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Extended Gompertz Distribution: Properties and Estimation under Complete and Censored Data

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Abstract: In this paper, a new flexible model with three-parameter alternative to exponential and Gompertz distributions is proposed. Some of its statistical properties are derived including quantities, moments, incomplete moments, moment of residual and reversed residual life. The parameters are estimated using the maximum likelihood method based on complete and Type II right censored data. We assess the performance of estimators in terms of bias and mean square error using simulation study. Finally, three real data sets are analyzed to illustrate the flexibility of the proposed model.

Keywords: Gompertz distribution, Hazard rate function, Maximum likelihood method, Censored samples.

1 Introduction

The thought of generating new extended distributions from classical ones has been of great concern among researchers in the past decades. In this article, we present a new generalized Gompertz model, which extends the Gompertz distribution by adding one extra shape parameter. Gompertz distribution force of mortality extends to the whole life span of populations with no observed mortality deceleration for low levels of infant mortality (see Vaupel, [1]).
To analyze the lifetime data, we often use probability models like exponential, Gompertz, Weibull, Gumbel etc. So, the researchers have been developing various univariate and bivariate extensions of these distributions for describing various phenomena in several fields. See for example, Willekens [2], Nadarajah and Kotz [3], Ali et al. [4], El-Gohary et al. [5,6,7], Khan et al. [8], El-Bassiouny et al. [9,10,11,12], Haitham et al. [13], El-Bassiouny and El-Morshedy [14], El-Morshedy et al. [15,16,17,18,19,15,21,22], Mohamed et al. [23], Eliwa et al. [24,25,26,27,28,29], Jehhan et al. [30], El-Morshedy and Eliwa [31], Alizadeh et al. [32], Eliwa and El-Morshedy [33,34,35,36], Salah et al. [37], among others. When the hazard function of the probability distribution is constant, increasing or decreasing, it cannot be used to model lifetime data with a bathtub shaped hazard rate function like machine life cycles. Thus, in this paper we propose an extension of the Gompertz distribution (GD) in the so-called extended Gompertz distribution (EGD) with three parameters. The CDF of the GD can be expressed as

$$G(x) = 1 - e^{-\frac{\lambda}{c}(e^{cx}-1)}; \quad x \geq 0,\ \lambda, c > 0. \tag{1}$$

The corresponding PDF to Equation (1) can be written as

$$g(x) = \lambda e^{cx} e^{-\frac{\lambda}{c}(e^{cx}-1)}; \quad x \geq 0,\ \lambda, c > 0. \tag{2}$$

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2 The EGD

The non-negative random variable X is said to have the EGD with vector parameters $= (\lambda, c, \theta)$ if its CDF is given by

$$F(x) = 1 - e^{-\frac{\lambda}{c}(e^{cx}-1)^{\theta}}; \quad x \geq 0, \ \lambda, c, \theta > 0. \tag{3}$$

The GD can be obtained as a special case when $\theta = 1$. Also, when the parameter c tends to zero and the parameter $\theta = 1$, the EGD tends to the exponential model. The corresponding PDF to Equation (3) can be written as follows

$$f(x) = \lambda\theta(e^{cx}-1)^{\theta-1}e^{cx}e^{-\frac{\lambda}{c}(e^{cx}-1)^{\theta}}; \quad x \geq 0, \ \lambda, c, \theta > 0. \tag{4}$$

Figure 1 shows the PDF for various values of the parameters. The PDF can take unimodal or decreasing-shaped.

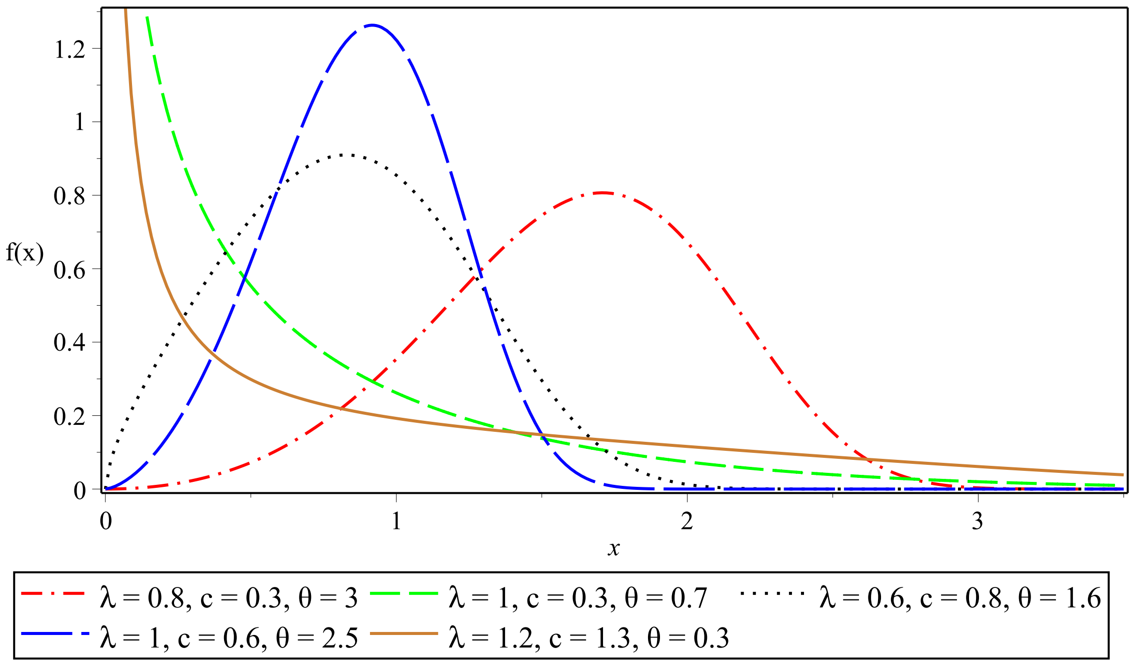

Fig. 1: The PDF of the EGD for different values of its parameters.

3 Statistical Properties

3.1 Quantile and mode

The quantile x_q of the EGD(Φ) can be expressed as a closed form where

$$x_q = \frac{1}{c}\ln(1 + \left[-\frac{c}{\lambda}\ln(1-q)\right]^{\frac{1}{\theta}}), \ 0 < q < 1. \tag{5}$$

Setting $q = \frac{1}{2}$ in (5), we get the median of EGD(Φ) as

$$Med(X) = \frac{1}{c}\ln(1 + \left[-\frac{c}{\lambda}\ln(\frac{1}{2})\right]^{\frac{1}{\theta}}). \tag{6}$$

On the other hand, we can derive the mode of EGD(Φ) by solving the following equation with respect to x

$$c + c(\theta-1)(e^{cx}-1)^{-1}e^{cx} - \lambda\theta(e^{cx}-1)^{\theta-1}e^{cx} = 0. \tag{7}$$

It must be obtained numerically.

