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Extended One-Step Methods for Solving Delay-Differential Equations

F. Ibrahim^{1,2,*}, *A. A. Salama*^{3}, *A. Ouazzi*^{1} *and S. Turek*^{1}

[1] Institut für Angewandte Mathematik, LS III, TU Dortmund, Vogelpothsweg 87, 44227 Dortmund, Gemany
[2] Department of Mathematics, Faculty of Science, South Valley University, Qena 83523 , Egypt
[3] Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt

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Abstract: We discuss extended one-step methods of order three for the numerical solution of delay-differential equations. A convergence theorem and the numerical studies regarding the convergence factor of these methods are given. Also, we investigate the stability properties of these methods. The results of the theoretical studies are illustrated by numerical examples.

Keywords: Delay-differential equations, Stability, Convergence, Numerical solutions

1 Introduction

Delay differential equations (DDEs) have a wide range of applications in science and engineering. They arise in models where the rate of change of process is not only determined by its present state, but also by a certain past state. As for instance, the natural delay in the population responding to overcrowding. Many examples of DDEs from practice can be found in Driver [7]. We consider the following initial-value problem for DDEs,

$$y\prime(x) = f(x, y(x), y(\alpha(x))), \qquad a \le x \le b, \qquad (1)$$

$$y(x) = g(x), \qquad v \le x \le a. \qquad (2)$$

Here f, α and g denote given functions with $\alpha(x) \le x$ for $x \ge a$, the function α is usually called the delay or lag function and y is unknown solution for $x > a$. Also, the function f and the initial function g satisfy the following conditions:

H1: The Lipschitz condition holds:

$$|f(x, y_1, z_1) - f(x, y_2, z_2)| \le L_1|y_1 - y_2| + L_2|z_1 - z_2|, \qquad (3)$$

where L_1 and L_2 are Lipschitz constants.

H2: For any $y \in C^1[v, b]$, the mapping $x \longrightarrow f(x, y(x), y(\alpha(x)))$ is continuous.

Under the conditions **H1** and **H2**, the problem (1, 2) has a unique solution Driver (see [7]).

Many methods have been proposed for the numerical approximation of problem (1, 2). Oberle and Pesch [18] introduced a class of numerical methods for the treatment of DDEs based on the well-known Runge-Kutta-Fehlberg methods. The retarded argument is approximated by an appropriate Hermite interpolation. The same methods are used by Arndt [2] with a different step size control mechanism. Bellen and Zennaro [4] developed a class of numerical methods to approximate the solution of DDEs. These methods are based on implicit Runge-Kutta methods. Paul and Baker [19] used explicit Runge-Kutta method for the numerical solution of singular DDEs. Torelli and Vermiglio [20] considered continuous numerical methods for differential equations with several constant delays. These methods are based on continuous quadrature rules. Hayashi [10] used continuous Runge-Kutta methods for the numerical solutions of retarded and neutral DDEs. Engelborghs et al. [6] presented collocation methods for the computation of periodic solution of DDEs. Hu and Cahlon [12] considered the numerical solution of initial-value discrete-delay systems.

The most obvious of the above methods for solving problem (1) numerically are θ-methods given in the following form

$$y_{n+1} = y_n + h\left[(1-\theta) f_n + \theta f_{n+1}\right], \qquad n = 0, 1, \ldots, N-1. \qquad (4)$$

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[*] Corresponding author e-mail: Fatma.Ibrahim@math.tu-dortmund.de

where θ is a parameter set to be $0 \le \theta \le 1$, N is the number of nodes, h is the uniform step length and y_n is an approximation to the exact solution $y(x_n)$ at the mesh point $x_n = a + nh$. Furthermore,

$$f_n = f(x_n, y_n, y(\alpha(x_n))) \tag{5}$$

where $y_h(x) = g(x)$ for $x \le a$ and $y_h(x)$ with $x > a$ is defined by piecewise linear interpolation, i.e.

$$\begin{aligned} y^h(x) = {} & \frac{x_{k+1}-x}{h} y_k \\ & + \frac{x-x_k}{h} y_{k+1}, \quad \text{for} \quad x_k \le x \le x_{k+1};\ k = 0,1,\dots \end{aligned} \tag{6}$$

In general the θ-methods described by (4), (6) and (5) are of first order and they are at most of second order for θ set to equal 0.5. The stability of θ-methods has been considered with respect to the following linear DDEs

$$\begin{aligned} y'(x) &= \lambda y(x) + \mu y(x-\tau), & x \ge 0 \\ y(x) &= g(x), & -\tau \le x \le 0 \end{aligned} \tag{7}$$

where λ and μ are complex numbers and $\tau > 0$. It is known, see Al-Mutib [1], that under the following two conditions

1. $g(x)$ is continuous
2. P-stability $|\mu| < -Re(\lambda)$,

the solution $y(x)$ of linear DDEs (7) tends to zero as x tends to infinity. The adaptation of θ-methods has already been considered in the literature, Calvo and Grande [5], Liu and Spijker [16], In't Hout and Spijker [13], Guglielmi [8] and Van Den Heuvel [21] and [22].

