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Analysis of a Priority Controllable Queue $M^{X_1}, M^{X_2}/G_1(a,b), G_2(a,b)/1$ With Multiple Vacations, Setup Times and Closedown Times

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Abstract: In this paper a priority queuing system $M^{X_1}, M^{X_2}/G_1(a,b), G_2(a,b)/1$ with multiple vacations, setup times with N-policy and closedown times has been deduced. Two types of customers, priority and nonpriority, are arrived and are served in this queuing situation. In priority schemes, customers with priority are selected for service ahead of those with nonpriority, independent of their time of arrival into a system, but with no preemption. On completion of service, if each of the number of priority customers ξ_1 and the number of nonpriority customers ξ_2 in the queue is less than "a" the server performs closedown work. Following closedown, the server leaves for multiple vacations of random length. When the server returns from a vacation and finds the number of customers of either type in the queue is less than "N", he leaves for another vacation and so on, until he finds at least "N" ($N \geqslant b$) customers of either type in the queue waiting for service. Then, he requires a setup time "R" to start service. After the setup he starts the service with a batch of "b" from the "N" priority customers, where $b \geq a$. After service, if the number of waiting priority customers $\xi_1 \geq a$, then the server serves a batch of min (ξ_1, b) customers of that type and so on until $\xi_1 < a$, then the server serves non-priority customers in the same way. The probability generating function of the queue size distribution at an arbitrary epoch and various characteristics of the queuing model are derived.

Keywords: Priority queue; N-policy; Closedown time; Setup time and Multiple vacations.

1 Introduction

Server vacation models are useful for the systems in which the server wants to utilize the idle time for different purposes. Application of server vacation models can be found in manufacturing systems, designing of local area networks and data communication systems. This paper concentrates on a vacation system with closedown times and setup times with N-policy. In practical situations, the closedown time corresponds to the time taken for closing down the service and setup time corresponds to the preparation time for starting the service.

The objective of this paper is to analyses a situation that exists in a pump manufacturing industry. A pump manufacturing industry manufactures two types of pumps, priority and nonpriority, which require shafts of various dimensions.

The partially finished pump shafts arrive at the copy turning centre from the turning centre. The operator starts the copy turning process only if required batch quantity of beams of either type is available, because the operating cost may increase otherwise. After processing, if the number of available shafts of either kind is less than the minimum batch quantity, then the operator will start doing other work such as making the templates for copy turning, checking the components. Hence, the operator always shuts down the machine and removes the templates before taking up other works.

When the operator returns from other work and finds that the shafts available from either type are more than the maximum batch quantity, the operator resumes the copy turning process, for which some amount of time is required to set up the template in the machine.

Otherwise the operator will continue with other work until he finds the required number of shafts. The above process can be modeled as M^{X_1}, M^{X_2}/ $G_1(a,b)$, $G_2(a,b)$/ 1 queuing system with multiple vacation, setup times with N-policy

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and closedown times. Various authors have analyzed queuing problems of server vacations with several combinations. A literature survey on queuing systems with server vacation can be found in Doshi [1]. Lee [2] has developed a procedure to calculate the system size probabilities for a bulk queuing model.

Several authors have analyzed the N-policy on queuing systems with vacation. Kella [3] provided detail discussions concerning N-policy queuing systems with vacations. Lee et al. [4,5] have analyzed a batch arrival queue with N-policy, but considered single service with a single vacation and multiple vacations.

Chae and Lee [6], studied a $M^X/G/1$ vacation model with N-policy and discussed the heuristic interpretation of mean waiting time. Reddy et al. [7] have analyzed a bulk queuing model and multiple vacations with setup time. They derived the expected number of customers in the queue at an arbitrary time epoch and obtained other measures. Recently, Ke [8] has analyzed the optimal policy for $M/G/1$ queuing systems of different vacation types with setup time. However, only very few have researched queuing systems with closedown time.

A $M/G/1$ queue is analyzed by Takagi [9], considering closedown time and setup time. The performance measures are also obtained. A $M^X/G(a,b)/1$ queue with multiple vacations including closedown time has been studied by Arumuganathan and Jeyakumar [10]. It is observed that most of the studies on vacation queues concentrated only either on a single server or an available arrival with single vacation.

Once the arrival occurs in bulk one can expect that the service can also be done in size. In practice, the a server may require some amount of time for closing the service after the service is completed and some amount of time for setup before the commencement of service.

By introducing N-policy in the system, the machine will continue to work for quite a long period so that it need not be shut down often.

In this paper, we consider a priority queuing system M^{X_1}, $M^{X_2}/\ G_1(a,b),\ G_2(a,b)/\ 1$ with multiple vacations, closedown time and setup time with N-policy. For this system, it is assumed that two types of customers, to be called priority and nonpriority customers arrive in batches according to compound Poisson processes with group arrival rates λ_1 for priority customers and λ_2 for nonpriority customers. The service discipline is FCFS within each type, but priority customers are always selected for service ahead of nonpriority customers, independent of their time of arrival into the system.

However, if priority customers arrive to find nonpriority customers in service, they can not preempt the nonpriority customers who are undergoing service and their service begin only on the completion of the use of the nonpriority customers.

That is, on completion of service, if each of the number of priority customers ξ_1 and the number of nonpriority customers ξ_2 in the queue is less than a, then the server performs closedown work.

Following closedown, the server leaves for multiple vacations of random length. When the server returns from a vacation and finds ξ_1 ,ξ_2 $< N$ (N is the threshold), then the server leaves for another vacation and so on, until he finds ξ_1 or ξ_2 $\geq N$ ($N \geq b$), then the server requires a setup time "R" to start the service.

After the setup, the server serves a batch of "b" from the "N." priority customers, where $b \geq a$. After a service completion of priority customers, if the number of waiting for priority customers in the queue $\xi_1 \geq a$, then the server serves a batch of min(ξ_1 ,b) priority customers, and so on until the number of priority customers becomes less than a, then the server starts to serve nonpriority customers in the same way.

The graph showing the sample path of the proposed queuing model is depicted in Fig. 1.

2 Notations and Definitions

The probabilities of the number of customers in the queue and service are defined as follows:

$$P_{i,m,n,1}(x,t) = P\{N_s(t) = i,\ N_1(t) = m,\ N_2(t) = n,\quad \text{priority customers are in service},\ x < S^0(t) \leq x+dt, Y(t) = 0\},\quad a \leq i \leq b,\ m,n \geq 0.$$

$$P_{i,m,n,2}(x,t) = P\{N_s(t) = i,\ N_1(t) = m,\ N_2(t) = n,\quad \text{nonpriority customers are in service},\ x < S^0(t) \leq x+dt,\ Y(t) = 0\}, a \leq i \leq b,\ m,n \geq 0.$$

$$Q_{j,m,n}(x,t) = P\{\ N_1(t) = m,\ N_2(t) = n,\ x < V^0(x) \leq x+dt,\ Y(t) = 2,\ Z(t) = j\},\qquad m,n \geq 0,\ \ j \geq 1.$$

$$C_{m,n}(x,t) = P\{\ N_1(t) = m,\ N_2(t) = n,\ x < C^0(x) \leq x+dt,\ Y(t) = 1\},\qquad m,n \geq 0.$$

$$R_{m,n}(x,t) = P\{N_1(t) = m, N_2(t) = n, x < R^0(x) \leq x+dt, Y(t) = 1\},\qquad m \geq N,\ n \geq 0\ or\ m \geq 0,\ n \geq N.$$