3.2 Moments and incomplete moments

If $X \sim EGD(\Phi)$, then the rth moment of X, is given as

$$\begin{aligned}\mu_r' = E(X^r) &= \lambda\theta\int_0^\infty x^r e^{cx}(e^{cx}-1)^{\theta-1}e^{-\frac{\lambda}{c}(e^{cx}-1)^\theta}\,dx\\ &= \sum_{i,l=0}^\infty \lambda\theta\frac{(-1)^i}{l!}\binom{\theta-1}{i}g_i\int_0^\infty x^{r+l}e^{-cx(i-\theta)}\,dx\\ &= \sum_{i,l=0}^\infty \nu_{i,l}^{(r)}\Gamma(r+l+1).\end{aligned} \tag{8}$$

where $\nu_{i,l}^{(r)} = \lambda\theta\frac{(-1)^i g_i}{l![c(i-\theta)]^{-(r+l+1)}}\binom{\theta-1}{i}$ and $g_i = \frac{d^i}{dx^i}e^{-\frac{\lambda}{c}(e^{cx}-1)^\theta}\,|_{x=0}$. The kurtosis and skewness measures can be calculated using the ordinary moments of X using well-known relationships. The nth central moment of X, say μ_n, is given by

$$\mu_n = E\left(X-\mu_1'\right)^n = \sum_{r=0}^n\binom{n}{r}\left(-\mu_1'\right)^{n-r}E\left(X^r\right), \tag{9}$$

where $\mu_1' = E(X)$. The cumulants (κ_n) of X are obtained from (8) as

$$\kappa_n = \mu_n' - \sum_{r=0}^{n-1}\binom{n-1}{r-1}(\kappa_r)\left(\mu_{n-r}'\right), \tag{10}$$

where $\kappa_1 = \mu_1'$thus $\kappa_2 = \mu_2' - \mu_1'^2, \kappa_3 = \mu_3' - 3\mu_2'\mu_1' + \mu_1'^3$ and so on. The sth incomplete moments, denoted by $\varphi_s(t)$, of the EGD is given by

$$\varphi_s(t) = \lambda\theta\int_0^t x^s(e^{cx}-1)^{\theta-1}e^{cx}e^{-\frac{\lambda}{c}(e^{cx}-1)^\theta}dx, \tag{11}$$

using the lower incomplete gamma function, we get

$$\varphi_s(t) = \sum_{i,l=0}^\infty \nu_{i,l}^{(r)}\gamma(r+l+1, ct(i-\theta)), \tag{12}$$

where $\gamma(a,y) = \int_0^t y^{a-1}e^{-y}dy$ is the lower incomplete gamma function.

4 Reliability Properties

4.1 Hazard rate function

If $X \sim EGD(\Phi)$, then the reliability function of X can be proposed as

$$R(x) = e^{-\frac{\lambda}{c}(e^{cx}-1)^\theta};\quad x\geq 0,\ \lambda, c, \theta > 0. \tag{13}$$

The hazard rate function of X can be written as

$$h(x) = \lambda\theta(e^{cx}-1)^{\theta-1}e^{cx};\quad x\geq 0,\ \lambda, c, \theta > 0. \tag{14}$$

From Equation (14), one can easily prove that, the hazard rate function of EGD is increasing function for $\theta \geq 1$ whereas bathtub-shaped when $\theta < 1$. To illustrate that, we plot the hazard rate function of EGD in Figure 2 for $\lambda = 1.5$, $c = 0.3$ and different values of θ.

Figure 2 shows that the hazard rate function can be take different shapes.

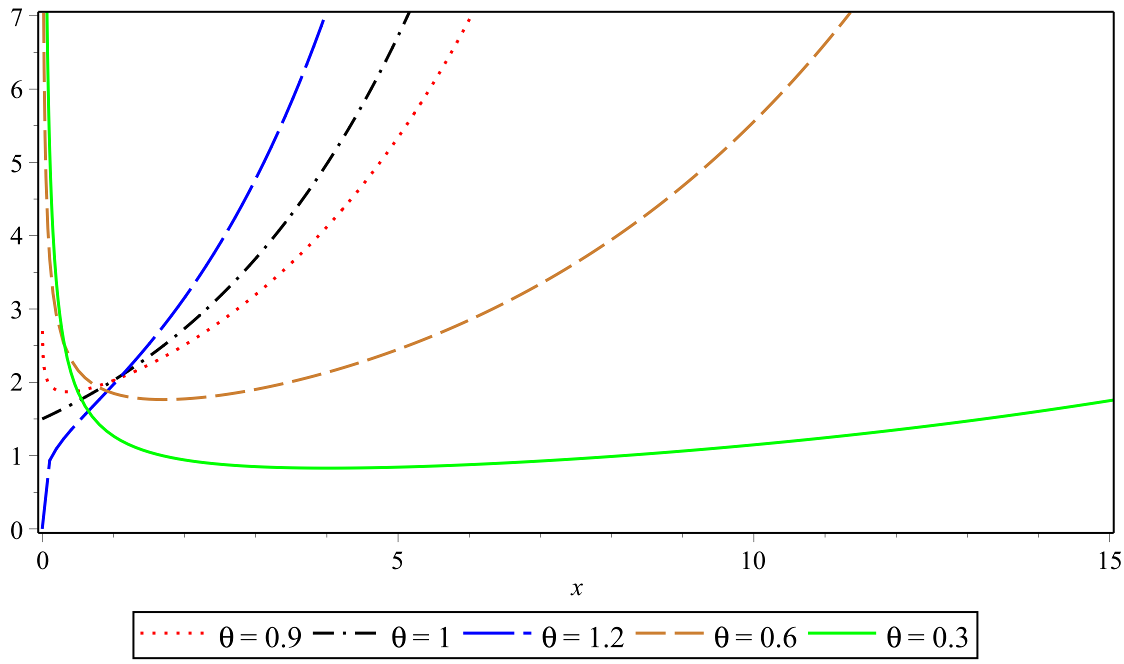

Fig. 2: Plots of the hazard rate function of EGD for $\lambda = 1.5$, $c = 0.3$ and different values of θ.

4.2 Entropies

In information theory, the Rényi entropy has numerous applications, for example distribution identification problems, average case analysis for random databases, image matching, econometrics etc. The Rényi entropy of a random variable X representing a measure of variation of the uncertainty. If $X \sim EGD(\Phi)$, then variation of the uncertainty to random variable X, is given as follows

$$\begin{aligned} I_\delta(X) &= \frac{1}{1-\delta}\log\int_0^\infty f^\delta(x)\,dx \\ &= \frac{1}{1-\delta}\log\left\{(\lambda\theta)^\delta\int_0^\infty e^{c\delta x}(e^{cx}-1)^{\delta(\theta-1)}e^{-\frac{\lambda\delta}{c}(e^{cx}-1)^\theta}\,dx\right\} \\ &= \frac{1}{1-\delta}\log\left\{(\lambda\theta)^\delta\sum_{i,l=0}^\infty\frac{(-1)^i}{l!}\binom{\delta\theta-\delta}{i}g_i\int_0^\infty x^l e^{-c\delta x(i-\theta)}\,dx\right\} \\ &= \frac{1}{1-\delta}\log\left\{(\lambda\theta)^\delta\sum_{i,l=0}^\infty\eta_{i,l}^{(\delta)}\,\Gamma(l+1)\right\}, \end{aligned} \tag{15}$$

where $\delta \in]0,\infty[-\{1\}$, $g_i = \frac{d^i}{dx^i}e^{-\frac{\lambda\delta}{c}(e^{cx}-1)^\theta}\,|_{x=0}$and $\eta_{i,l}^{(\delta)} = \frac{(-1)^i g_i}{l![c\delta(i-\theta)]^{-(l+1)}}\binom{\delta\theta-\delta}{i}$. The δ-entropy, say $H_\delta(X)$, is defined by

$$H_\delta(X) = \frac{1}{\delta-1}\log\{1-[(1-\delta)I_\delta(X)]\}. \tag{16}$$

4.3 Moments of the residual life (RL) and mean residual life (MRL)

The nth moment of the RL of X is given by

$$\begin{aligned} m_n(t) &= E[(X-t)^n \mid X>t],\ n=1,2,\ldots \\ &= \frac{1}{1-F(t)}\int_t^\infty (x-t)^n dF(x). \end{aligned} \tag{17}$$