Our work aims to extend the current θ-methods to be of third order. Moreover, as these methods depend on a free parameter, we determine the range of the free parameter which guarantees the stability of these methods. The paper is organized as follows: In the next Section, we derive our third order methods for solving DDEs. Section 3 is devoted to the investigation the stability of the methods and the determination of the stability regions. In Section 4, we determine the convergence factor of the present methods. The proof of convergence of the present methods is given in Section 5. Finally, in Section 6, we test numerically our extended methods and make numerical comparison with the θ-methods.

2 Extended one-step third-order methods

In this section, we extend the work of Usmani and Agarwal [23], Jacques [14] and Kondrat and Jacques [15] to derive the present methods. We start with the following discretization for the numerical solution of (1)

$$\begin{aligned} y_{n+1} = {} & y_n \\ & + h\left[\alpha_0 f_n + \alpha_1 f_{n+1} + \alpha_2 \hat{f}_{n+2}\right], \quad n = 0,1,\dots,N-1 \end{aligned} \tag{8}$$

where

$$\hat{f}_{n+2} = f(x_{n+2}, \hat{y}_{n+2}, y_h(\alpha(x_{n+2}))) \tag{9}$$

In order to determine the coefficients α_0, α_1 and α_2, we rewrite (8) in the exact form

$$\begin{aligned} y(x_{n+1}) = y(x_n) + h[& \alpha_0 f(x_n, y(x_n), y(\alpha(x_n))) \\ & + \alpha_1 f(x_{n+1}, y(x_{n+1}), y(\alpha(x_{n+1}))) \\ & + \alpha_2 f(x_{n+2}, y(x_{n+2}), y(\alpha(x_{n+2})))] \\ & + t(x_{n+1}). \end{aligned} \tag{10}$$

We expand the left and right sides of (10) in the Taylor series at the point x_{n+1}, equate the coefficients up to the third order terms $O(h^3)$ and solving the resulting system of equations, we obtain

$$\alpha_0 = \frac{5}{12}, \quad \alpha_1 = \frac{2}{3}, \quad \alpha_2 = -\frac{1}{12} \tag{11}$$

and

$$t(x_{n+1}) = \frac{h^4}{24} y^{(4)}(\xi) \tag{12}$$

where $x_n < \xi < x_{n+2}$. Substituting the alpha coefficients from (11) into (8), we obtain

$$y_{n+1} = y_n + \frac{h}{12}\left[5 f_n + 8 f_{n+1} - \hat{f}_{n+2}\right] \tag{13}$$

Here, and in the following

$$y^h(x) = g(x) \quad \text{for} \quad x \le a \tag{14}$$

and $y^h(x)$ with $x > a$ is defined by

$$\begin{aligned} y^h(x) = {} & \beta_0 y_k + \beta_1 y_{k+1} \\ & + h[\beta_2 f_k + \beta_3 f_{k+1}], \text{ for } x_k < x \le x_{k+1};\ k = 0,1,\dots \end{aligned} \tag{15}$$

In order to determine the coefficients β_0, β_1, β_2 and β_3, we rewrite (15) in the exact form

$$\begin{aligned} y(x) = {} & \beta_0 y(x_k) + \beta_1 y(x_{k+1}) \\ & + h[\beta_2 f(x_k, y(x_k), y(\alpha(x_k))) \\ & + \beta_3 f(x_{k+1}, y(x_{k+1}), y(\alpha(x_{k+1})))] + t(x_{k+1}) \end{aligned} \tag{16}$$

