n(x;1)dx=(-1)^k\int_{0}^{1}x^k(1-x)^{n-k}dx. \tag{11}$$

By using the following well-known result, which is related to the Beta function and gamma function [11, 20], for $0<x<1$ and $\alpha>0,\beta>0$,

$$B(\alpha,\beta)=\int_{0}^{1}x^{\alpha-1}(1-x)^{\beta-1}dx=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{12}$$

in (11), we arrive at the desired result. If we integrate equation (3) from -1 to 0, we easily get proof of (9), so we omit it.□

Binomial coefficients play an important role in many branches of Mathematics and Mathematical Physics, especially including statistics, probability and analytic number theory.

By using (9) and (10), we arrive at the following theorem:

Theorem 4.3.

$$\sum_{j=0}^{n-k}(-1)^{n-j}\binom{n-k}{j}\frac{1}{n-j+1}=\frac{(-1)^k}{(n+1)\binom{n}{k}}.$$

5 Differentiating of the q-beta polynomials

In this section we give derivatives of the q-beta polynomials. Taking derivative of (6), with respect to x, we obtain the following PDE:

$$\frac{\partial F_{q,k}(t,x)}{\partial x}=\frac{kq^x\ln(q)}{[1+x](q-1)}\left(F_{q,k-1}(t,x)-qF_{q,k}(t,x)\right)+\frac{tq^{x+1}\ln(q)}{(q-1)}F_{q,k}(t,x). \tag{13}$$

By using this equation, we obtain derivative formula for the q-beta polynomials by the following theorem.

Theorem 5.1.

$$\frac{d}{dx}\mathfrak{B}_k^n(x;q)=\frac{kq^x\ln(q)}{[1+x](q-1)}\left(\mathfrak{B}_{k-1}^n(x;q)-q\mathfrak{B}_k^n(x;q)\right)+\frac{nq^{x+1}\ln(q)}{(q-1)}\mathfrak{B}_k^{n-1}(x;q).$$

Proof. By substituting the right-hand side of (5) into (13), we obtain

$$\sum_{n=0}^{\infty}\frac{d}{dx}\mathfrak{B}_k^n(x;q)\frac{t^n}{n!}$$

$$=\frac{kq^x\ln(q)}{[1+x](q-1)}\sum_{n=0}^{\infty}\left(-\mathfrak{B}_k^n(x;q)+q\mathfrak{B}_{k-1}^n(x;q)\right)\frac{t^n}{n!}+\frac{q^{x+1}\ln(q)}{(q-1)}\sum_{n=0}^{\infty}\mathfrak{B}_k^n(x;q)\frac{t^{n+1}}{n!}.$$

Thus, we have

$$\sum_{n=0}^{\infty}\frac{d}{dx}\mathfrak{B}_k^n(x;q)\frac{t^n}{n!}$$

$$=\frac{kq^x\ln(q)}{[1+x](q-1)}\sum_{n=0}^{\infty}\left(-\mathfrak{B}_k^n(x;q)+q\mathfrak{B}_{k-1}^n(x;q)\right)\frac{t^n}{n!}+\frac{q^{x+1}\ln(q)}{(q-1)}\sum_{n=0}^{\infty}n\mathfrak{B}_k^{n-1}(x;q)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above equation, we arrive at the desired result.□

We also derive the following second order PDE as follows:

$$\frac{\partial^2F_{q,k}(t,x)}{\partial t\partial x}=\frac{q^x\ln q}{q-1}F_{q,k-1}(t,x)+\frac{q^{x+1}\ln q}{q-1}[x]\,tF_{q,k-1}(t,x)+(k-1)\frac{q^x\ln q}{q-1}\left(F_{q,k-1}(t,x)-qF_{q,k}(t,x)\right).$$

Using the above equation, we also obtain other derivative formula for the q-beta polynomials by the following theorem.

Theorem 5.2.

$$\frac{d\mathfrak{B}_k^n(x;q)}{dx} = \frac{q^x \ln q}{q-1}\mathfrak{B}_{k-1}^n(x;q) + n\frac{q^{x+1}\ln q}{q-1}[x]\,\mathfrak{B}_{k-1}^{n-1}(x;q) + (k-1)\frac{q^x \ln q}{q-1}\left(\mathfrak{B}_{k-1}^n(x;q) - q\mathfrak{B}_k^n(x;q)\right).$$

Proof of this theorem is same as that of Theorem 5.1, so we omit it.