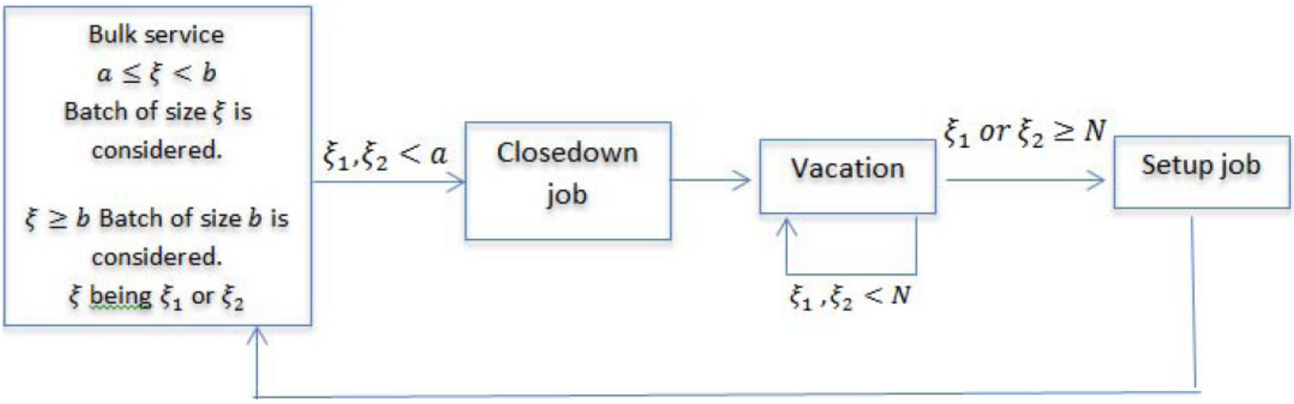

Fig. 1: Schematic representation of the queuing model

N	threshold
λ_1,λ_2	group arrival rates for priority and nonpriority customers, respectively.
X_1,X_2	arrival group size random variables for priority and nonpriority customers, respectively.
$g_i(k)$	probability that k customers arrive in a batch, i=1,2.
$X(y),X(z)$	$p.g.f'sofX_1,X_2$ respectively.
$S(.),V(.),R(.),C(.)$	c.d.f's of service time, vacation time, setup time and closedown time, respectively.
$s(x),v(x),r(x),c(x)$	p.d.f's of the service time, vacation time, setup time and closed own time, respectively.
$S^*(\theta),V^*(\theta),R^*(\theta),C^*(\theta)$	laplace transforms of s(x), v(x), r(x) and c(x), respectively.
$S^\circ(t),V^\circ(t),R^\circ(t),C^\circ(t)$	remaining service time, remaining vacation time, remaining setup time and remaining closedown time at time t, respectively.
$Y(t)=0$	if the server is busy at time t.
$Y(t)=1$	if the server is on closedown job or on setup job at time t.
$Y(t)=2$	if the server is on vacation at time t.
$Z(t)=j$	if the server is on jth vacation at time t.
$N_s(t)$	number of customers in service at time t.
$N_1(t),N_2(t)$	number of priority customers, number of nonpriority customers in the queue at time t, respectively.

3 Analysis of Queue Size Distribution

The steady state queue size equations are obtained as

$$-P'_{i,0,0,1}(x) = -(\lambda_1+\lambda_2)P_{i,0,0,1}(x) + \sum_{l=a}^{b}[P_{l,i,0,1}(0)+P_{l,i,0,2}(0)]s(x),\ a\leq i\leq b. \tag{1}$$

$$-P'_{i,0,0,2}(x) = -(\lambda_1+\lambda_2)P_{i,0,0,2}(x) + \sum_{l=a}^{b}[P_{l,0,i,1}(0)+P_{l,0,i,2}(0)]s(x),\ a\leq i\leq b. \tag{2}$$

$$-P'_{i,m,0,1}(x) = -(\lambda_1+\lambda_2)P_{i,m,0,1}(x) + \sum_{k=1}^{m}P_{i,m-k,0,1}(x)\lambda_1 g_1(k),\ a\leq i\leq b-1,\ m\geq 1. \tag{3}$$

$$-P'_{i,0,n,2}(x) = -(\lambda_1+\lambda_2)P_{i,0,n,2}(x) + \sum_{k=1}^{n}P_{i,0,n-k,2}(x)\lambda_2 g_2(k),\ a\leq i\leq b-1,\ n\geq 1. \tag{4}$$

$$-P'_{b,m,0,1}(x) = -(\lambda_1+\lambda_2)P_{b,m,0,1}(x) + \sum_{l=a}^{b}[P_{l,m+b,0,1}(0)+P_{l,m+b,0,2}(0)]s(x) + \sum_{k=1}^{m}P_{b,m-k,0,1}(x)\lambda_1 g_1(k),\quad 1\leq m\leq N-b-1. \tag{5}$$

$$-P'_{b,0,n,2}(x) = -(\lambda_1+\lambda_2)P_{b,0,n,2}(x) + \sum_{l=a}^{b}[P_{l,0,n+b,1}(0)+P_{l,0,n+b,2}(0)]s(x) + \sum_{k=1}^{n}P_{b,0,n-k,2}(x)\lambda_2 g_2(k),\quad 1\leq n\leq N-b-1. \tag{6}$$

$$-P'_{b,m,0,1}(x) = -(\lambda_1+\lambda_2)P_{b,m,0,1}(x) + \sum_{l=a}^{b}[P_{l,m+b,0,1}(0)+P_{l,m+b,0,2}(0)]s(x) + \sum_{k=1}^{m}P_{b,m-k,0,1}(x)\lambda_1 g_1(k) + R_{m+b,0}(0)s(x), \quad m \geq N-b. \tag{7}$$

$$-P'_{b,0,n,2}(x) = -(\lambda_1+\lambda_2)P_{b,0,n,2}(x) + \sum_{l=a}^{b}[P_{l,0,n+b,1}(0)+P_{l,0,n+b,2}(0)]s(x) + \sum_{k=1}^{n}P_{b,0,n-k,2}(x)\lambda_2 g_2(k) + R_{0,n+b}(0)s(x), \quad n \geq N-b. \tag{8}$$

$$-P'_{i,m,n,1}(x) = -(\lambda_1+\lambda_2)P_{i,m,n,1}(x) + \sum_{k=1}^{m}P_{i,m-k,n,1}(x)\lambda_1 g_1(k) + \sum_{k=1}^{n}P_{i,m,n-k,1}(x)\lambda_2 g_2(k), \quad 1 \leq i \leq b-1,\ m,n \geq 1. \tag{9}$$

$$-P'_{i,m,n,2}(x) = -(\lambda_1+\lambda_2)P_{i,m,n,2}(x) + \sum_{k=1}^{n}P_{i,m,n-k,2}(x)\lambda_2 g_2(k) + \sum_{k=1}^{m}P_{i,m-k,n,2}(x)\lambda_1 g_1(k), \quad a \leq i \leq b-1,\ m,n \geq 1. \tag{10}$$

$$-P'_{b,m,n,1}(x) = -(\lambda_1+\lambda_2)P_{b,m,n,1}(x) + \sum_{l=a}^{b}[P_{l,m+b,n,1}(0)+P_{l,m+b,n,2}(0)]s(x) + \sum_{k=1}^{m}P_{b,m-k,n,1}(x)\lambda_1 g_1(k) + \sum_{k=1}^{n}P_{b,m,n-k,1}(x)\lambda_2 g_2(k), \quad 1 \leq m \leq N-b-1,\ n \geq 1. \tag{11}$$

$$-P'_{b,m,n,2}(x) = -(\lambda_1+\lambda_2)P_{b,m,n,2}(x) + \sum_{l=a}^{b}[P_{l,m,n+b,1}(0)+P_{l,m,n+b,2}(0)]s(x) + \sum_{k=1}^{n}P_{b,m,n-k,2}(x)\lambda_2 g_2(k) + \sum_{k=1}^{m}P_{b,m-k,n,2}(x)\lambda_1 g_1(k), \quad m \geq 1, 1 \leq n \leq N-b-1. \tag{12}$$

$$\begin{aligned}-P'_{b,m,n,1}(x) =& -(\lambda_1+\lambda_2)P_{b,m,n,1}(x) + \sum_{l=a}^{b}[P_{l,m+b,n,1}(0)+P_{l,m+b,n,2}(0)]s(x) + \sum_{k=1}^{m}P_{b,m-k,n,1}(x)\lambda_1 g_1(k) + \\ &\sum_{k=1}^{n}P_{b,m,n-k,1}(x)\lambda_2 g_2(k) + R_{m+b,n}(0)s(x), \qquad m \geq N-b, n \geq 1.\end{aligned} \tag{13}$$

$$\begin{aligned}-P'_{b,m,n,2}(x) =& -(\lambda_1+\lambda_2)P_{b,m,n,2}(x) + \sum_{l=a}^{b}[P_{l,m,n+b,1}(0)+P_{l,m,n+b,2}(0)]s(x) + \sum_{k=1}^{n}P_{b,m,n-k,2}(x)\lambda_2 g_2(k) + \\ &\sum_{k=1}^{m}P_{b,m-k,n,2}(x)\lambda_1 g_1(k) + R_{m,n+b}(0)s(x), \qquad m \geq 1, n \geq N-b.\end{aligned} \tag{14}$$

$$-C'_{m,n}(x) = -(\lambda_1+\lambda_2)C_{m,n}(x) + \sum_{l=a}^{b}[P_{l,m,n,1}(0)+P_{l,m,n,2}(0)]c(x) + \sum_{k=1}^{m}C_{m-k,n}(x)\lambda_1 g_1(k) + \sum_{k=1}^{n}C_{m,n-k}(x)\lambda_2 g_2(k), \quad m,n < a. \tag{15}$$

$$-C'_{m,n}(x) = -(\lambda_1+\lambda_2)C_{m,n}(x) + \sum_{k=1}^{m-a}C_{m-k,n}(x)\lambda_1 g_1(k) + \sum_{k=1}^{n}C_{m,n-k}(x)\lambda_2 g_2(k), \qquad m \geq a, n < a. \tag{16}$$