Similarly, we expand the left and right sides of (16) with Taylor series at point x_{k+1} and equate the coefficients up to the terms of second order $O(h^2)$. We obtain the resulting system of equations

$$\left\{ \begin{aligned} \beta_0 + \beta_1 &= 1 \\ \beta_0 - \beta_2 - \beta_3 &= -\delta(x) \\ \beta_0 - 2\beta_2 &= \delta^2(x) \end{aligned} \right. \tag{17}$$

where

$$\delta(x) = \frac{1}{h}(x - x_{k+1}) \tag{18}$$

The solution of the above system (17) is

$$\begin{cases} \beta_0 = 1-\beta_1 \\ \beta_2 = \dfrac{1}{2}(1-\beta_1-\delta^2(x)) \\ \beta_3 = \dfrac{1}{2}(\delta^2(x)+2\delta(x)-\beta_1+1) \end{cases} \tag{19}$$

and

$$t(x_{k+1}) = \frac{h^3}{12}(2\delta^3(x)+3\delta^2(x)+\beta_1-1)y^{(3)}(\eta) \tag{20}$$

where β_1 is a free parameter and $x_k < \eta < x_{k+1}$. Substituting from (19) into (15), we obtain

$$\begin{aligned} y^h(x) = {} & (1-\beta_1)y_k+\beta_1 y_{k+1} \\ & +\frac{h}{2}\left[(1-\beta_1-\delta^2(x))f_k\right. \\ & \left.+(\delta^2(x)+2\delta(x)-\beta_1+1)f_{k+1}\right], \\ & \qquad \text{for} \quad x_k < x \le x_{k+1};\ k=0,1,\ldots \end{aligned} \tag{21}$$

Finally, from (21), the approximation $\hat{y}_{n+2}$ is determined in the form

$$\hat{y}_{n+2} = (1-\beta_1)y_n+\beta_1 y_{n+1}-\frac{h}{2}\left[\beta_1 f_n+(\beta_1-4)f_{n+1}\right] \tag{22}$$

Equations (13), (21) and (22) are the basis of the present methods.

3 Stability definitions and results

The stability investigations are based on the linear equation (7) and the concept of P-stability introduced by Barwell [3]

Definition 1.1. (P-stability region) Given a numerical method for solving (7), the P-stability region of the method is the set S_P of the pairs (X,Y), $X=\lambda h$ and $Y=\mu h$, such that the numerical solution of (7) asymptotically vanishes for step-lengths h satisfying

$$h = \frac{\tau}{m} \tag{23}$$

with m is positive integer.

Definition 1.2. (P-stability) A numerical method for (7) is said to be P-stable if

$$S_P \supseteq R,$$

where

$$R = \{(X,Y) : Y < -X\}.$$

In order to solve the Problem (7), the present methods are written as follows

$$\begin{aligned} y_{n+1} = {} & \\ & [24+2\lambda h(4+\beta_1) \\ & +(\lambda h)^2\beta_1]\,y_n+\mu h\left[(10+\lambda h\beta_1)y(x_n-\tau)\right. \\ & \left.+(16-\lambda h(4-\beta_1))y(x_{n+1}-\tau)-2\mu h y(x_{n+2}-\tau)\right] \end{aligned} \tag{24}$$

with a constant step size h satisfying the constraint (23). The characteristic polynomial associated with (24) takes the form

$$\begin{aligned} W_m(z) = {} & \left[24-2X(8-\beta_1)+X^2(4-\beta_1)\right]z^{m+1} \\ & -\left[24+2X(4+\beta_1)+X^2\beta_1\right]z^m \\ & -Y\left[10+X\beta_1+(16-X(4-\beta_1))z-2z^2\right] \\ & =0, \quad m=1,2,\ldots \end{aligned} \tag{25}$$

It is clear that $(X,Y) \in S_P$ if and only if all roots of the polynomials W_m are inside the unit disc for $m=1,2,\ldots$. Let

$$\begin{aligned} P(z) := {} & \left[24-2X(8-\beta_1)+X^2(4-\beta_1)\right]z^{m+1} \\ & -\left[24+2X(4+\beta_1)+X^2\beta_1\right]z^m, \\ Q(z) := {} & -Y\left[10+X\beta_1+(16-X(4-\beta_1))z-2z^2\right] \end{aligned} \tag{26}$$

and z^* denotes the only nonzero root of $P(z)$. It follows from Rouche's theorem, see Marden [17], that $(X,Y) \in S_P$ if $[z^*] < 1$ and $|P(z)| > |Q(z)|$ on the unit circle. Furthermore, on the unit circle we have

$$\begin{aligned} |P(z)| \ge {} & \left|\left(|24-2X(8-\beta_1)+X^2(4-\beta_1)|\right)\right. \\ & \left.-\left(|24+2X(4+\beta_1)+X^2\beta_1|\right)\right|, \\ |Q(z)| \le {} & |Y|\left(|10+X\beta_1|\right. \\ & \left.+|16-X(4-\beta_1)|+2\right). \end{aligned} \tag{27}$$