6 Recurrence Relation

In this section we give recurrence relations for the q-beta polynomials. Taking derivative of the generating functions for the q-beta polynomials with respect to x, we obtain the following PDEs:

$$\frac{\partial^v F_{q,k}(t,x)}{\partial t^v} = [x]^v F_{q,k-v}(t,x) \tag{14}$$

and

$$\frac{\partial^v F_{q,k}(t,x)}{\partial t^v} = [1+x]^v F_{q,k}(t,x).$$

Theorem 6.1.

$$\mathfrak{B}_k^{n+v}(x;q) = [x]^v \mathfrak{B}_{k-v}^n(x;q)$$

and

$$\mathfrak{B}_k^{n+v}(x;q) = [1+x]^v \mathfrak{B}_k^n(x;q).$$

Proof. By substituting the right-hand side of (5) into (14), we obtain

$$\sum_{n=v}^{\infty} \mathfrak{B}_k^n(x;q)\frac{t^{n-v}}{(n-v)!} = [x]^v \sum_{n=0}^{\infty} \mathfrak{B}_k^n(x;q)\frac{t^n}{n!}.$$

Thus we have

$$\sum_{n=0}^{\infty} \mathfrak{B}_k^{n+v}(x;q)\frac{t^n}{n!} = [x]^v \sum_{n=0}^{\infty} \mathfrak{B}_k^n(x;q)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above equation, we arrive at the desired result.□

7 Relations between q-Beta polynomials, Bernoulli polynomials, Euler polynomials and Stirling numbers

In this section, we derive some identities related to the q-Beta polynomials, the Bernoulli polynomials, the Euler polynomials and the Stirling numbers of the second kind.

The Bernoulli polynomials $B_n^{(v)}(x)$ of higher-order, the Euler polynomials $E_n(x)$ and the Stirling numbers of the second kind are defined by means of the following generating functions, respectively: Let v be a positive integer.

$$\frac{t^v e^{tx}}{(e^t-1)^v} = \sum_{n=0}^{\infty} B_n^{(v)}(x)\frac{t^n}{n!}, (|t| < 2\pi) \tag{15}$$

$$\frac{2e^{tx}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, (|t| < \pi) \tag{16}$$

and

$$\frac{(e^t-1)^v}{v!} = \sum_{n=0}^{\infty} S(n,v)\frac{t^n}{n!} \tag{17}$$

which of course

$$B_n^{(v)}(0) = B_n^{(v)}$$

and

$$E_n(0) = E_n$$

where $B_n^{(v)}$ and E_n are denoted the Bernoulli numbers of higher-order and the Euler numbers, respectively [2, 8–10, 12, 13, 20, 21].

Theorem 7.1.

$$\mathfrak{B}_k^n(x;q) = \frac{k!}{(n)_k}\frac{[x]^k}{[x+1]^k}\sum_{j=k}^{n}\binom{n}{j}B_{n-j}^{(k)}([1+x])\,S(j,k).$$

Proof. By using (5), (15) and (17), we get

$$\sum_{n=0}^{\infty} \mathfrak{B}_k^n(x;q)\frac{t^n}{n!} = \frac{k!\,[x]^k}{(t\,[x+1])^k}\left(\sum_{n=0}^{\infty} B_n^{(k)}([1+x])\frac{t^n}{n!}\right) \times \left(\sum_{n=0}^{\infty} S(n,k)\frac{t^n}{n!}\right).$$

Thus, by using the Cauchy product in the above equation and then equating the coefficients of $\frac{t^n}{n!}$ on both sides of the resulting equation, we get the desired result.□

Theorem 7.2.

$$\mathfrak{B}_k^n(x;q) = \frac{[x]^k}{2\,[x+1]^k}\left(E_n(1+[1+x]) + E_n([1+x])\right).$$