If $X \sim EGD(\Phi)$, we get

$$m_n(t) = \frac{1}{1-F(t)} \sum_{i,l=0}^{\infty} \sum_{r=0}^{n} \nu_{i,l,r}^{(n)} \gamma(r+l+1, ct(i-\theta)), \tag{18}$$

where

$$\nu_{i,l,r}^{(n)} = \nu_{i,l}^{(n)} \binom{n}{r} (-t)^{n-r} \text{ and } \gamma(\zeta, y) = \int_b^{\infty} y^{\zeta-1} e^{-y} dy.$$

The MRL at age t is defined as $m_1(t) = E[(X-t) \mid X > t]$, which represents the expected additional life length for a unit which is alive at age t. The MRL of X can be obtained by setting $n = 1$ in $m_n(t)$ equation.

4.4 Moments of the reversed residual life (RRL) and mean inactivity time (MIT)

The nth moment of the RRL of X is given by

$$\begin{aligned} M_n(t) &= E[(t-X)^n \mid X \le t] \text{ for } t > 0 \text{ and } n = 1, 2, \ldots \\ &= \frac{1}{F(t)} \int_0^t (t-x)^n dF(x). \end{aligned} \tag{19}$$

If $X \sim EGD(\Phi)$, we get

$$M_n(t) = \frac{1}{F(t)} \sum_{i,l=0}^{\infty} \sum_{r=0}^{n} \nu_{i,l,r}^{(n)\bigstar} \gamma(r+l+1, ct(i-\theta)), \tag{20}$$

where

$$\nu_{i,l,r}^{(n)\bigstar} = \nu_{i,l}^{(n)} (-1)^r \binom{n}{r} t^{n-r}. \tag{21}$$

The MIT is given by $M_1(t) = E[(t-X) \mid X \le t]$. It represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0, t)$. By setting $n = 1$ in the $M_n(t)$ equation we get the MIT of EGD.

5 Parameters Estimation

5.1 Maximum likelihood estimators based on complete sample

Assume $X_1, X_2, \ldots, X_n$ be a random sample of size n from EGD, then the likelihood function ℓ is

$$\ell = \prod_{i=1}^{n} f(x_i; \lambda, c, \theta). \tag{22}$$

Substituting from (4) into (22), we get

$$\ell = \prod_{i=1}^{n} \lambda\theta (e^{cx_i} - 1)^{\theta-1} e^{cx_i} e^{-\frac{\lambda}{c}(e^{cx_i}-1)^{\theta}}. \tag{23}$$

The log-likelihood function L is given by

$$L = n\ln(\lambda\theta) + c\sum_{i=1}^{n} x_i - \frac{\lambda}{c} \sum_{i=1}^{n} (e^{cx_i} - 1)^{\theta} + (\theta - 1) \sum_{i=1}^{n} \ln(e^{cx_i} - 1). \tag{24}$$

By differentiating L with respect to λ, c and θ, we get

$$\frac{n}{\hat{\lambda}} - \frac{1}{\hat{c}} \sum_{i=1}^{n} (e^{\hat{c}x_i} - 1)^{\hat{\theta}} = 0, \tag{25}$$

$$\sum_{i=1}^{n} x_i - \frac{\hat{\lambda}}{\hat{c}^2} \sum_{i=1}^{n} (e^{\hat{c}x_i} - 1)^{\hat{\theta}} \left[\hat{c}\hat{\theta} g(x_i; \hat{c}) - 1 \right] + (\hat{\theta} - 1) \sum_{i=1}^{n} g(x_i; \hat{c}) = 0 \tag{26}$$

and

$$\frac{n}{\hat{\theta}}+\sum_{i=1}^{n}\left(\ln\left(e^{\hat{c}x_i}-1\right)\left[1-\frac{\hat{\lambda}}{\hat{c}}(e^{\hat{c}x_i}-1)^{\hat{\theta}}\right]\right)=0, \tag{27}$$

where

$$g(x_i;\hat{c})=\frac{x_i e^{\hat{c}x_i}}{e^{\hat{c}x_i}-1}. \tag{28}$$

From (25), we can be obtained the MLE of $\hat{\lambda}$ for a given $\hat{c}$ and $\hat{\theta}$ as the following form

$$\hat{\lambda}=\frac{n\hat{c}}{\sum_{i=1}^{n}(e^{\hat{c}x_i}-1)^{\hat{\theta}}}. \tag{29}$$

Substituting from (29) into (26) and (27), we get the MLE of c and θ by solving the two equations.

5.1.1 Asymptotic confidence intervals (CI)

We derive the asymptotic CI of the parameters λ, c and θ using Fisher matrix I_0^{-1}, see Lawless [38] where I_0^{-1} is

$$I_0^{-1}=-\begin{pmatrix}\frac{\partial^2 L}{\partial\lambda^2} & \frac{\partial^2 L}{\partial\lambda\partial c} & \frac{\partial^2 L}{\partial\lambda\partial\theta}\\ \frac{\partial^2 L}{\partial c\partial\lambda} & \frac{\partial^2 L}{\partial c^2} & \frac{\partial^2 L}{\partial c\partial\theta}\\ \frac{\partial^2 L}{\partial\theta\partial\lambda} & \frac{\partial^2 L}{\partial\theta\partial c} & \frac{\partial^2 L}{\partial\theta^2}\end{pmatrix}^{-1}=\begin{pmatrix}Var(\hat{\lambda}) & Cov(\hat{\lambda},\hat{c}) & Cov(\hat{\lambda},\hat{\theta})\\ Cov(\hat{c},\hat{\lambda}) & Var(\hat{c}) & Cov(\hat{c},\hat{\theta})\\ Cov(\hat{\theta},\hat{\lambda}) & Cov(\hat{\theta},\hat{c}) & Var(\hat{\theta})\end{pmatrix} \tag{30}$$