Therefore, $(X,Y) \in S_P$ if the following set of inequalities are satisfied

$$\begin{aligned} & \left|\left(|24-2X(8-\beta_1)+X^2(4-\beta_1)|\right)\right. \\ & \left.-\left(|24+2X(4+\beta_1)+X^2\beta_1|\right)\right| \ge \\ & |Y|\left(|10+X\beta_1|+|16-X(4-\beta_1)|+2\right), \end{aligned} \tag{28}$$

and

$$\left|\frac{24+2X(4+\beta_1)+X^2\beta_1}{24-2X(8-\beta_1)+X^2(4-\beta_1)}\right| < 1. \tag{29}$$

It can be seen that $X \in S_A$ where S_A is the A-stability region of the present methods for solving ordinary differential equation if and only if (29) is satisfied, we refer to Hairer et al. [9] for more details concerning the A-stability concept. It is easy to see that (29) is satisfied if $\beta_1 \in (-\infty,2]$. Moreover, the P-stability region for various values of $\beta_1 \in (-\infty,2]$ is determined by solving the system of inequalities (28) and (29). Thus we establish the following.

Theorem 1. *For the present methods, the region of P-stability satisfies the relation*

$$S_P \cap R = \{(X,Y) : |Y| < -X \text{ and } |Y| < \phi(X)\}$$

where

$$\phi(X) = \begin{cases} \dfrac{X^2-6X}{7-X} & \text{for} \quad X \ge -3 \\ \dfrac{X^2-2X+12}{7-X} & \text{for} \quad X < -3 \end{cases}$$

for $\beta_1 = 0$.

Proof. The proof follows immediately from inequality (28).

Theorem 2. *For the present methods the region of P-stability satisfy the relation*

$$S_P \cap R = \{(X,Y) : Y < -X \, and \; |Y| < \phi(X)\},$$

where

$$\phi(X) = \begin{cases} \dfrac{(2-\beta_1)X^2 - 12X}{14-(2-\beta_1)X}, & if \quad X \geq \dfrac{-10}{\beta_1}, \\ \dfrac{(2-\beta_1)X^2 - 12X}{2(2-X)}, & if \quad X < \dfrac{-10}{\beta_1}, \end{cases}$$

for $\beta_1 \in (0,2]$.

Proof. The proof follows immediately from inequality (28).

Theorem 3. *For the present methods the region of P-stability satisfy the relation*

$$S_P \cap R = \{(x,y) : |Y| < -X \; and Y < \phi(X)\},$$

where

$$\phi(X) = \begin{cases} \dfrac{(\beta_1-2)X^2 - 12X}{(1-\beta_1)X - 14}, \; if X > -\left(\dfrac{\beta_1+4}{\beta_1}\right) \\ \qquad\qquad -\sqrt{\left(\dfrac{\beta_1+4}{\beta_1}\right)^2 - \dfrac{24}{\beta_1}}, \\ \dfrac{2X^2-(1-2\beta_1)X+24}{14-X(1-\beta_1)}, \; if X < -\left(\dfrac{\beta_1+4}{\beta_1}\right) \\ \qquad\qquad -\sqrt{\left(\dfrac{\beta_1+4}{\beta_1}\right)^2 - \dfrac{24}{\beta_1}}, \end{cases}$$

for $\beta_1 \in (-\infty, 0)$.

Proof. The proof follows immediately from inequality (28).

The **Fig. 1** shows the different regions of the *P*-stability with respect to different values of β_1.

4 Convergence factor for the present methods

In this section, we present the main result concerning the convergence factor of the methods (13), (21) and (22) with $\beta_1 = 0$. This case may be expressed in the form

$$y_{n+1}^{(j+1)} = y_n + \frac{h}{12}\Big[5f_n + 8f(x_{n+1}, y_{n+1}^{(j)}, y^{h(j)}(\alpha(x_{n+1}))) \\ - f(x_{n+2}, \hat{y}_{n+2}^{(j)}, y^{h(j)}(\alpha(x_{n+2})))\Big] \quad j = 1, \ldots$$

$$y^{h(j)}(x) = y_k + \frac{h}{2}\left[(1-\delta^2(x))f_k + (1+\delta(x))^2 f_{k+1}\right], \\ \text{for} \quad x_k < x \leq x_{k+1};\; k = 0,1,\ldots \tag{30}$$

where $y_{n+1}^{(0)}$ is an initial approximation to the solution y at x_{n+1} and $y_{n+1}^{(j)}$, $j \geq 1$ are Picard iterations.