Proof. By using (5) and (16), we obtain

$$\sum_{n=0}^{\infty} \mathfrak{B}_k^n(x;q)\frac{t^n}{n!} = \frac{[x]^k}{2[x+1]^k}\sum_{n=0}^{\infty}\left(E_n\left(1+[1+x]\right)+E_n\left([1+x]\right)\right)\frac{t^n}{n!}.$$

Thus, equating the coefficients of $\frac{t^n}{n!}$ on both sides of the resulting equation, we get the desired result.□

8 Beta distribution

In this section we give some remarks and comments on the q-Beta polynomials and the beta distribution. The beta distribution has the following probability density function

$$f(x) = \frac{(x-a)^{\beta}(b-x)^{\alpha}}{(b-a)^{\alpha+\beta+1}B(\beta+1,\alpha+1)} \tag{18}$$

where $a \leq x \leq b$ and $B(u,v)$ is denoted the beta function, which is given by equation (12).
The above formula is the work of Xiu and Karniadakis [22].
By substituting $\beta = n-k$, $n \geq k$, $n,k \in \mathbb{N}$, $\alpha = k$, $a = -1$ and $b = 0$ into (18), domain of $f(x)$ is $-1 \leq x \leq 0$. Therefore, we now give the following probability density function associated with the beta polynomials.

Let X be a continuous random variable, defined on the interval $(-1,0)$. A continuous random variable X has the following distribution with parameters k and n:

$$f_{\mathfrak{B}}(x;k,n) = \frac{(-1)^k\mathfrak{B}_k^n(x;1)}{B(n-k+1,k+1)} \tag{19}$$

$-1 < x < 0$. One can easily see that

$$\int_{-1}^{0} f_{\mathfrak{B}}(x;k,n)dx = 1.$$

A moment generating function of a random variable X is defined as follows:

Definition 8.1. The moment generating function of a random variable X, denoted by $M_X(t)$, is defined as

$$M_X(t) = E\left(e^{xt}\right)$$

provided that the expectation is finite for $|t| < a$ with some $a > 0$ ([11, p. 79, Definition 2.3.3]).

Due to definition of the expectation, one can easily see that

$$M_X(0) = 1.$$

The moment generating function has many applications in probability theory and statistics.

The rth moment η_r of X is given by the following formula:

$$\eta_r = \frac{d^r}{dt^r}M_X(t)\,|_{t=0}. \tag{20}$$

By using (19) and (20), one can easily see that

$$\frac{d^r}{dt^r}M_X(t) = \frac{d^r}{dt^r}\int_{-1}^{0} e^{xt} f_{\mathfrak{B}}(x;k,n)dx,$$

and then it becomes clear that, by using differentiation under the integral sign, $\frac{d^r}{dt^r}M_X(t)$ when evaluated at $t=0$ will coincide with

$$\eta_r = \int_{-1}^{0} x^r f_{\mathfrak{B}}(x;k,n)dx.$$

From the above rth moment formula, we now compute some moments as follows:

$$\eta_1 = \int_{-1}^{0} x f_{\mathfrak{B}}(x;k,n)dx = -\frac{B(k+2,n-k+1)}{B(n-k+1,k+1)}$$

$$= -\frac{k+1}{n+2},$$

and

$$\eta_2 = \int_{-1}^{0} x^2 f_{\mathfrak{B}}(x;k,n)dx = \frac{B(k+3,n-k+1)}{B(n-k+1,k+1)}$$

$$= \frac{k^2+3k+2}{n+3}.$$

By using η_1 and η_2, we find expectation $E(X)$ and variance σ^2 of the probability density function $f_{\mathfrak{B}}(x;k,n)$ as follows, respectively:

$$E(X) = \eta_1 = -\frac{k+1}{n+2}$$

and

$$\sigma^2 = \eta_2 - \eta_1^2$$

$$= \frac{k^2+3k+2}{n+3} - \left(\frac{k+1}{n+2}\right)^2.$$

Remark 8.1. The q-beta polynomials may be related to (q-) beta distribution in q-probability density function.

9 Application of some integral transforms to the generating function

In this section, by applying the Mellin transform, the Fourier transform and the Laplace transform to the generating functions for the q-Beta polynomials, we obtain not only interpolation function, but also some interesting series representations for the q-Beta polynomials.