$$-C'_{m,n}(x) = -(\lambda_1+\lambda_2)C_{m,n}(x) + \sum_{k=1}^{m}C_{m-k,n}(x)\lambda_1 g_1(k) + \sum_{k=1}^{n-a}C_{m,n-k}(x)\lambda_2 g_2(k), \quad m < a, n \geq a. \tag{17}$$

$$-C'_{m,n}(x) = -(\lambda_1+\lambda_2)C_{m,n}(x) + \sum_{k=1}^{m-a}C_{m-k,n}(x)\lambda_1 g_1(k) + \sum_{k=1}^{n-a}C_{m,n-k}(x)\lambda_2 g_2(k), \quad m,n \geq a. \tag{18}$$

$$-Q'_{1,0,0}(x) = -(\lambda_1+\lambda_2)Q_{1,0,0}(x) + C_{0,0}(0)v(x). \tag{19}$$

$$-Q'_{1,m,0}(x) = -(\lambda_1+\lambda_2)Q_{1,m,0}(x) + \sum_{k=1}^{m}Q_{1,m-k,0}(x)\lambda_1 g_1(k) + C_{m,0}(0)v(x), m \geq 1. \tag{20}$$

$$-Q'_{1,0,n}(x) = -(\lambda_1+\lambda_2)Q_{1,0,n}(x) + \sum_{k=1}^{n} Q_{1,0,n-k}(x)\lambda_2 g_2(k) + C_{0,n}(0)v(x), n \geq 1. \tag{21}$$

$$-Q'_{1,m,n}(x) = -(\lambda_1+\lambda_2)Q_{1,m,n}(x) + \sum_{k=1}^{m} Q_{1,m-k,n}(x)\lambda_1 g_1(k) + \sum_{k=1}^{n} Q_{1,m,n-k}(x)\lambda_2 g_2(k) + C_{m,n}(0)v(x), \quad m,n \geq 1. \tag{22}$$

$$-Q'_{j,0,0}(x) = -(\lambda_1+\lambda_2)Q_{j,0,0}(x) + Q_{j-1,0,0}(0)v(x), \qquad j \geq 2. \tag{23}$$

$$-Q'_{j,m,0}(x) = -(\lambda_1+\lambda_2)Q_{j,m,0}(x) + Q_{j-1,m,0}(0)v(x) + \sum_{k=1}^{m} Q_{j,m-k,0}(x)\lambda_1 g_1(k), \quad j \geq 2, m \geq 1. \tag{24}$$

$$-Q'_{j,0,n}(x) = -(\lambda_1+\lambda_2)Q_{j,0,n}(x) + Q_{j-1,0,n}(0)v(x) + \sum_{k=1}^{m} Q_{j,0,n-k}(x)\lambda_2 g_2(k), \quad j \geq 2, n \geq 1. \tag{25}$$

$$-Q'_{j,m,n}(x) = -(\lambda_1+\lambda_2)Q_{j,m,n}(x) + Q_{j-1,m,n}(0)v(x) + \sum_{k=1}^{m} Q_{j,m-k,n}(x)\lambda_1 g_1(k) + \sum_{k=1}^{n} Q_{j,m,n-k}(x)\lambda_2 g_2(k), \quad j \geq 2, m,n \geq 1. \tag{26}$$

$$-Q'_{j,m,n}(x) = -(\lambda_1+\lambda_2)Q_{j,m,n}(x) + \sum_{k=1}^{m} Q_{j,m-k,n}(x)\lambda_1 g_1(k) + \sum_{k=1}^{n} Q_{j,m,n-k}(x)\lambda_2 g_2(k), \quad j \geq 2, m \geq N, n \geq 1. \tag{27}$$

$$-Q'_{j,m,n}(x) = -(\lambda_1+\lambda_2)Q_{j,m,n}(x) + \sum_{k=1}^{m} Q_{j,m-k,n}(x)\lambda_1 g_1(k) + \sum_{k=1}^{n} Q_{j,m,n-k}(x)\lambda_2 g_2(k), \quad j \geq 2, m \geq 1, n \geq N. \tag{28}$$

$$-R'_{m,n}(x) = -(\lambda_1+\lambda_2)R_{m,n}(x) + \sum_{j=1}^{\infty} Q_{j,m,n}(0)r(x) + \sum_{k=1}^{m-N} R_{m-k,n}(x)\lambda_1 g_1(k) + \sum_{k=1}^{n} R_{m,n-k}(x)\lambda_2 g_2(k), \; m \geq N, n \geq 0. \tag{29}$$

$$-R'_{m,n}(x) = -(\lambda_1+\lambda_2)R_{m,n}(x) + \sum_{j=1}^{\infty} Q_{j,m,n}(0)r(x) + \sum_{k=1}^{m} R_{m-k,n}(x)\lambda_1 g_1(k) + \sum_{k=1}^{n-N} R_{m,n-k}(x)\lambda_2 g_2(k), \; m \geq 0, n \geq N. \tag{30}$$

In order to find the probability generating function (PGF) of queue size at an arbitrary time epoch, we define the following:

$$P_i^{\sim}(y,z,\theta) = P_{i,1}^{\sim}(y,z,\theta) + P_{i,2}^{\sim}(y,z,\theta),$$

where

$$P_{i,1}^{\sim}(y,z,\theta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_{i,m,n,1}^{\sim}(\theta)y^m z^n \quad \text{and} \quad P_{i,2}^{\sim}(y,z,\theta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_{i,m,n,2}^{\sim}(\theta)y^m z^n$$

$$P_i(y,z,0) = P_{i,1}(y,z,0) + P_{i,2}(y,z,0)$$

where

$$P_{i,1}(y,z,0) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_{i,m,n,1}(0)y^m z^n \quad \text{and} \quad P_{i,2}(y,z,0) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_{i,m,n,2}(0)y^m z^n,$$

$$C^{\sim}(y,z,\theta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_{m,n}^{\sim}(\theta)y^m z^n \qquad C(y,z,0) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_{m,n}(0)y^m z^n$$

$$Q_j^{\sim}(y,z,\theta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} Q_{j,m,n}^{\sim}(\theta)y^m z^n \qquad Q_j(y,z,0) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} Q_{j,m,n}(0)y^m z^n$$

$$\tilde{R}(y,z,\theta) = \tilde{R}_1(y,z\theta) + \tilde{R}_2(y,z\theta)$$

where

$$\tilde{R}_1(y,z,\theta) = \sum_{n=0}^{\infty}\sum_{m=N}^{\infty}\tilde{R}_{m,n}(\theta)y^m z^n \quad \text{and} \quad \tilde{R}_2(y,z,\theta) = \sum_{n=N}^{\infty}\sum_{m=0}^{\infty}\tilde{R}_{m,n}(\theta)y^m z^n$$
$$R(y,z,0) = R_1(y,z,0) + R_2(y,z,0) \tag{31}$$

where

$$R_1(y,z,0) = \sum_{n=0}^{\infty}\sum_{m=N}^{\infty}R_{m,n}(0)y^m z^n \quad \text{and} \quad R_2(y,z,0) = \sum_{n=N}^{\infty}\sum_{m=0}^{\infty}R_{m,n}(0)y^m z^n$$

Taking LT on both sides of Eqs. (1)-(14), we get

$$\theta P^{\sim}_{i,0,0,1}(\theta) - P_{i,0,0,1}(0) = (\lambda_1+\lambda_2)P^{\sim}_{i,0,0,1}(\theta) - \sum_{l=a}^{b}[P_{l,i,0,1}(0) + P_{l,i,0,2}(0)]s^{\sim}(\theta), a \le i \le b. \tag{32}$$

$$\theta P^{\sim}_{i,0,0,2}(\theta) - P_{i,0,0,2}(0) = (\lambda_1+\lambda_2)P^{\sim}_{i,0,0,2}(\theta) - \sum_{l=a}^{b}[P_{l,0,i,1}(0) + P_{l,0,i,2}(0)]s^{\sim}(\theta), a \le i \le b. \tag{33}$$

$$\theta P^{\sim}_{i,m,0,1}(\theta) - P_{i,m,0,1}(0) = (\lambda_1+\lambda_2)P^{\sim}_{i,m,0,1}(\theta) - \sum_{k=1}^{m}P^{\sim}_{i,m-k,0,1}(\theta)\lambda_1 g_1(k), a \le i \le b-1, m \ge 1. \tag{34}$$

$$\theta P^{\sim}_{i,0,n,2}(\theta) - P_{i,0,n,2}(0) = (\lambda_1+\lambda_2)P^{\sim}_{i,0,n,2}(\theta) - \sum_{k=1}^{n}P^{\sim}_{i,0,n-k,2}(\theta)\lambda_2 g_2(k), a \le i \le b-1, n \ge 1. \tag{35}$$