Now, we state and prove the following theorem.

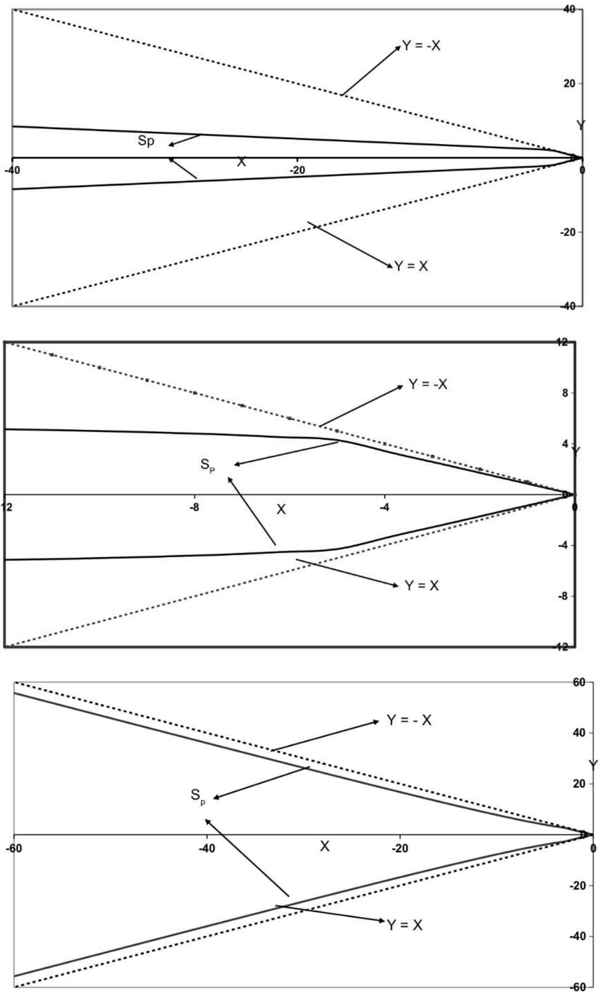

Fig. 1: The *P*-stability region for β_1 parameter set equal to $-10, 2$ and 0 (Top-Bottom).

Theorem 4. *If the sequence* $\{y_{n+1}^{(j)}\}$ *given by* (30) *is bounded by a constant C and the condition*

$$\begin{aligned} hL &< \frac{-2R_1 + 2\sqrt{R_1^2 + 6R_2}}{R_2} \\ R_1 &= 4 + 3r_1^2 \\ R_2 &= r_2^2 + 2r_2 + 5 \end{aligned} \tag{31}$$

is satisfied, where $r_1, r_2 \in (0,1]$ *and* $L = \max\{L_1, L_2\}$. *Then, the method* (30) *is convergent.*

Proof. From (21) with $\beta_1 = 0$, when $\alpha(x_{n+1}) \in (x_k, x_{k+1}]$, $k = 0,1,\ldots,n-1$, we have

$$y^h(\alpha(x_{n+1})) = y_k + \frac{h}{2}\left[1-\delta^2(\alpha(x_{n+1}))^2 f_k \right. \\ \left. + (1+\delta(\alpha(x_{n+1})))^2 f_{k+1}\right] \tag{32}$$

Moreover, if $\alpha(x_{n+1}) \in (x_n, x_{n+1}]$, we put

$$y^h(\alpha(x_{n+1})) = y_n + \frac{h}{2}\left[(1-\delta^2(\alpha(x_{n+1})))f_n \right. \\ \left. + (1+\delta(\alpha(x_{n+1})))^2 f_{n+1}\right] \tag{33}$$

and

$$y^{h(j)}(\alpha(x_{n+1})) = y_n + \frac{h}{2}\left[(1-\delta^2(\alpha(x_{n+1})))f_n \right. \\ \left. +(1+\delta(\alpha(x_{n+1})))^2 f(x_{n+1}, y^{(j)}_{n+1}, y^{h(j)}(\alpha(x_{n+1})))\right]. \tag{34}$$

Since $\alpha(x_{n+1}) - x_n \le h$, we let $\alpha(x_{n+1}) - x_n = r_1 h$ with $r_1 \in (0,1]$. Then, from (33) and (34) we obtain

$$\left|y^{h(j)}(\alpha(x_{n+1})) - y^h(\alpha(x_{n+1}))\right| = \\ \frac{h}{2}(1+\delta(\alpha(x_{n+1})))^2 \left| f(x_{n+1}, y^{(j)}_{n+1}, y^{h(j)}(\alpha(x_{n+1}))) \right. \\ \left. - f(x_{n+1}, y_{n+1}, y^h(\alpha(x_{n+1})))\right|. \tag{35}$$