In terms of the generating function $F_{q,k}(t,x)$ occurring in (6), integral representation for the interpolation function of the q-Beta polynomials, which involves the Mellin transformation is given by

$$S_q(x,n,k) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} F_{q,k}(-t,x)dt,$$

where

$$\Re(s) > 0$$

the additional constraint $|q| < 1$ and $1+x > 0$ are required for the convergence of the above integral. Therefore, we assume that $|q[x]| < 1$, then we obtain

$$S_q(x,s,k) = \sum_{j=0}^\infty (-1)^j \binom{j+s+k-1}{s+k-1} q^j [x]^{k+j}.$$

We modify the function $S_q(x,s,k)$ as follows:

$$S_q(x,s,k) = \frac{[x]^k}{[1+x]^{s+k}}, \tag{21}$$

where $|q| < 1$ and $x \in (-1,0]$ and k is a nonnegative integer.

The function $S_q(x,s,k)$ is a meromorphic function. One can see that $S_q(-1,s,k) = \infty$. This function has also a zero of order k at $x = 0$.

Upon substituting $s = -n$, $(n \in \mathbb{N})$ into (21), we easily find that

$$S_q(x,-n,k) = \mathfrak{B}_k^n(x;q).$$

By using (6), we get the following functional equation:

$$F_{q,k}(t,x)e^{-\frac{t}{1-q}} = \left(\frac{[x]}{[1+x]}\right)^k e^{-\frac{tq^{1+x}}{1-q}}. \tag{22}$$

Combining (5) and (22), we obtain

$$\sum_{n=0}^\infty \mathfrak{B}_k^n(x;q)\frac{t^n}{n!}e^{-\frac{t}{1-q}} = \left(\frac{[x]}{[1+x]}\right)^k e^{-\frac{tq^{1+x}}{1-q}}. \tag{23}$$

Integrating this equation with respect to t from 0 to ∞, we get

$$\sum_{n=0}^\infty \frac{\mathfrak{B}_k^n(x;q)}{n!} \int_0^\infty t^n e^{-\frac{t}{1-q}} dt = \left(\frac{[x]}{[1+x]}\right)^k \int_0^\infty e^{-\frac{tq^{1+x}}{1-q}} dt,$$

where the additional constraint $0 < q < 1$ and $1+x > 0$ are required for the convergence of the above integral. By using the Laplace transform in the above equation, we arrive at the following theorem:

Theorem 9.1. Let $0 < q < 1$ and $x \in [-1,0]$. Then we have

$$\sum_{n=0}^\infty \mathfrak{B}_k^n(x;q)(1-q)^n = q^{-(1+x)}\left(\frac{[x]}{[1+x]}\right)^k.$$

By applying the Fourier transform to (22), we get

$$\sum_{n=0}^\infty \frac{\mathfrak{B}_k^n(x;q)}{n!} \int_0^\infty t^n e^{-\frac{t}{1-q}} e^{-ist} dt$$

$$= \left(\frac{[x]}{[1+x]}\right)^k \int_0^\infty e^{-\frac{tq^{1+x}}{1-q}} e^{-ist} dt.$$

After some elementary calculations from this equation, we arrive at the following theorem:

Theorem 9.2. Let $0 < q < 1$ and $x \in [-1,0]$ and $s \in \mathbb{R}$. Then we have

$$\sum_{n=0}^\infty \mathfrak{B}_k^n(x;q)\frac{(1-q)^n}{(1+(1-q)is)^{n+1}}$$

$$= \frac{1}{q^{1+x}+(1-q)is}\left(\frac{[x]}{[1+x]}\right)^k.$$

where $\left|\frac{1-q^{1+x}}{1+(1-q)is}\right| < 1$.

By using (22), we find the following identities related to the q-Beta polynomials.