The second partial derivatives included in I_0 are given as follows:

$$\frac{\partial^2 L}{\partial\lambda^2}=-\frac{n}{\lambda^2},\ \frac{\partial^2 L}{\partial\lambda\partial c}=\sum_{i=1}^{n}\frac{1}{c^2}(e^{cx_i}-1)^{\theta}\left[1-c\theta g(x_i;c)\right],\ \frac{\partial^2 L}{\partial\lambda\partial\theta}=\sum_{i=1}^{n}\ln\left(e^{cx_i}-1\right)\frac{1}{c}(e^{cx_i}-1)^{\theta}, \tag{31}$$

$$\frac{\partial^2 L}{\partial c^2}=-\frac{\lambda\theta}{c^2}\sum_{i=1}^{n}(e^{cx_i}-1)\,g(x_i;c)\left[cx_i-1\right]-\frac{2\lambda}{c^3}\sum_{i=1}^{n}(e^{cx_i}-1)^{\theta}+\frac{\lambda\theta}{c^2}\sum_{i=1}^{n}g(x_i;c)+(\theta-1)\sum_{i=1}^{n}g(x_i;c)\left[x_i-g(x_i;c)\right], \tag{32}$$

$$\frac{\partial^2 L}{\partial c\partial\theta}=\frac{\lambda}{c}\sum_{i=1}^{n}(e^{cx_i}-1)^{\theta}\ g(x_i;c)\left[\theta\ln\left(e^{cx_i}-1\right)+1\right]-\sum_{i=1}^{n}g(x_i;c)-\frac{\lambda}{c^2}\sum_{i=1}^{n}(e^{cx_i}-1)^{\theta}\ln\left(e^{cx_i}-1\right) \tag{33}$$

and

$$\frac{\partial^2 L}{\partial\theta^2}=-\frac{n}{\theta^2}+\frac{\lambda}{c}\sum_{i=1}^{n}(e^{cx_i}-1)^{\theta}\left[\ln\left(e^{cx_i}-1\right)\right]^2. \tag{34}$$

5.2 *Maximum likelihood estimation based on Type II right censored data*

In reliability, medical and engineering research, censoring is a case in which the value of data is only partially known. There are many applications ranging from accelerated life testing through to floods, sea currents, earthquakes, rain fall, wind speeds etc, see Kotz and Nadarajah [39]. There are several types of censored samples. One of the most common censoring is Type II right censored data. In a Type II right censored data, if we have a set number of items and stops the experiment when a fixed number are observed to have failed, the residual items are then right censored. Many distributions are studied based on Type II right censored, see Lawless [38], Balakrishnan and Aggarwala [40], Lin et al. [41], Castro-Kuriss [42] and Eliwa et al. [26]. The likelihood function is

$$l=\frac{n!}{(n-k)!}\,(R(x_k))^{n-k}\prod_{i=1}^{k}f(x_i). \tag{35}$$

If $X_1, X_2, \ldots, X_n$ is random sample from EGD and $X_{(1)}, X_{(2)}, \ldots, X_{(k)}$, $k \leq n$ represent the ordered sample obtained from Type II right censored sample, then the log - likelihood function is

$$L = \ln(\frac{n!}{(n-k)!}) + (n-k)\ln(R(x_k)) + \sum_{i=1}^{k} \ln(f(x_i))$$

$$= \ln(\frac{n!}{(n-k)!}) - \frac{\lambda(n-k)}{c}(e^{cx_k}-1)^{\theta} + k\ln\lambda + k\ln\theta - \frac{\lambda}{c}\sum_{i=1}^{k}(e^{cx_i}-1)^{\theta} + c\sum_{i=1}^{k}x_i + (\theta-1)\sum_{i=1}^{k}(e^{cx_i}-1). \tag{36}$$

Differentiating Equation (36) with respect to λ, c and θ, we get

$$\frac{\partial L}{\partial \lambda} = -\frac{(n-k)}{c}(e^{cx_k}-1)^{\theta} + \frac{k}{\lambda} - \frac{1}{c}\sum_{i=1}^{k}(e^{cx_i}-1)^{\theta}, \tag{37}$$

$$\frac{\partial L}{\partial c} = \frac{\lambda(n-k)}{c^2}(e^{cx_k}-1)^{\theta} - \frac{\lambda\theta(n-k)x_k}{c}e^{cx_k}(e^{cx_k}-1)^{\theta} + \frac{\lambda}{c^2}\sum_{i=1}^{k}(e^{cx_i}-1)^{\theta} - \frac{\lambda\theta}{c}\sum_{i=1}^{k}x_i e^{cx_i}(e^{cx_i}-1)^{\theta-1}$$
$$+\sum_{i=1}^{k}x_i + (\theta-1)\sum_{i=1}^{k}\frac{x_i e^{cx_i}}{(e^{cx_i}-1)} \tag{38}$$

and

$$\frac{\partial L}{\partial \theta} = -\frac{\lambda(n-k)}{c}(e^{cx_k}-1)^{\theta}\ln(e^{cx_k}-1) + \frac{k}{\theta} + \sum_{i=1}^{k}\ln(e^{cx_i}-1) - \frac{\lambda}{c}\sum_{i=1}^{k}(e^{cx_i}-1)^{\theta}\ln(e^{cx_i}-1). \tag{39}$$

6 Simulation

In this section, we present the results of simulation studies. We consider the EGD and generate $N = 1000$ random samples with different set of parameters for $n = 15, 17, 19, \ldots, 60$. In each random sample, we calculate the average value of MLE(s) and the mean squared errors (MSE) to every parameter. The results are given in Figures 3 - 6.

We note that the bias and MSE are reduced as the sample size is increased.

7 Data Analysis

7.1 Complete Sample

We provide two applications of the EGD to show empirically its potentiality. we shall compare the fits of the EGD with those of other competitive models, namely: the exponential distribution (ED) and Gompertz distribution (GD) whose pdfs are well known. In order to compare the fits of the EGD with other competing distributions, we consider K-S test (see, Kolmogorov and Smirnov [43]), (Corrected) Akaike Information Criterion (C)AIC (see, Hurvich and Tsai [44]), Bayesian Information Criterion (BIC) (see, Zhao [45]), HQIC (see Hannan and Quinn [46]), A^* (see, Anderson-Darling [47]) and W^* (see, Cramér and Mises [48]) statistics. The smaller these statistics are, the better the fit.

Data set I:

The data represent the lifetimes of 50 devices, see Aarset [49]. Using the EGD and comparing it with the other fitted models like ED and GD using K-S distance test statistic and its corresponding p-value and the log-likelihood values (L), AIC, CAIC, BIC, HQIC, A^* and W^* are calculated for this data, which are given in Tables 1 and 2 respectively.

Table 1. The MLEs of the parameters, K-S test statistic and corresponding p-values.