Using the Lipschitz condition, it follows that

$$\left|y^{h(j)}(\alpha(x_{n+1})) - y^h(\alpha(x_{n+1}))\right| \le \frac{hLr_1^2}{2-hLr_1^2}\left|y^{(j)}_{n+1} - y_{n+1}\right|. \tag{36}$$

Similarly, let $\alpha(x_{n+2}) - x_{n+1} = r_2 h$ with $r_2 \in (0,1]$, we get

$$\left|y^{h(j)}(\alpha(x_{n+2})) - y^h(\alpha(x_{n+2}))\right| \le \frac{hL(1+r_2)^2}{2-hLr_1^2}\left|y^{(j)}_{n+1} - y_{n+1}\right|. \tag{37}$$

From (37), it follows

$$\left|\hat{y}^{(j)}_{n+2} - y_{n+2}\right| \le \frac{4hL}{2-hLr_1^2}\left|y^{(j)}_{n+1} - y_{n+1}\right|. \tag{38}$$

Using (36), (37) and (38), we obtain

$$\left|y^{(j+1)}_{n+1} - y_{n+1}\right| \le C\left|y^{(j)}_{n+1} - y_{n+1}\right|, \ j = 0, 1, \ldots \tag{39}$$

where

$$C = \frac{hL\left(16 + 4hL + hL(1+r_2)^2\right)}{12(2-hLr_1^2)}$$

The constant C is referred as the convergence factor. Thus, the iterative process (30) is convergent if $C < 1$, or if hL satisfies the condition (31). This completes proof of the theorem.

*Remark.*In the same manner, one can determine the convergence factor for different values of β_1.

5 Error estimate

We state and prove the error estimate for the methods (13), (21) and (22). Our error estimate is given by the following theorem:

Theorem 5. *Let y_n be obtained by the methods (13), (21) and (22). Then, at each mesh point x_n, we have the following error estimate:*

$$e_n = |y(x_n) - y_n| \le C_1\, h^3, \ n = 1, 2, \ldots, N \tag{40}$$

where C_1 is independent of n and h.

Proof. Without loose of generality, we take $\beta_1 = 0$. Subtracting (13) from (10) with the coefficients in (11), we obtain

$$\begin{aligned} y(x_{n+1}) - y_{n+1} = {} & y(x_n) - y_n \\ & + \frac{h}{12}\left[(5(f(x_n, y(x_n)), y(\alpha(x_n)))\right. \\ & - f(x_n, y_n, y^h(\alpha(x_n)))) \\ & + 8(f(x_{n+1}, y(x_{n+1}), y(\alpha(x_{n+1}))) \\ & - f(x_{n+1}, y_{n+1}, y^h(\alpha(x_{n+1})))) \\ & - (f(x_{n+2}, y(x_{n+2}), y(\alpha(x_{n+2}))) \\ & \left. - f(x_{n+2}, \hat{y}_{n+2}, y^h(\alpha(x_{n+2}))))\right] \\ & + t(x_{n+1}) \end{aligned} \tag{41}$$

From the definition of e_n in (40) and the Lipschitz condition (3), we obtain

$$\begin{aligned} e_{n+1} \le e_n + \frac{h}{12} & \left[5(L_1 e_n + L_2 e_{\alpha_n})\right. \\ & + 8(L_1 e_{n+1} + L_2 e_{\alpha_{n+1}}) \\ & \left. + L_1 \hat{e}_{n+2} + L_2 \hat{e}_{\alpha_{n+2}}\right] \\ & + |t(x_{n+1})| \end{aligned} \tag{42}$$

where

$$e_{\alpha_n} = \left|y(\alpha(x_n)) - y^h(\alpha(x_n))\right| \tag{43}$$

and

$$\hat{e}_{n+2} = |y(x_{n+2}) - \hat{y}_{n+2}|. \tag{44}$$

Form (15), the inequality (42) can be rewritten as follows

$$\begin{aligned} e_{n+1} \le e_n + \frac{h}{12} & \left[5(L_1 e_n + L_2 e_{\alpha_n})\right. \\ & + 8(L_1 e_{n+1} + L_2 e_{\alpha_{n+1}}) \\ & \left. + L_1 \hat{e}_{n+2} + L_2 e_{\alpha_{n+2}}\right] + O(h^4). \end{aligned} \tag{45}$$