$$\begin{aligned}\theta P^{\sim}_{b,m,0,1}(\theta) - P_{b,m,0,1}(0) =& (\lambda_1+\lambda_2)P^{\sim}_{b,m,0,1}(\theta) - \sum_{l=a}^{b}[P_{l,m+b,0,1}(0) + P_{l,m+b,0,2}(0)]s^{\sim}(\theta) \\ &- \sum_{k=1}^{m}P^{\sim}_{b,m-k,0,1}(\theta)\lambda_1 g_1(k), \qquad 1 \le m \le N-b-1.\end{aligned} \tag{36}$$

$$\begin{aligned}\theta P^{\sim}_{b,0,n,2}(\theta) - P_{b,0,n,2}(0) =& (\lambda_1+\lambda_2)P^{\sim}_{b,0,n,2}(\theta) - \sum_{l=a}^{b}[P_{l,0,n+b,1}(0) + P_{l,0,n+b,2}(0)]s^{\sim}(\theta) \\ &- \sum_{k=1}^{n}P^{\sim}_{b,0,n-k,2}(\theta)\lambda_2 g_2(k), \qquad 1 \le n \le N-b-1.\end{aligned} \tag{37}$$

$$\begin{aligned}\theta P^{\sim}_{b,m,0,1}(\theta) - P_{b,m,0,1}(0) =& (\lambda_1+\lambda_2)P^{\sim}_{b,m,0,1}(\theta) - \sum_{l=a}^{b}[P_{l,m+b,0,1}(0) + P_{l,m+b,0,2}(0)]s^{\sim}(\theta) \\ &- \sum_{k=1}^{m}P^{\sim}_{b,m-k,0,1}(\theta)\lambda_1 g_1(k) - R_{m+b,0}(0)s^{\sim}(\theta), \qquad m \ge N-b.\end{aligned} \tag{38}$$

$$\begin{aligned}&\theta P^{\sim}_{b,0,n,2}(\theta) - P_{b,0,n,2}(0) = (\lambda_1+\lambda_2)P^{\sim}_{b,0,n,2}(\theta) - \sum_{l=a}^{b}[P_{l,0,n+b,1}(0) + P_{l,0,n+b,2}(0)]s^{\sim}(\theta) \\ &- \sum_{k=1}^{n}P^{\sim}_{b,0,n-k,2}(\theta)\lambda_2 g_2(k) - R_{0,n+b}(0)s^{\sim}(\theta), \qquad n \ge N-b.\end{aligned} \tag{39}$$

$$\begin{aligned}\theta P^{\sim}_{i,m,n,1}(\theta) - P_{i,m,n,1}(0) =& (\lambda_1+\lambda_2)P^{\sim}_{i,m,n,1}(\theta) - \sum_{k=1}^{m}P^{\sim}_{i,m-k,n,1}(\theta)\lambda_1 g_1(k) \\ &- \sum_{k=1}^{n}P^{\sim}_{i,m,n-k,1}(\theta)\lambda_2 g_2(k), \ 1 \le i \le b-1, m,n \ge 1.\end{aligned} \tag{40}$$

$$\begin{aligned}\theta P^{\sim}_{i,m,n,2}(\theta)-P_{i,m,n,2}(0)=&(\lambda_1+\lambda_2)P^{\sim}_{i,m,n,2}(\theta)-\sum_{k=1}^{m}P^{\sim}_{i,m-k,n,2}(\theta)\lambda_1 g_1(k)\\&-\sum_{k=1}^{n}P^{\sim}_{i,m,n-k,2}(\theta)\lambda_2 g_2(k),\ a\le i\le b-1, m,n\ge 1.\end{aligned}\tag{41}$$

$$\begin{aligned}&\theta P^{\sim}_{b,m,n,1}(\theta)-P_{b,m,n,1}(0)=(\lambda_1+\lambda_2)P^{\sim}_{b,m,n,1}(\theta)-\sum_{l=a}^{b}[P_{l,m+b,n,1}(0)+P_{l,m+b,n,2}(0)]s^{\sim}(\theta)\\&-\sum_{k=1}^{m}P^{\sim}_{b,m-k,n,1}(\theta)\lambda_1 g_1(k)-\sum_{k=1}^{n}P^{\sim}_{b,m,n-k,1}(\theta)\lambda_2 g_2(k),\\&\qquad 1\le m\le N-b-1, n\ge 1.\end{aligned}\tag{42}$$

$$\begin{aligned}&\theta P^{\sim}_{b,m,n,2}(\theta)-P_{b,m,n,2}(0)=\\&(\lambda_1+\lambda_2)P^{\sim}_{b,m,n,2}(\theta)-\sum_{l=a}^{b}[P_{l,m,n+b,1}(0)+P_{l,m,n+b,2}(0)]s^{\sim}(\theta)-\sum_{k=1}^{m}P^{\sim}_{b,m-k,n,2}(\theta)\lambda_1 g_1(k)\\&-\sum_{k=1}^{n}P^{\sim}_{b,m,n-k,2}(\theta)\lambda_2 g_2(k),\qquad 1\le n\le N-b-1, m\ge 1.\end{aligned}\tag{43}$$

$$\begin{aligned}&\theta P^{\sim}_{b,m,n,1}(\theta)-P_{b,m,n,1}(0)=\\&(\lambda_1+\lambda_2)P^{\sim}_{b,m,n,1}(\theta)-\sum_{l=a}^{b}[P_{l,m+b,n,1}(0)+P_{l,m+b,n,2}(0)]s^{\sim}(\theta)-\sum_{k=1}^{m}P^{\sim}_{b,m-k,n,1}(\theta)\lambda_1 g_1(k)\\&-\sum_{k=1}^{n}P^{\sim}_{b,m,n-k,1}(\theta)\lambda_2 g_2(k)-R_{m+b,n}(0)s^{\sim}(\theta),\qquad m\ge N-b, n\ge 1.\end{aligned}\tag{44}$$

$$\begin{aligned}&\theta P^{\sim}_{b,m,n,2}(\theta)-P_{b,m,n,2}(0)=\\&(\lambda_1+\lambda_2)P^{\sim}_{b,m,n,2}(\theta)-\sum_{l=a}^{b}[P_{l,m,n+b,1}(0)+P_{l,m,n+b,2}(0)]s^{\sim}(\theta)-\sum_{k=1}^{m}P^{\sim}_{b,m-k,n,2}(\theta)\lambda_1 g_1(k)\\&-\sum_{k=1}^{n}P^{\sim}_{b,m,n-k,2}(\theta)\lambda_2 g_2(k)-R_{m,n+b}(0)s^{\sim}(\theta),\qquad n\ge N-b, m\ge 1.\end{aligned}\tag{45}$$

Multiplying (32) by y^0z^0, (34) by y^mz^0 $(m\ge 1)$ and (40) by y^mz^n $(m,n\ge 1)$, summing up from $n=0$ to ∞ and $m=0$ to ∞ and using (31), we get

$$[\theta-(\lambda_1+\lambda_2)+\lambda_1X_1(y)+\lambda_2X_2(z)]P^{\sim}_{i,1}(y,z,\theta)=P_{i,1}(y,z,0)-s^{\sim}(\theta)\sum_{l=a}^{b}[P_{l,i,0,1}(0)+P_{l,i,0,2}(0)].\ \ a\le i\le b-1\tag{46}$$

Multiplying (32) by y^0z^0 $(i=b)$, (36) by y^mz^0 $(1\le m\le N-b-1)$, (38) by y^mz^0 $(m\ge N-b)$, (42) by y^mz^n $(1\le m\le N-b-1,\ n\ge 1)$ and (44) by y^mz^n $(m\ge N-b,\ n\ge 1)$, summing up from $n=0$ to ∞ and $m=0$ to ∞ and using 31, we obtain

$$\begin{aligned}&[\theta-(\lambda_1+\lambda_2)+\lambda_1X_1(y)+\lambda_2X_2(z)]P^{\sim}_{b,1}(y,z,\theta)=\\&P_{b,1}(y,z,0)-\frac{s^{\sim}(\theta)}{y^b}\{\sum_{l=a}^{b}[P_{l,1}(y,z,0)-\sum_{m=0}^{b-1}\sum_{n=0}^{\infty}P_{l,m,n,1}(0)y^mz^n+P_{l,2}(y,z,0)-\sum_{m=0}^{b-1}\sum_{n=0}^{\infty}P_{l,m,n,2}(0)y^mz^n]+R_1(y,z,0)\}.\end{aligned}\tag{47}$$

Multiplying (33) by y^0z^0 $(a\le i\le b)$, (35) by y^0z^n $(n\ge 1,\ a\le i\le b-1)$ and (41) by y^mz^n $(a\le i\le b-1,\ m,n\ge 1)$, summing up from $n=0$ to ∞ and $m=0$ to ∞ and using (31), we get

$$[\theta-(\lambda_1+\lambda_2)+\lambda_1X_1(y)+\lambda_2X_2(z)]P^{\sim}_{i,2}(y,z,\theta)=P_{i,2}(y,z,0)-s^{\sim}(\theta)\sum_{l=a}^{b}[P_{l,0,i,1}(0)+P_{l,o,i,2}(0)],\ a\le i\le b-1\tag{48}$$