The model	MLE(s)	K-S	p-value
$E(\lambda)$	$\hat{\lambda} = 0.022$	0.191	0.048
$G(\lambda, c)$	$\hat{\lambda} = 9.7\times10^{-3}, \hat{c} = 2.03\times10^{-2}$	0.169	0.113
$EG(\lambda, c, \theta)$	$\hat{\lambda} = 0.014, \hat{c} = 0.078, \hat{\theta} = 0.385$	0.133	0.339

(40)

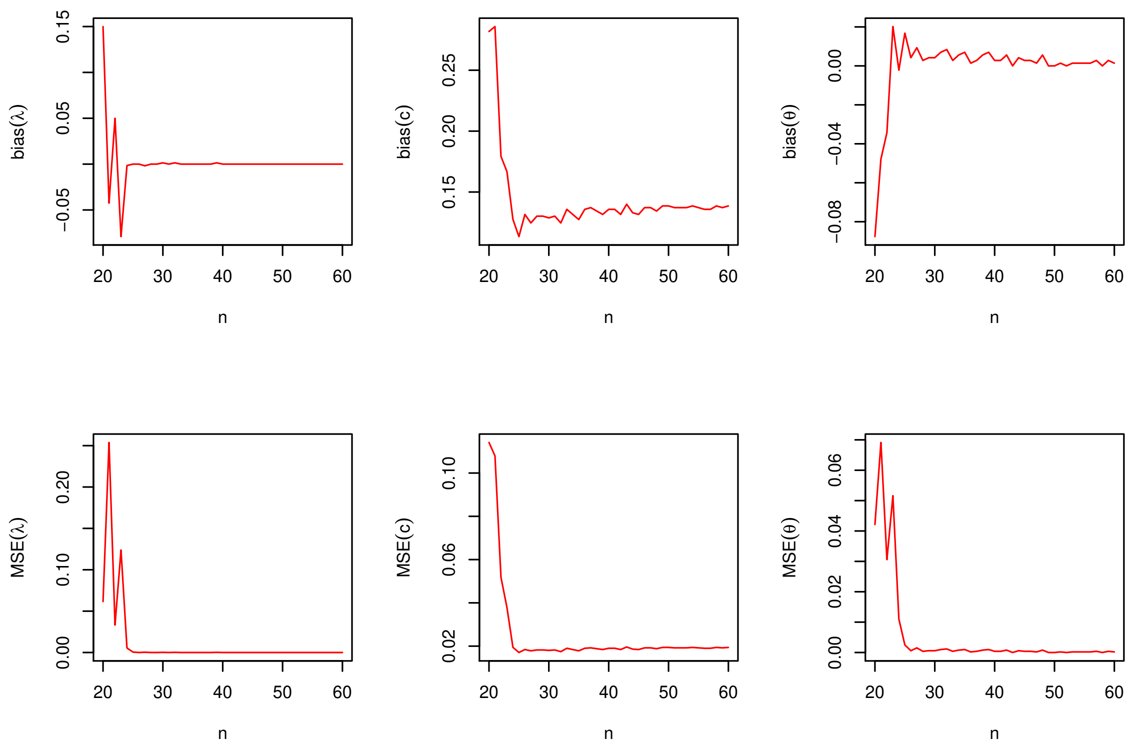

Fig. 3: The bias and MSE of $(\lambda = 0.3, c = 0.2, \theta = 1.4)$.

Table 2. The log-likelihood, AIC, CAIC, BIC, HQIC, A^* and W^* values.

The model	-L	AIC	CAIC	BIC	HQIC	A*	W*
$E(\lambda)$	241.1	484.2	484.3	486.1	484.9	0.488	2.962
$G(\lambda, c)$	235.3	474.7	474.9	478.5	476.1	0.290	1.897
$EG(\lambda, c, \theta)$	225.7	457.3	457.8	463.03	459.5	0.212	1.479

(41)

From Table 2, it is observed that the EGD fits the data better than the other tested models. More information is provided by a visual comparison of the empirical of the data set I with the fitted distributions. Figures 7 and 8 show the empirical and estimated reliability functions of ED, EGD and GD and the hazard rate functions for this data as well as the P-P plot.

Estimation of the variance covariance matrix for data set I given as the following:

$$I_0^{-1} = \begin{pmatrix} 0.0014 & -0.0055 & 1.4 \times 10^{-4} \\ -0.0055 & 0.0242 & -7.5 \times 10^{-4} \\ 1.4 \times 10^{-4} & -7.5 \times 10^{-4} & 3.3 \times 10^{-5} \end{pmatrix}. \tag{42}$$

The approximate 95% two sided CI of the parameters λ, c and θ are given respectively as

$$[0.0027, 0.0253], \ [0.0056, 0.1504], \ [0.0801, 0.6899]. \tag{43}$$

The profiles of the log-likelihood function of λ, c and θ for data set I are plotted in Figure 9, which show that the likelihood equations have a unique solution.

Data set II:

The data set represents the times of failure and running times for 30 devices studied, see Meeker and Escobar [50]. Using the EGD and comparing it with the other fitted models like ED and GD using K-S and its corresponding p-value

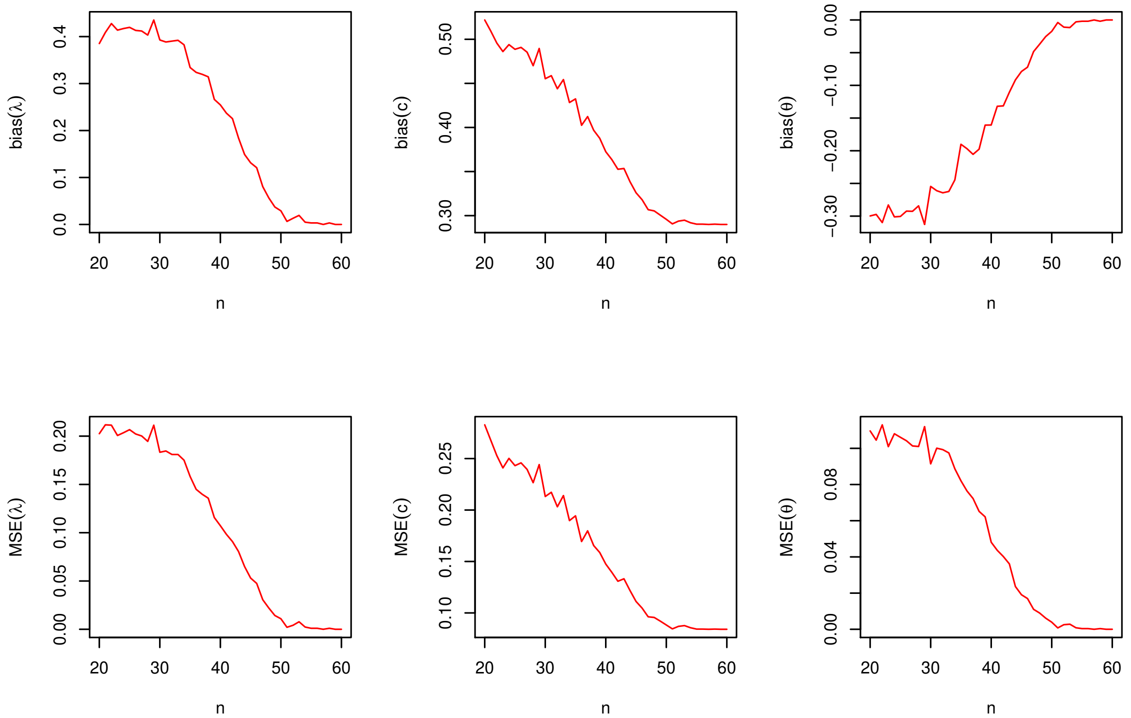

Fig. 4: The bias and MSE of $(\lambda = 0.4, c = 0.3, \theta = 2.9)$.

and the log-likelihood values (L), AIC, CAIC, BIC, HQIC, A^* and W^* are calculated for this data, which are given in Tables 3 and 4 respectively.