Now, we estimates the quantities e_{α_n}, $e_{\alpha_{n+1}}$, $e_{\alpha_{n+2}}$ and $\hat{e}_{n+2}$ in (42). From (16) and (21) with the coefficient in (19), we obtain

$$\begin{aligned} e_{\alpha_n} \le e_k & + g_1(x_n)(L_1 e_k + L_2 e_{\alpha_k}) \\ & + g_2(x_n)(L_1 e_{k+1} + L_2 e_{\alpha_{k+1}}) \\ & + |t(x_{k+1})|, \\ & \text{for } x_k < \alpha(x_n) \le x_{k+1};\, k = 0, 1, \ldots \end{aligned} \tag{46}$$

where

$$g_1(x) = \frac{h}{2}(1 - \delta^2(\alpha(x))) \tag{47}$$

and

$$g_2(x) = \frac{h}{2}(1+\delta(\alpha(x)))^2. \tag{48}$$

Let us consider

$$E_n = \max_{0\le k\le n} e_k$$

and

$$E_{\alpha_n} = \max_{0\le k\le n} e_{\alpha_k}.$$

Then (46) with (20) leads to the following estimation

$$\begin{aligned} E_{\alpha_{n+j}} \le E_n &+ g_1(x_{n+j})(L_1E_n + L_2E_{\alpha_n}) \\ &+ g_2(x_{n+j})(L_1E_{n+1} + L_2E_{\alpha_{n+1}}) \\ &+ O(h^3); \quad j = 0,1,2 \end{aligned} \tag{49}$$

and

$$\hat{E}_{n+2} \le E_n + 2h(L_1E_{n+1} + L_2E_{\alpha_{n+1}}) + O(h^3), \tag{50}$$

where

$$\hat{E}_{n+2} = \max_{0\le k\le n+2} \hat{e}_k.$$

Assume that h is sufficiently small to satisfy $g_1(x)L_2 < 1$ and $g_2(x)L_2 < 1$. Using (49), we rewrite $E_{\alpha_{n+j}}$ in terms of E_{n+j}, for $j = 0,1$, as follows

$$E_{\alpha_n} \le \frac{1}{g_3(x_n)}(g_4(x_n)E_n + g_2(x_n)L_1E_{n+1}) + O(h^3) \tag{51}$$

and

$$E_{\alpha_{n+1}} \le \frac{1}{g_3(x_n)}(g_5(x_n)E_n + g_6(x_n)E_{n+1}) + (h^3), \tag{52}$$

where

$$\begin{aligned} g_3(x) &= 1 - g_1(x)L_2 - g_2(x+h)L_2 \\ &\quad + g_1(x)g_2(x+h)L_2^2 - g_1(x+h)g_2(x)L_2^2, \\ g_4(x) &= 1 + g_1(x)L_1 - g_2(x+h)L_2 \\ &\quad - g_1(x)g_2(x+h)L_1L_2 + g_2(x)L_2 \\ &\quad + g_1(x+h)g_2(x)L_1L_2, \\ g_5(x) &= 1 - g_1(x)L_2 + g_1(x+h)(L_1+L_2), \\ g_6(x) &= \ g_2(x+h)L_1 - g_1(x)g_2(x+h)L_1L_2 \\ &\quad + g_1(x+h)g_2(x)L_1L_2. \end{aligned} \tag{53}$$

Substituting (51) and (52) into (49) for $j = 2$, we get

$$\begin{aligned} E_{\alpha_{n+2}} \le (1 &+ g_1(x_{n+2})L_1)E_n + g_2(x_{n+2})L_1E_{n+1} \\ &+ \frac{L_2}{g_3(x_n)}[g_4(x_n)g_1(x_{n+2})E_n \\ &\qquad + g_5(x_n)g_2(x_{n+2})E_n \\ &\qquad + g_2(x_n)g_1(x_{n+2})L_1E_{n+1} \\ &\qquad + g_6(x_n)g_2(x_{n+2})E_{n+1}] + O(h^3), \end{aligned} \tag{54}$$

and

$$\begin{aligned} \hat{E}_{n+2} \le E_n &+ 2hL_1E_{n+1} \\ &+ \frac{2hL_2}{g_3(x_n)}[g_5(x_n)E_n + g_6(x_n)E_{n+1}] + O(h^3). \end{aligned} \tag{55}$$