Multiplying (33) by y^0z^0, (37) by y^0z^n $(1 \le n \le N-b-1)$, (39) by y^0z^n $(n \ge N-b)$, (43) by y^mz^n $(m \ge 1, 1 \le n \le N-b-1)$ and (45) by y^mz^n $(m \ge 1, n \ge N-b)$, summing up from $n=0$ to ∞ and $m=0$ to ∞ and using (31), we get

$$[\theta-(\lambda_1+\lambda_2)+\lambda_1X_1(y)+\lambda_2X_2(z)]P^{\sim}_{b,2}(y,z,\theta)= P_{b,2}(y,z,0)-\frac{s^{\sim}(\theta)}{z^b}[\sum_{l=a}^{b}[P_{l,1}(y,z,0)-\sum_{m=0}^{\infty}\sum_{n=0}^{b-1}P_{l,m,n,1}(0)y^mz^n+P_{l,2}(y,z,0)-\sum_{m=0}^{\infty}\sum_{n=0}^{b-1}P_{l,m,n,2}(0)y^mz^n]+R(y,z,0)] \tag{49}$$

Taking LT on both sides of Eqs. (15) - (18), we get

$$\theta C^{\sim}_{m,n}(\theta)-C_{m,n}(0)=(\lambda_1+\lambda_2)C^{\sim}_{m,n}(\theta)-\sum_{l=a}^{b}[P_{l,m,n,1}(0)+P_{l,m,n,2}(0)]C^{\sim}(\theta) -\sum_{k=1}^{m}C^{\sim}_{m-k,n}(\theta)\lambda_1g_1(k)-\sum_{k=1}^{n}C^{\sim}_{m,n-k}(\theta)\lambda_2g_2(k), \quad m,n<a. \tag{50}$$

$$\theta C^{\sim}_{m,n}(\theta)-C_{m,n}(0)=(\lambda_1+\lambda_2)C^{\sim}_{m,n}(\theta)-\sum_{k=1}^{m-a}C^{\sim}_{m-k,n}(\theta)\lambda_1g_1(k)-\sum_{k=1}^{n}C^{\sim}_{m,n-k}(\theta)\lambda_2g_2(k), \qquad m\ge a,n<a. \tag{51}$$

$$\theta C^{\sim}_{m,n}(\theta)-C_{m,n}(0)=(\lambda_1+\lambda_2)C^{\sim}_{m,n}(\theta)-\sum_{k=1}^{m}C^{\sim}_{m-k,n}(\theta)\lambda_1g_1(k)-\sum_{k=1}^{n-a}C^{\sim}_{m,n-k}(\theta)\lambda_2g_2(k), \qquad m<a,n\ge a. \tag{52}$$

Multiplying (50) by y^mz^n $(m,n<a)$, (51) by y^mz^n $(m\ge a,\ n<a)$, (52) by y^mz^n $(m<a,\ n\ge a)$, summing up from $n=0$ to ∞ and $m=0$ to ∞ and using (31), we get

$$[\theta-(\lambda_1+\lambda_2)+\lambda_1X_1(y)+\lambda_2X_2(z)]C^{\sim}(y,z,\theta)=C(y,z,0)-c^{\sim}(\theta)\sum_{l=a}^{b}\sum_{m=0}^{a-1}\sum_{n=0}^{a-1}y^mz^n[P_{l,m,n,1}(0)+P_{l,m,n,2}(0)]. \tag{53}$$

Taking LT on both sides of Eqs. (19)- (28), we get

$$\theta Q^{\sim}_{1,0,0}(\theta)-Q_{1,0,0}(0)=(\lambda_1+\lambda_2)Q^{\sim}_{1,0,0}(\theta)-C_{0,0}(0)v^{\sim}(\theta). \tag{54}$$

$$\theta Q^{\sim}_{1,m,0}(\theta)-Q_{1,m,0}(0)=(\lambda_1+\lambda_2)Q^{\sim}_{1,m,0}(\theta)-\lambda_1\sum_{k=1}^{m}Q^{\sim}_{1,m-k,0}(\theta)g_1(k)-C_{m,0}(0)v^{\sim}(\theta), \qquad m\ge 1. \tag{55}$$

$$\theta Q^{\sim}_{1,0,n}(\theta)-Q_{1,0,n}(0)=(\lambda_1+\lambda_2)Q^{\sim}_{1,0,n}(\theta)-\lambda_2\sum_{k=1}^{n}Q^{\sim}_{1,0,n-k}(\theta)g_2(k)-C_{0,n}(0)v^{\sim}(\theta), \quad n\ge 1. \tag{56}$$

$$\theta Q^{\sim}_{1,m,n}(\theta)-Q_{1,m,n}(0)=(\lambda_1+\lambda_2)Q^{\sim}_{1,m,n}(\theta)-\lambda_1\sum_{k=1}^{m}Q^{\sim}_{1,m-k,n}(\theta)g_1(k)-\lambda_2\sum_{k=1}^{n}Q^{\sim}_{1,m,n-k}(\theta)g_2(k)-C_{m,n}(0)v^{\sim}(\theta),\ m,n\ge 1. \tag{57}$$

$$\theta Q^{\sim}_{j,0,0}(\theta)-Q_{j,0,0}(0)=(\lambda_1+\lambda_2)Q^{\sim}_{j,0,0}(\theta)-Q_{j-1,0,0}(0)v^{\sim}(\theta), \quad j\ge 2. \tag{58}$$

$$\theta Q^{\sim}_{j,m,0}(\theta)-Q_{j,m,0}(0)=(\lambda_1+\lambda_2)Q^{\sim}_{j,m,0}(\theta)-Q_{j-1,m,0}(0)v^{\sim}(\theta)-\lambda_1\sum_{k=1}^{m}Q^{\sim}_{j,m-k,0}(\theta)g_1(k), \quad j\ge 2,m\ge 1. \tag{59}$$

$$\theta Q^{\sim}_{j,0,n}(\theta)-Q_{j,0,n}(0)=(\lambda_1+\lambda_2)Q^{\sim}_{j,0,n}(\theta)-Q_{j-1,0,n}(0)v^{\sim}(\theta)-\lambda_2\sum_{k=1}^{m}Q^{\sim}_{j,0,n-k}(\theta)g_2(k), \quad j\ge 2,n\ge 1. \tag{60}$$

$$\theta Q^{\sim}_{j,m,n}(\theta)-Q_{j,m,n}(0)=(\lambda_1+\lambda_2)Q^{\sim}_{j,m,n}(\theta)-Q_{j-1,m,n}(0)v^{\sim}(\theta)-\lambda_1\sum_{k=1}^{m}Q^{\sim}_{j,m-k,n}(\theta)g_1(k)-\lambda_2\sum_{k=1}^{n}Q^{\sim}_{j,m,n-k}(\theta)g_2(k), \quad j\ge 2,\ m,n\ge 1. \tag{61}$$

$$\theta Q^{\sim}_{j,m,n}(\theta)-Q_{j,m,n}(0)=(\lambda_1+\lambda_2)Q^{\sim}_{j,m,n}(\theta)-\lambda_1\sum_{k=1}^{m}Q^{\sim}_{j,m-k,n}(\theta)g_1(k)-\lambda_2\sum_{k=1}^{n}Q^{\sim}_{j,m,n-k}(\theta)g_2(k),\quad j\geq 2,\ m\geq N,n\geq 1. \tag{62}$$

$$\theta Q^{\sim}_{j,m,n}(\theta)-Q_{j,m,n}(0)=(\lambda_1+\lambda_2)Q^{\sim}_{j,m,n}(\theta)-\lambda_1\sum_{k=1}^{m}Q^{\sim}_{j,m-k,n}(\theta)g_1(k)-\lambda_2\sum_{k=1}^{n}Q^{\sim}_{j,m,n-k}(\theta)g_2(k),\quad j\geq 2,m\geq 1,\ n\geq N. \tag{63}$$

Multiplying (54) by y^0z^0, (55) by y^mz^0 $(m\geq 1)$, (56) by y^0z^n $(n\geq 1)$ and (57) by y^mz^n $(m,n\geq 1)$, summing up from $n=0$ to ∞ and $m=0$ to ∞ and using (31), we get

$$[\theta-(\lambda_1+\lambda_2)+\lambda_1X_1(y)+\lambda_2X_2(z)]Q^{\sim}_1(y,z,\theta)=Q_1(y,z,0)-C(y,z,0)v^{\sim}(\theta). \tag{64}$$