Table 3. The MLEs of the parameters, K-S test statistic and corresponding p-values.

The model	MLE(s)	K-S	p-value
$E(\lambda)$	$\hat{\lambda} = 5.65 \times 10^{-3}$	0.216	0.121
$G(\lambda, c)$	$\hat{\lambda} = 1.85 \times 10^{-3}, \hat{c} = 7.4 \times 10^{-3}$	0.189	0.235
$EG(\lambda, c, \theta)$	$\hat{\lambda} = 2.34 \times 10^{-3}, \hat{c} = 0.0187, \hat{\theta} = 0.499$	0.173	0.332

(44)

Table 4. The log-likelihood, AIC, CAIC, BIC, HQIC, A^* and W^* values.

The model	-L	AIC	CAIC	BIC	HQIC	A^*	W^*
$E(\lambda)$	185.29	372.58	372.72	373.98	373.03	1.906	0.322
$G(\lambda, c)$	179.50	363.00	363.45	365.81	363.89	1.356	0.209
$EG(\lambda, c, \theta)$	177.16	360.24	361.24	364.52	361.66	1.303	0.193

(45)

From Table 4, it is obsarved that the EGD fits the data better than the other tested models. Figures 10 and 11 show the empirical and estimated reliability functions of ED, EGD and GD and the hazard rate functions for this data as well as the P-P plot.

Estimation of the variance covariance matrix for data set II given as the following:

$$I_0^{-1} = \begin{pmatrix} 7.142 \times 10^{-5} & -1.398 \times 10^{-3} & 2.644 \times 10^{-6} \\ -1.398 \times 10^{-3} & 3.323 \times 10^{-2} & -1.193 \times 10^{-4} \\ 2.644 \times 10^{-6} & -1.193 \times 10^{-4} & 1.058 \times 10^{-6} \end{pmatrix} \tag{46}$$

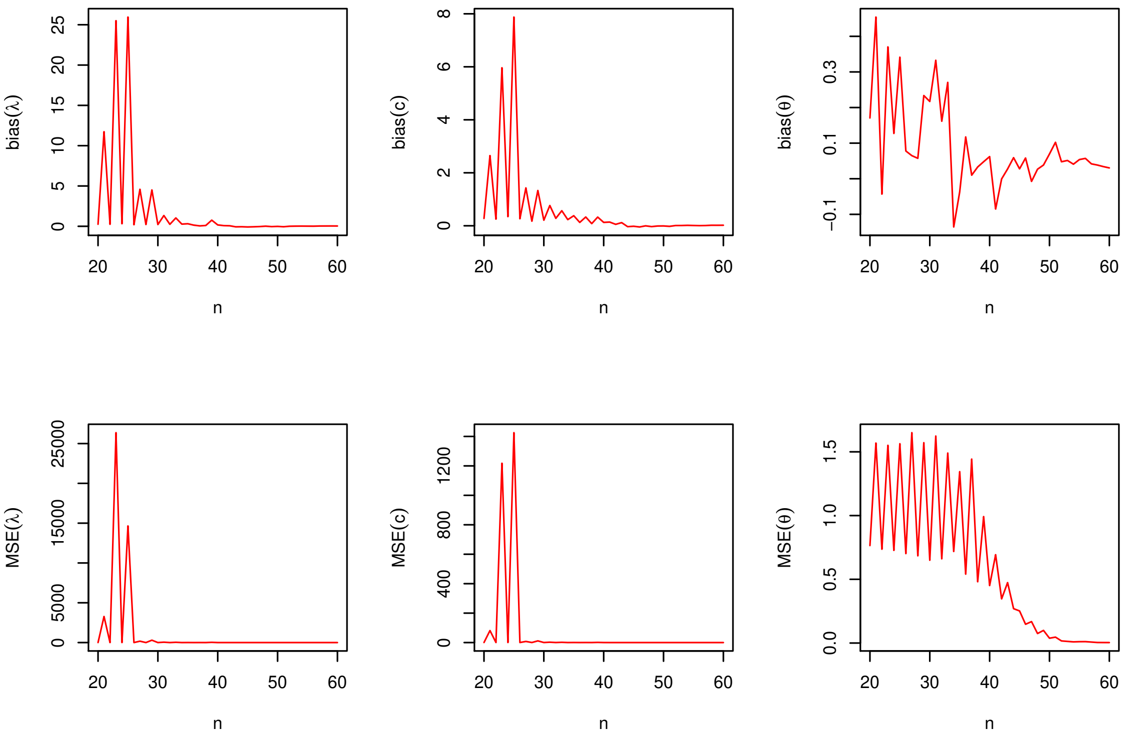

Fig. 5: The bias and MSE of ($\lambda = 0.7, c = 0.6, \theta = 1.1$).

The approximate 95% two sided CI of the parameters λ, c and θ are given respectively as

$$[3.241 \times 10^{-4}, 4.356 \times 10^{-3}],\ [2.136 \times 10^{-3}, 3.526 \times 10^{-2}],\ [0.1413, 0.8559]. \tag{47}$$

The profiles of the log-likelihood function of λ, c and θ for data set II are plotted in Figure 12.

7.2 Type II right censored sample

The following censored data have been obtained from McCool [51] which obtains the fatigue life of 10 bearing of a certain type in hours.

$$152.7, 172.0, 172.5, 173.3, 193.0, 204.7, 216.5, 234.9, 262.6, 422.6. \tag{48}$$

In this case, if we assume type II right censored sample of size $k = 8$, we obtain the MLE(s) of the unknown parameter(s), K-S and p-value for three different models are given in following Table. We find that the EGD provides a good fit for this data because it has the smallest value of K-S.

Table 5. The MLE(s) of the parameter(s), L, (K-S) value and p-value. (49)

The model	**MLE(s)**	**-L**	**K-S**	**p-value**
$ED(\lambda)$	$\overset{\wedge}{\lambda} = 4.021 \times 10^{-3}$	52.129	0.459	0.0176
$GD(\lambda, c)$	$\overset{\wedge}{\lambda} = 4.169 \times 10^{-3}, \hat{c} = 5.433 \times 10^{-5}$	52.098	0.472	0.0134
$EGD(\lambda, c, \theta)$	$\overset{\wedge}{\lambda} = 4.634 \times 10^{-4}, \hat{c} = 3.903 \times 10^{-3}, \overset{\wedge}{\theta} = 4.221$	30.541	0.3749	0.0851

Depending on p-value, we perform the following testing hypotheses:

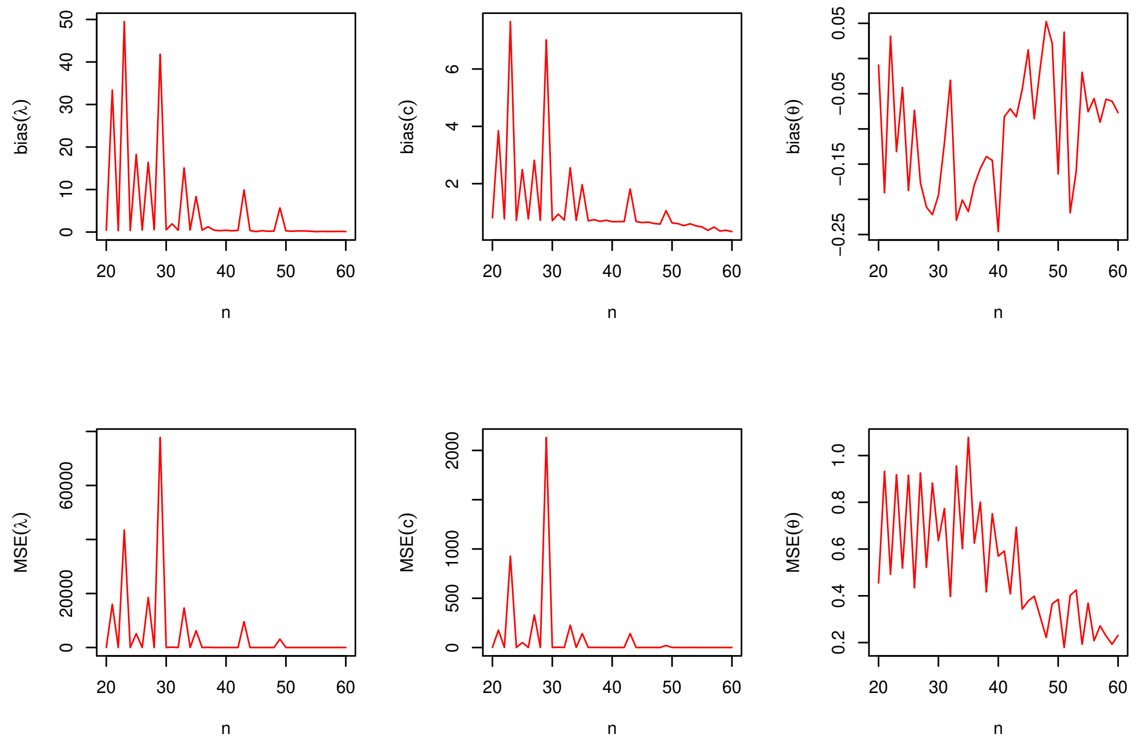

Fig. 6: The bias and MSE of $(\lambda = 0.7, c = 0.6, \theta = 1.7)$.

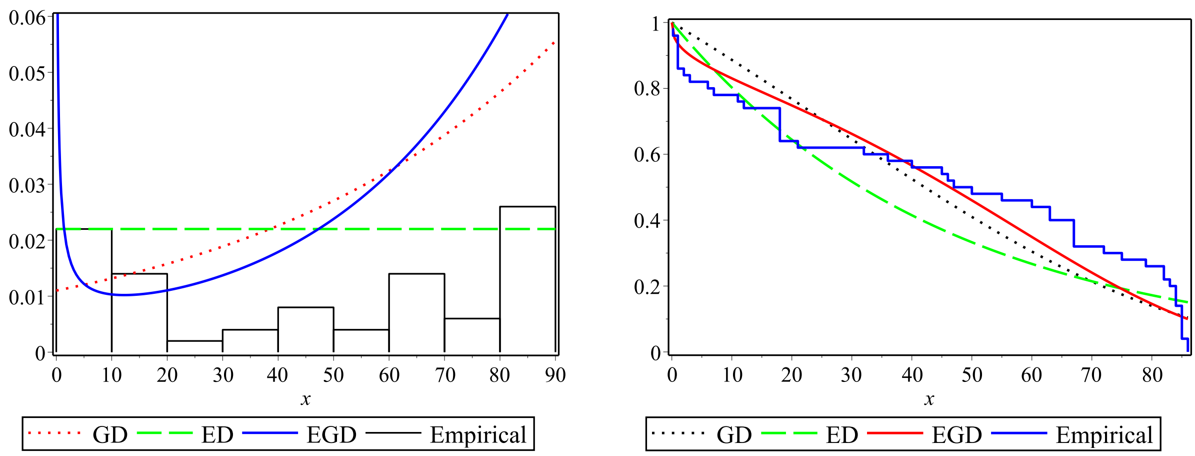

Fig. 7: The hazard rate functions (left panel) and the estimated reliability functions (right panel) of ED, EGD and GD for data set I.

1. H_0 : The sample of $x_1, x_2, ..., x_n \sim$ **ED.**
2. H_0 : The sample of $x_1, x_2, ..., x_n \sim$ **GD.**
3. H_0 : The sample of $x_1, x_2, ..., x_n \sim$ **EGD.**

So, it is clear that we reject the hypotheses 1 and 2 when the level of significance $\alpha = 0.05$ and keep the hypothese 3. Figure 13 shows the empirical and estimated reliability functions of ED, EGD and GD for censored data.

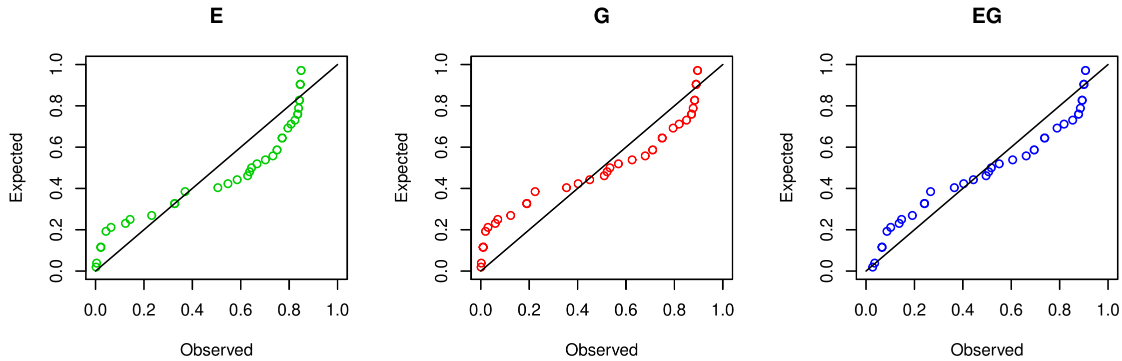

Fig. 8: The P-P plots for data set I.

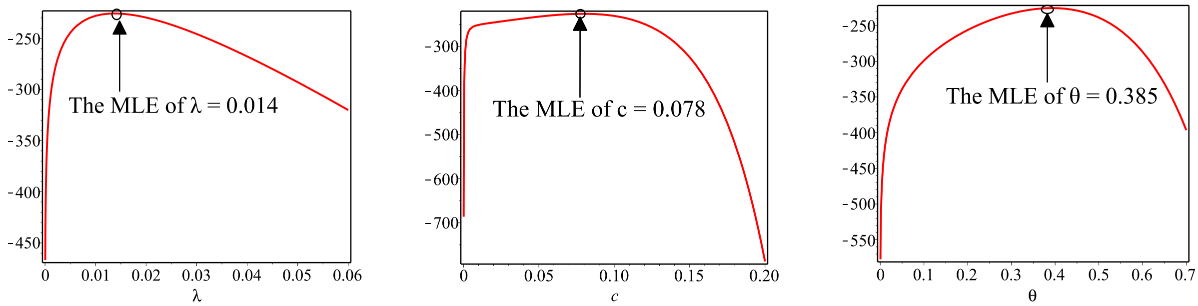

Fig. 9: The profile of the log-likelihood functions of λ, c and θ for data set I.