Theorem 9.3.

$$\sum_{j=0}^n \binom{n}{j}(q-1)^j \mathfrak{B}_k^j(x;q) = \left(\frac{[x]}{[1+x]}\right)^k q^{n(1+x)},$$

and

$$\mathfrak{B}_k^n(x;q) = \left(\frac{[x]}{[1+x]}\right)^k \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{q^{j(1+x)}}{(1-q)^n}.$$

10 Applications the p-adic Volkenborn integral to the q-beta polynomials

By using the p-adic Volkenborn integrals on $\mathbb{Z}_p$, we derive some identities associated with the q-beta polynomials, the Carlitz's q-Bernoulli numbers, and the Kim's q-Euler numbers.

In order to derive the main results in this section, we recall some well known results related to the p-adic Volkenborn integral.

Let p be a fixed prime. It is known that

$$\mu_q(x+dp^N\mathbb{Z}_p) = \frac{q^x}{[dp^N]}$$

is a distribution on $\mathbb{Z}_p$ for $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ [8, 9, 21]. Let $UD(\mathbb{Z}_p)$ be the set of uniformly differentiable functions on $\mathbb{Z}_p$. The p-adic q-Volkenborn integral of the function $f \in UD(\mathbb{Z}_p)$ was defined by Kim [9] as follows:

$$\int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]} \sum_{x=0}^{p^N-1} f(x)q^x \tag{24}$$

see also [8, 21].
By using the above integral, we have the Witt's formula for the Carlitz's q-Bernoulli numbers $\beta_{n,q}$ as follows:

$$\int_{\mathbb{Z}_p} [x]^n \, d\mu_q(x) = \beta_{n,q} \tag{25}$$

[8, 9, 21]. These numbers are given explicitly as follows:

$$\beta_{0,q} = 1$$

and

$$q(q\beta+1)^n - \beta = \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases}$$

with the usual convention of replacing β^j by β_j [3, 21].

The Witt's formula for the q-Euler numbers $K_{n,q}$ was given by Kim [9] as follows:

$$\int_{\mathbb{Z}_p} [x]^n \, d\mu_{-q}(x) = K_{n,q}. \tag{26}$$

These numbers are defined by

$$K_{n,q} = [2]\left(\frac{1}{1-q}\right)^n \sum_{j=0}^{n}(-1)^j \binom{n}{j} \frac{1}{q^{j+1}+1}$$

[8, 9].

Observe that

$$\lim_{q\to 1} K_{n,q} = E_n,$$

E_n is denoted the Euler numbers, which are defined by means of the following generating function:

$$\frac{2}{e^t+1} = \sum_{m=0}^{\infty} E_n \frac{t^n}{n!}$$

[8, 9].

Theorem 10.1.

$$\int_{\mathbb{Z}_p} \mathfrak{B}_k^n(x;q)d\mu_q(x) = \sum_{j=0}^{n-k} \binom{n-k}{j} q^{n-k-j} \beta_{n-j,q}.$$

Proof. By using (1), we get

$$\mathfrak{B}_k^n(x;q) = \sum_{j=0}^{n-k} \binom{n-k}{j} q^{n-k-j} [x]^{n-j}. \tag{27}$$

By applying the p-adic q-Volkenborn integral to the above equation, we obtain

$$\int_{\mathbb{Z}_p} \mathfrak{B}_k^n(x;q)d\mu_q(x) = \sum_{j=0}^{n-k} \binom{n-k}{j} q^{n-k-j} \int_{\mathbb{Z}_p} [x]^{n-j} \, d\mu_q(x).$$

By using (25) in the above equation, we arrive at the desired result.□

Theorem 10.2.

$$\int_{\mathbb{Z}_p} \mathfrak{B}_k^n(x;q)d\mu_{-q}(x) = \sum_{j=0}^{n-k} \binom{n-k}{j} q^{n-k-j} K_{n-j,q}.$$

Proof. By replacing q by $-q$ in (24), one has the p-adic fermionic q-Volkenborn integral. By applying this integral to (27), we get

$$\int_{\mathbb{Z}_p} \mathfrak{B}_k^n(x;q)d\mu_{-q}(x) = \sum_{j=0}^{n-k} \binom{n-k}{j} q^{n-k-j} \int_{\mathbb{Z}_p} [x]^{n-j} \, d\mu_{-q}(x).$$

By using (26) in the above equation, we arrive at the desired result.□

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