Multiplying (58) by $y^0z^0(j\geq 2)$, (59) by y^mz^0 $(j\geq 2,\ m\geq 1)$, (60) by y^0z^n $(j\geq 2,\ n\geq 1)$, (61) by y^mz^n $(j\geq 2,m,n\geq 1)$, (62) by y^mz^n $(j\geq 2,m\geq N,n\geq 1)$, and (63) by $y^mz^n(j\geq 2,m\geq 1,n\geq N)$, summing up from $n=0$ to ∞ and $m=0$ to ∞ and using (31), we get

$$[\theta-(\lambda_1+\lambda_2)+\lambda_1X_1(y)+\lambda_2X_2(z)]Q^{\sim}_j(y,z,\theta)=Q_j(y,z,0)-\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}Q_{j-1,m,n}(0)v^{\sim}(\theta)y^mz^n,\qquad j\geq 2. \tag{65}$$

Taking LT on both sides of Eqs. (29)-(30), we get

$$\theta R^{\sim}_{m,n}(\theta)-R_{m,n}(0)=(\lambda_1+\lambda_2)R^{\sim}_{m,n}(\theta)-\sum_{j=1}^{\infty}Q_{j,m,n}(0)r^{\sim}(\theta)-\sum_{k=1}^{m-N}R^{\sim}_{m-k,n}(\theta)\lambda_1g_1(k)-\sum_{k=1}^{n}R^{\sim}_{m,n-k}(\theta)\lambda_2g_2(k),\ m\geq N,n\geq 0. \tag{66}$$

$$\theta R^{\sim}_{m,n}(\theta)-R_{m,n}(0)=(\lambda_1+\lambda_2)R^{\sim}_{m,n}(\theta)-\sum_{j=1}^{\infty}Q_{j,m,n}(0)r^{\sim}(\theta)-\sum_{k=1}^{m}R^{\sim}_{m-k,n}(\theta)\lambda_1g_1(k)-\sum_{k=1}^{n-N}R^{\sim}_{m,n-k}(\theta)\lambda_2g_2(k),\ m\geq 0,n\geq N \tag{67}$$

Multiplying (66) by $y^mz^n(m\geq N,n\geq 0)$, summing up from $m=N$ to ∞ and $n=0$ to ∞ and using (31), we get

$$[\theta-(\lambda_1+\lambda_2)+\lambda_1X_1(y)+\lambda_2X_2(z)]R^{\sim}_1(y,z,\theta)=R_1(y,z,0)-R^{\sim}(\theta)\sum_{j=1}^{\infty}[Q_j(y,z,0)-\sum_{m=0}^{N-1}\sum_{n=0}^{\infty}Q_{j,m,n}(0)y^mz^n]. \tag{68}$$

Similarly, multiplying (67) by $y^mz^n(m\geq 0,n\geq N)$, summing if from $m=0$ to ∞ and $n=N$ to ∞ and using (**??**) to get

$$[\theta-(\lambda_1+\lambda_2)+\lambda_1x_1(y)+\lambda_2x_2(z)]\tilde{R}_2(y,z,\theta)=R_2(y,z,0)-\tilde{R}(\theta)\sum_{j=1}^{\infty}[Q_j(y,z,0)-\sum_{m=0}^{\infty}\sum_{n=0}^{N-1}Q_{j,m,n}(0)y^mz^z] \tag{69}$$

By substituting $\theta=(\lambda_1+\lambda_2)-(\lambda_1X_1(y)+\lambda_2X_2(z))$ in the (46), (47),(48), (49), (53), (64), (65), (68) and (69), we get

$$P_{i,1}(y,z,0)=s^{\sim}[(\lambda_1+\lambda_2)-(\lambda_1X_1(y)+\lambda_2X_2(z))]\sum_{l=a}^{b}[P_{l,i,0,1}(0)+P_{l,i,0,2}(0)],\qquad a\leq i\leq b-1. \tag{70}$$

$$\begin{aligned}P_{b,1}(y,z,0)=&\frac{s^{\sim}[(\lambda_1+\lambda_2)-(\lambda_1X_1(y)+\lambda_2X_2(z))]}{y^b}\left[\sum_{l=a}^{b}[P_{l,1}(y,z,0)-\sum_{m=0}^{b-1}\sum_{n=0}^{\infty}P_{l,m,n,1}(0)y^mz^n\right.\\&\left.+P_{l,2}(y,z,0)-\sum_{m=0}^{b-1}\sum_{n=0}^{\infty}P_{l,m,n,2}(0)y^mz^n]+R_1(y,z,0)\right].\end{aligned} \tag{71}$$

$$P_{i,2}(y,z,0)=s^{\sim}[(\lambda_1+\lambda_2)-(\lambda_1X_1(y)+\lambda_2X_2(z))]\sum_{l=a}^{b}[P_{l,0,i,1}(0)+P_{l,o,i,2}(0)],\qquad a\leq i\leq b-1. \tag{72}$$

$$P_{b,2}(y,z,0) = \frac{s^{\sim}[(\lambda_1+\lambda_2)-(\lambda_1 X_1(y)+\lambda_2 X_2(z))]}{z^b}\left[\sum_{l=a}^{b}[P_{l,1}(y,z,0)-\sum_{m=0}^{\infty}\sum_{n=0}^{b-1}P_{l,m,n,1}(0)y^m z^n \right.$$
$$\left. +P_{l,2}(y,z,0)-\sum_{m=0}^{\infty}\sum_{n=0}^{b-1}P_{l,m,n,2}(0)y^m z^n]+R_2(y,z,0)\right]. \tag{73}$$

$$C(y,z,0) = c^{\sim}[(\lambda_1+\lambda_2)-(\lambda_1 X_1(y)+\lambda_2 X_2(z))]\sum_{m=0}^{a-1}\sum_{n=0}^{a-1}\sum_{l=a}^{b} y^m z^n [P_{l,m,n,1}(0)+P_{l,m,n,2}(0)]. \tag{74}$$

$$Q_1(y,z,0) = v^{\sim}[(\lambda_1+\lambda_2)-(\lambda_1 X_1(y)+\lambda_2 X_2(z))]C(y,z,0). \tag{75}$$

$$Q_j(y,z,0) = v^{\sim}[(\lambda_1+\lambda_2)-(\lambda_1 X_1(y)+\lambda_2 X_2(z))]\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}Q_{j-1,m,n}(0)y^m z^n, \quad j\geq 2. \tag{76}$$

$$R_1(y,z,0) = R^{\sim}[(\lambda_1+\lambda_2)-(\lambda_1 X_1(y)+\lambda_2 X_2(z))]\sum_{j=1}^{\infty}[Q_j(y,z,0)-\sum_{m=0}^{N-1}\sum_{n=0}^{\infty}Q_{j,m,n}(0)y^m z^n]. \tag{77}$$

$$R_2(y,z,0) = R^{\sim}[(\lambda_1+\lambda_2)-(\lambda_1 X_1(y)+\lambda_2 X_2(z))]\sum_{j=1}^{\infty}[Q_j(y,z,0)-\sum_{m=0}^{\infty}\sum_{n=0}^{N-1}Q_{j,m,n}(0)y^m z^n]. \tag{78}$$

Equation (71) can be written as

$$P_{b,1}(y,z,0) = \frac{s^{\sim}[A][f_1(y,z)]}{y^b - s^{\sim}[A]}. \tag{79}$$

where,

$$A = (\lambda_1+\lambda_2)-(\lambda_1 X_1(y)+\lambda_2 X_2(z)),$$

and

$$f_1(y,z) = \sum_{l=a}^{b-1}P_{l,1}(y,z,0)-\sum_{l=a}^{b}\sum_{m=0}^{b-1}\sum_{n=0}^{\infty}P_{l,m,n,1}(0)y^m z^n+\sum_{l=a}^{b}P_{l,2}(y,z,0)-\sum_{l=a}^{b}\sum_{m=0}^{b-1}\sum_{n=0}^{\infty}P_{l,m,n,2}(0)y^m z^n+R_1(y,z,0),$$

Similarly from Eqn. (73), we find

$$P_{b,2}(y,z,0) = \frac{s^{\sim}[A][f_2(y,z)]}{z^b - s^{\sim}[A]}. \tag{80}$$

where,

$$f_2(y,z) = \sum_{l=a}^{b}P_{l,1}(y,z,0)-\sum_{l=a}^{b}\sum_{m=0}^{\infty}\sum_{n=0}^{b-1}P_{l,m,n,1}(0)y^m z^n+\sum_{l=a}^{b-1}P_{l,2}(y,z,0)-\sum_{l=a}^{b}\sum_{m=0}^{\infty}\sum_{n=0}^{b-1}P_{l,m,n,2}(0)y^m z^n+R_2(y,z,0)$$

Now, using (75) in (64), we get

$$Q_1^{\sim}(y,z,\theta) = \frac{Q_1(y,z,0)-v^{\sim}(\theta)C(y,z,0)}{[\theta-(\lambda_1+\lambda_2)+\lambda_1 X_1(y)+\lambda_2 X_2(z)]} = \frac{v^{\sim}[A]C(y,z,0)-v^{\sim}(\theta)C(y,z,0)}{[\theta-A]} = \frac{[v^{\sim}[A]-v^{\sim}(\theta)]C(y,z,0)}{[\theta-A]}. \tag{81}$$