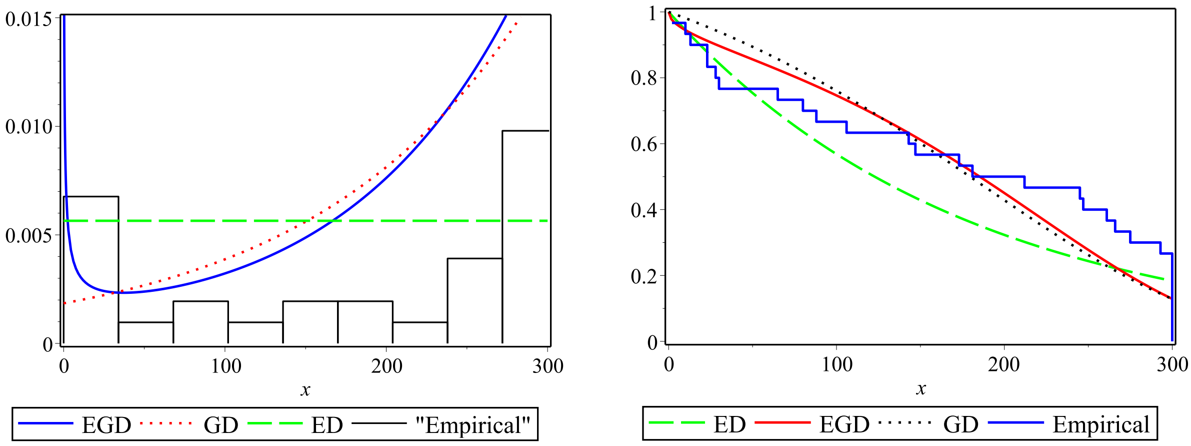

Fig. 10: The hazard rate functions (left panel) and the estimated reliability functions (right panel) of ED, EGD and GD for data set II.

8 Conclusions

A new flexible model alternative to exponential and Gompertz distributions has been proposed. Some of its statistical properties have been discussed in detail. The model parameters have been estimated by utilizing the maximum likelihood approach based on complete and Type II right censored samples. A simulation has been performed to assess the

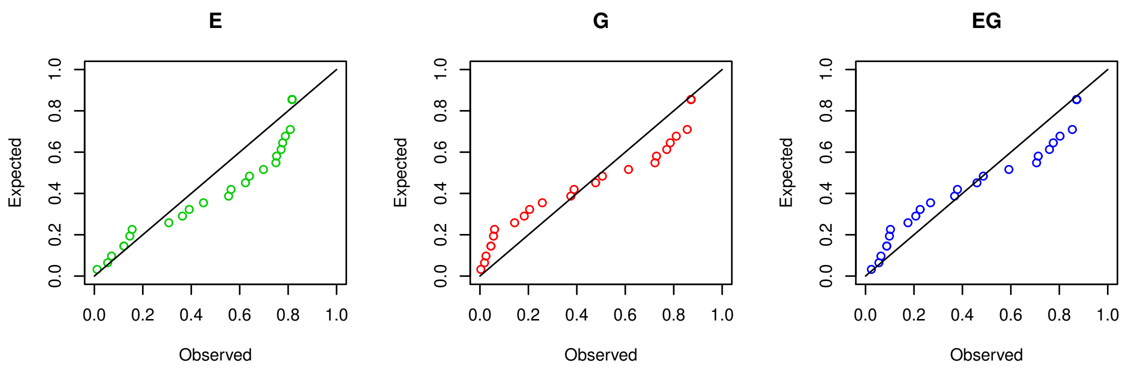

Fig. 11: The P-P plots for data set II.

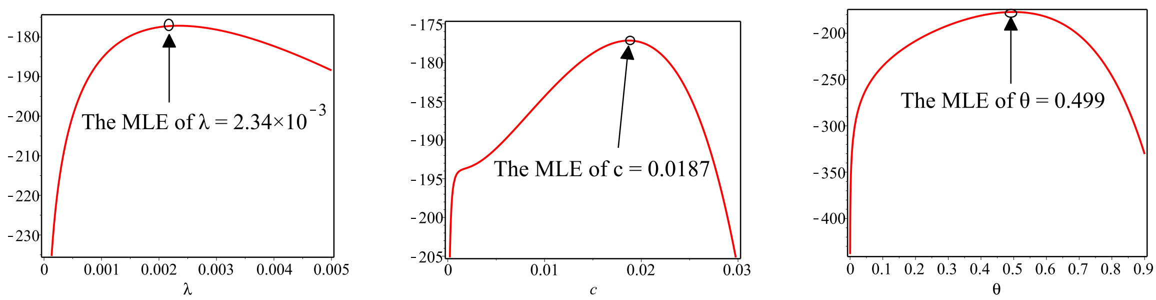

Fig. 12: The profile of the log-likelihood functions of λ, c and θ for data set II.

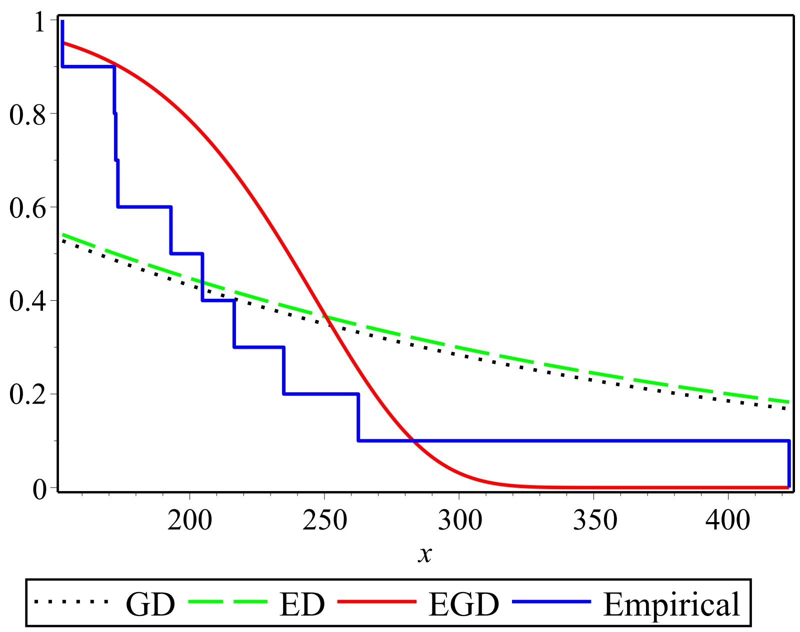

Fig. 13: The empirical and estimated reliability functions of ED, EGD and GD for Mc Cool data.

performance of estimators in terms of bias and mean square error. Finally, three real data sets have been analyzed to illustrate the flexibility of the proposed model.

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