Substituting (76) in (65), we get

$$Q_j^{\sim}(y,z,\theta) = \frac{Q_j(y,z,0)-v^{\sim}(\theta)\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}Q_{j-1,m,n}(0)y^m z^n}{[\theta-A]} = \frac{v^{\sim}(A)\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}Q_{j-1,m,n}(0)y^m z^n - v^{\sim}(\theta)\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}Q_{j-1,m,n}(0)y^m z^n}{[\theta-A]}$$
$$= \frac{[v^{\sim}(A)-v^{\sim}(\theta)]\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}Q_{j-1,m,n}(0)y^m z^n}{[\theta-A]}, \quad j\geq 2. \tag{82}$$

Using (74) in (53), we get

$$C^{\sim}(y,z,\theta) = \frac{\{C(y,z,0) - c^{\sim}(\theta)\sum_{m=0}^{a-1}\sum_{n=0}^{a-1}\sum_{l=a}^{b} y^m z^n [P_{l,m,n,1}(0) + P_{l,m,n,2}(0)]\}}{[\theta - A]}$$
$$= \frac{\{C^{\sim}(A)\sum_{m=0}^{a-1}\sum_{n=0}^{a-1}\sum_{l=a}^{b} y^m z^n [P_{l,m,n,1}(0) + P_{l,m,n,2}(0)]}{[\theta - A]} - \frac{C^{\sim}(\theta)\sum_{m=0}^{a-1}\sum_{n=0}^{a-1}\sum_{l=a}^{b} y^m z^n [P_{l,m,n,1}(0) + P_{l,m,n,2}(0)]\}}{[\theta - A]} \tag{83}$$
$$= \frac{\{[C^{\sim}(A) - C^{\sim}(\theta)]\sum_{m=0}^{a-1}\sum_{n=0}^{a-1}\sum_{l=a}^{b} y^m z^n [P_{l,m,n,1}(0) + P_{l,m,n,2}(0)]\}}{[\theta - A]}$$

Substituting (77) in (68), we get

$$R_1^{\sim}(y,z,\theta) = \frac{R_1(y,z,0) - R^{\sim}(\theta)\sum_{j=1}^{\infty}[Q_j(y,z,0) - \sum_{m=0}^{N-1}\sum_{n=0}^{\infty} Q_{j,m,n}(0) y^m z^n]}{[\theta - A]}$$
$$= \frac{[R^{\sim}(A) - R^{\sim}(\theta)]\sum_{j=1}^{\infty}[Q_j(y,z,0) - \sum_{m=0}^{N-1}\sum_{n=0}^{\infty} Q_{j,m,n}(0) y^m z^n]}{[\theta - A]}. \tag{84}$$

Substituting (79) in (70), we get

$$R_2^{\sim}(y,z,\theta) = \frac{[R^{\sim}(A) - R^{\sim}(\theta)]\sum_{j=1}^{\infty}[Q_j(y,z,0) - \sum_{m=0}^{\infty}\sum_{n=0}^{N-1} Q_{j,m,n}(0) y^m z^n]}{[\theta - A]}. \tag{85}$$

Substituting (70) in (46), we get

$$P_{i,1}^{\sim}(y,z,\theta) = \frac{[s^{\sim}(A) - s^{\sim}(\theta)]\sum_{l=a}^{b}[P_{l,i,0,1}(0) + P_{l,i,0,2}(0)]}{[\theta - A]}, \quad a \le i \le b-1. \tag{86}$$

Substituting (72) in (48), we get

$$P_{i,2}^{\sim}(y,z,\theta) = \frac{[s^{\sim}(A) - s^{\sim}(\theta)]\sum_{l=a}^{b}[P_{l,0,i,1}(0) + P_{l,0,i,2}(0)]}{[\theta - A]}, \; a \le i \le b-1. \tag{87}$$

From (79) and (47), we have

$$P_{b,1}^{\sim}(y,z,\theta) = \frac{[s^{\sim}(A) - s^{\sim}(\theta)]f_1(y,z)}{[y^b - s^{\sim}(A)][\theta - A]}. \tag{88}$$

Where

$$f_1(y,z) = s^{\sim}(A)\sum_{i=a}^{b-1} p_{i1} - \sum_{m=0}^{b-1}\sum_{n=0}^{\infty} p_{m,n} y^m z^n + s^{\sim}(A)\sum_{i=a}^{b} p_{i2} + R^{\sim}(A)[v^{\sim}(A)c^{\sim}(A)\sum_{m=0}^{a-1}\sum_{n=0}^{a-1} p_{m,n} y^m z^n + \{v^{\sim}(A) - 1\}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} q_{m,n} y^m z^n].$$

From (80) and (49), we have

$$P_{b,2}^{\sim}(y,z,\theta) = \frac{[s^{\sim}(A) - s^{\sim}(\theta)]f_2(y,z)}{[z^b - s^{\sim}(A)][\theta - A]}. \tag{89}$$

Where

$$f_2(y,z) = s^{\sim}(A)\sum_{i=a}^{b} p_{i1} - \sum_{m=0}^{\infty}\sum_{n=0}^{b-1} p_{m,n} y^m z^n + s^{\sim}(A)\sum_{i=a}^{b-1} p_{i2} + R^{\sim}(A)[v^{\sim}(A)c^{\sim}(A)\sum_{m=0}^{a-1}\sum_{n=0}^{a-1} p_{m,n} y^m z^n + \{v^{\sim}(A) - 1\}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} q_{m,n} y^m z^n].$$

Let $P(y,z)$ be the PGF of the queue size at an arbitrary time epoch, then

$$P(y,z)=\sum_{i=a}^{b-1}P_{i,1}^{\sim}(y,z,0)+P_{b,1}^{\sim}(y,z,0)+\sum_{i=a}^{b-1}P_{i,2}^{\sim}(y,z,0)+P_{b,2}^{\sim}(y,z,0)+C^{\sim}(y,z,0)+\sum_{j=1}^{\infty}Q_{j}^{\sim}(y,z,0)+R^{\sim}(y,z,0). \quad (90)$$

Let

$$\left.\begin{aligned} p_{i1}&=\sum_{l=a}^{b}[P_{l,i,0,1}(0)+P_{l,i,0,2}(0)]\\ p_{i2}&=\sum_{l=a}^{b}[P_{l,0,i,1}(0)+P_{l,0,i,2}(0)]\\ q_{m,n}&=\sum_{j=1}^{\infty}Q_{j,m,n}(0)\end{aligned}\right\} \quad (91)$$

Using the Eqns. (81)- (89) in (90) with $\theta=0$, we get

$$\begin{aligned} P(y,z)=&\{\sum_{m=0}^{a-1}\sum_{n=0}^{a-1}y^m z^n p_{m,n}\{[y^b-s^{\sim}(A)][z^b-s^{\sim}(A)][1-R^{\sim}(A)v^{\sim}(A)c^{\sim}(A)]\\ &+[R^{\sim}(A)v^{\sim}(A)c^{\sim}(A)][1-s^{\sim}(A)][[z^b-s^{\sim}(A)]+[y^b-s^{\sim}(A)]]\}\\ &+R^{\sim}(A)[1-v^{\sim}(A)]\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}y^m z^n q_{m,n}\{[z^b-s^{\sim}(A)][y^b-s^{\sim}(A)]\\ &-[y^b-s^{\sim}(A)][1-s^{\sim}(A)]\}+y^b[z^b-s^{\sim}(A)][1-s^{\sim}(A)]\sum_{i=a}^{b-1}p_{i1}\\ &+z^b[y^b-s^{\sim}(A)][1-s^{\sim}(A)]\sum_{i=a}^{b-1}p_{i2}\\ &+[1-s^{\sim}(A)]\{[z^b-s^{\sim}(A)][s^{\sim}(A)\sum_{i=a}^{b}p_{i2}-\sum_{m=0}^{b-1}\sum_{n=0}^{\infty}y^m z^n p_{m,n}]\\ &+[y^b-s^{\sim}(A)][s^{\sim}(A)\sum_{i=a}^{b}p_{i1}-\sum_{m=0}^{\infty}\sum_{n=0}^{b-1}y^m z^n p_{m,n}]\}\}/A[y^b-s^{\sim}(A)][z^b-s^{\sim}(A)].\end{aligned} \quad (92)$$

Special case:

As a special case, it may by noted that when considering only one type of customers (priority ones), then the PGF obtained in (92) reduces to the following form

$$\begin{aligned} P(z)=&\{[S^{\sim}(\lambda-\lambda X(z))-1]\sum_{i=a}^{b-1}(z^b-z^i)p_i+(z^b-1)[R^{\sim}(\lambda-\lambda X(z))C^{\sim}(\lambda-\lambda X(z))V^{\sim}(\lambda-\lambda X(z))-1]\sum_{i=0}^{a-1}p_i z^i\\ &+(z^b-1)R^{\sim}(\lambda-\lambda X(z))[V^{\sim}(\lambda-\lambda X(z))-1]\sum_{n=0}^{N-1}q_n z^n\}/\{[-\lambda+\lambda X(z)][z^b-S^{\sim}(\lambda-\lambda X(z))]\}.\end{aligned}$$

4 Some Characteristics and Performance Measures for the Queuing System

4.1 Expected lengths of idle period

Let I be the idle period random variable, then the expected length of idle a period is given by

$$E(I)=E(C)\ +E(I_1)+E(R)$$

where, Define a random variable U as

I_1 is the idle period due to multiple vacation process.
$E(C)$ is the expected closedown time.
$E(R)$ is the expected setup time.

$U = 0$, if the server finds at least N customers of either type (priority or non-priority) after first vacation,
$= 1$, if the server finds less than N customers of either type (priority or non-priority) after first vacation.

Now,

$$E(I_1) = E(I_1/U = 0)P(U = 0) + E(I_1/U = 1)P(U = 1) = E(V)P(U = 0) + [E(V) + E(I_1)]P(U = 1).$$

Solving for $E(I_1)$, we get

$$E(I_1) = \frac{E(V)}{1 - P(U = 1)}. \tag{93}$$

Therefore the expected length of idle period $E(I_1)$ is obtained as

$$E(I) = E(C) + \frac{E(V)}{P(U = 0)} + E(R) \tag{94}$$

4.2 Expected lengths of busy period

Let B be the busy period random variable and let us define the random variable J as
$J = 0$, if the server finds less than "a" customers of both types (priority and non-priority) after first service,
$J = 1$, if the server finds at least "a" customers of either type (priority and non-priority) after the first service.
Now,

$$E(B) = E(B/J = 0)P(J = 0) + E(B/J = 1)P(J = 1) = E(S)P(J = 0) + [E(S) + E(B)]P(J = 1).$$

Solving for $E(B)$ in the above, we get the expected length of the busy period as

$$E(B) = \frac{E(S)}{\sum\limits_{m=0}^{a-1}\sum\limits_{n-0}^{a-1} p_{mn}}. \tag{95}$$

4.3 Expected queue length

The expected queue length $E(Q) = E(Q_1) + E(Q_2)$ at an arbitrary time epoch, where $E(Q_1)$ the expected number of priority customers in the queue is obtained by differentiating $P(y,z)$ with respect to y evaluated at $y = 1, z = 1$, $E(Q_2)$ the expected number of non-priority customers in the queue is obtained by differentiating $P(y,z)$ with respect to z evaluated at $y = 1, z = 1$, and by applying L'HÔpital's rule four times we get

$$E(Q_1) = \{f_{11}\sum_{i=a}^{b-1}[b(b-1) - i(i-1)]p_{i1} + f_{21}\sum_{i=a}^{b-1}(b-1)p_{i2} + f_{31}\sum_{m=0}^{a-1}\sum_{n=0}^{a-1}p_{m,n} + f_{41}[\sum_{m=0}^{b-1}\sum_{n=0}^{\infty}p_{m,n} + \sum_{m=0}^{N-1}\sum_{n=0}^{N-1}q_{m,n}]$$
$$+ f_{51}\sum_{m=0}^{a-1}\sum_{n=0}^{a-1}mp_{m,n} + f_{61}[\sum_{m=0}^{\infty}\sum_{n=0}^{b-1}mp_{m,n} + \sum_{m=0}^{N-1}\sum_{n=0}^{\infty}mq_{m,n}]\}/2[\lambda_1 E(X_1)(b - S_{11})]^2,$$

where,

$$\begin{aligned}
f_{11} &= \lambda_1 E(X_1)(b-S_{11})S_{11},\\
f_{21} &= \lambda_1 E(X_1)(b-S_{11})S_{21} - T_1 S_{11},\\
f_{31} &= \lambda_1 E(X_1)(b-S_{11})[b(b-1)(E(C)+E(R)) + b(R_{21}+C_{21}) + 2b[E(C)+E(R)+E(V)]] - T_1 b(C_{11}+R_{11}),\\
f_{41} &= \lambda_1 E(X_1)(b-S_{11})[2bE(V)E(R) + b(b-1)E(V) + bV_{21}] - T_1 bE(V),\\
f_{51} &= \lambda_1 E(X_1)(b-S_{11})[b(E(C)+E(R))],\\
f_{61} &= \lambda_1 E(X_1)(b-S_{11})bE(V),\\
S_{11} &= \lambda_1 E(X_1)E(S),\\
V_{11} &= \lambda_1 E(X_1)E(V),\\
R_{11} &= \lambda_1 E(X_1)E(R),\\
C_{11} &= \lambda_1 E(X_1)E(C),\\
S_{21} &= 4\lambda_1 E(S) + \lambda_1^2 E(X_1)^2 E(S^2),\\
V_{21} &= 4\lambda_1 E(V) + \lambda_1^2 E(X_1)^2 E(V^2),\\
R_{21} &= 4\lambda_1 E(R) + \lambda_1^2 E(X_1)^2 E(R^2),\\
C_{21} &= 4\lambda_1 E(C) + \lambda_1^2 E(X_1)^2 E(C^2),\\
X_{21} &= X_1'(1),\\
T_1 &= \lambda_1 E(X_1)[b(b-1) - S_{21}] + \lambda_1 X_{21}(b-S_{11}).
\end{aligned}$$

and

$$\begin{aligned}
E(Q_2) = \{ & f_{12}\sum_{i=a}^{b-1}[b(b-1)-i(i-1)]p_{i2} + f_{22}\sum_{i=a}^{b-1}(b-1)p_{i1} + f_{32}\sum_{m=0}^{a-1}\sum_{n=0}^{a-1}p_{m,n} + f_{42}[\sum_{m=0}^{\infty}\sum_{n=0}^{b-1}p_{m,n} + \sum_{m=0}^{N-1}\sum_{n=0}^{N-1}q_{m,n}]\\
&+ f_{52}\sum_{m=0}^{a-1}\sum_{n=0}^{a-1}np_{m,n} + f_{62}[\sum_{m=0}^{b-1}\sum_{n=0}^{\infty}np_{m,n} + \sum_{m=0}^{\infty}\sum_{n=0}^{N-1}nq_{m,n}]\}/2[\lambda_2 E(X_2)(b-S_{12})]^2,
\end{aligned}$$

where,

$$\begin{aligned}
f_{12} &= \lambda_2 E(X_2)(b-S_{12})S_{12},\\
f_{22} &= \lambda_2 E(X_2)(b-S_{12})S_{22} - T_2 S_{12},\\
f_{32} &= \lambda_2 E(X_2)(b-S_{12})[b(b-1)(E(C)+E(R)) + b(R_{12}+C_{12}) + 2b[E(C)+E(R)+E(V)]] - T_2 b(C_{12}+R_{12}),\\
f_{42} &= \lambda_2 E(X_2)(b-S_{12})[2bE(V)E(R) + b(b-1)E(V) + bV_{12}] - T_2 bE(V),\\
f_{52} &= \lambda_2 E(X_2)(b-S_{12})[b(E(C)+E(R))],\\
f_{62} &= \lambda_2 E(X_2)(b-S_{12})bE(V),\\
S_{12} &= \lambda_2 E(X_2)E(S),\\
V_{12} &= \lambda_2 E(X_2)E(V),\\
R_{12} &= \lambda_2 E(X_2)E(R),\\
C_{12} &= \lambda_2 E(X_2)E(C),\\
S_{22} &= 4\lambda_2 E(S) + \lambda_2^2 E(X_2)^2 E(S^2),\\
V_{12} &= 4\lambda_2 E(V) + \lambda_2^2 E(X_2)^2 E(V^2),\\
R_{12} &= 4\lambda_2 E(R) + \lambda_2^2 E(X_2)^2 E(R^2),\\
C_{12} &= 4\lambda_2 E(C) + \lambda_2^2 E(X_2)^2 E(C^2),\\
X_{12} &= X_2'(1),\\
T_2 &= \lambda_2 E(X_2)[b(b-1) - S_{22}] + \lambda_2 X_{12}(b-S_{12}).
\end{aligned}$$

then

$$E(Q) = E(Q_1) + E(Q_2). \tag{96}$$

Arumuganathana et al. [11], is considered a particular case from our results

5 Conclusion

A priority queuing system $M^{X_1}, M^{X_2}/G_1(a,b), G_2(a,b)/1$ with multiple vacations, setup times with N-policy and closedown times has been studied with two types of customers, priority customers and nonpriority customers.

The PGF of queue size at an arbitrary time epoch is obtained. Some characteristics and performance measures are also derived, such as (Expected length of idle period, Expected length of a busy period and Expected queue length). A special case of the model, when considering only one type of customers, is also presented.

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