Using (49-52) and (42), we obtain

$$E_{n+1} \le (1 + h\hat{g}_1(x_n))E_n + h\hat{g}_2(x_n)E_{n+1} + Bh^4, \tag{56}$$

where B is a nonnegative constant and the $\hat{g}_1$ and $\hat{g}_2$ are defined as follows

$$\begin{aligned} \hat{g}_1(x) &= \frac{1}{12}[6L_1 + L_2 + L_1L_2g_1(x+2h)] \\ &\quad + \frac{1}{12g_3(x)}[(5 + g_1(x+2h)L_2)g_4(x)L_2 \\ &\qquad\qquad + (8 + 2hL_1 + g_2(x+2h)L_2)g_5(x)L_2], \\ \hat{g}_2(x) &= \frac{1}{12}[8L_1 + 2hL_1^2 + g_2(x+2h)L_1L_2] \\ &\quad + \frac{1}{12g_3(x)}[(5 + g_1(x+2h)L_2^2)g_2(x)L_1 \\ &\qquad\qquad + (8 + 2hL_1)g_6(x)L_2]. \end{aligned} \tag{57}$$

Thus, (56) can be rewritten as

$$E_{n+1} \le \frac{1 + h\hat{g}_1(x_n)}{1 - h\hat{g}_2(x_n)}E_n + \frac{B}{1 - h\hat{g}_2(x_n)}h^4. \tag{58}$$

Assume that h is sufficiently small to ensure $h\hat{g}_2(x) < 1$. Then, there exists two positive constants w_1 and w_2 such that

$$\begin{aligned} \frac{1 + h\hat{g}_1(x_n)}{1 - h\hat{g}_2(x_n)} &\le 1 + hw_1, \\ \frac{h^4}{1 - h\hat{g}_2(x_n)} &\le h^4 w_2. \end{aligned} \tag{59}$$

Then,

$$E_{n+1} \le (1 + hw_1)E_n + Bw_2h^4. \tag{60}$$

Applying Henrici Lemma [11] to the inequality (60) yields

$$E_n \le E_0 + \frac{h^3}{w_1}w_2B(e^{nhw_1} - 1). \tag{61}$$

Since $E_0 = 0$, then

$$E_n \le \frac{h^3}{w_1}w_2B(e^{nhw_1} - 1). \tag{62}$$

This complete the proof of the Theorem 5.

6 Numerical tests

In this section, we validate our methods (13), (21) and (22) numerically for different values of β_1. The comparison with θ-methods for different values of θ is considered as well. We restrict our study to equidistant steps size time.

Example 1

$$y'(x) = \frac{1}{2}e^{\frac{x}{2}}y(\frac{x}{2}) + \frac{x}{2}y(x), 0 \le x \le 1 \qquad (63)$$
$$y(0) = 1$$

The exact solution is $y(x) = e^x$.

Table 1: Comparison of Extended one-step method with parameter β set to equal 0 and 1 with the θ-methods with θ set to equal 0, 0.5 and 1 for Example 1.

θ-methods

	θ=0		θ=0.5		θ=1	
N	E^N	R^N	E^N	R^N	E^N	R^N
10	5.83E-02		6.93E-04		6.28E-02	
20	2.97E-02	0.97	1.73E-04	2.00	3.08E-02	1.03
40	1.50E-02	0.98	4.33E-05	2.00	1.52E-02	1.01
80	7.52E-03	0.99	1.08E-05	2.00	7.59E-03	1.01
160	3.77E-03	0.99	2.70E-06	2.00	3.78E-03	1.00

Extended one-step methods

	β=0		β=1	
N	E^N	R^N	E^N	R^N
10	5.37E-06		1.07E-05	
20	6.63E-07	3.02	1.32E-06	3.01
40	8.24E-08	3.01	1.65E-07	3.01
80	1.03E-08	3.00	2.05E-08	3.00
160	1.28E-09	3.00	2.56E-09	3.00

Example 2

$$\begin{aligned} y_1'(x) &= y_1(x-1) + y_2(x), \quad x \ge 0 \\ y_2'(x) &= y_1(x) - y_1(x-1) \\ y_1(x) &= e^x, \quad x \le 0 \\ y_2(0) &= 1 - e^{-1} \end{aligned} \qquad (64)$$

The exact solution is

$$\begin{aligned} y_1(x) &= e^x, & x \ge 0 \\ y_2(x) &= 1 - e^{x-1}, & x \ge 0 \end{aligned} \qquad (65)$$

The tables **Table 1** and **Table 2** show the error reduction, E^N, with respect to time step size refinement, $h = 1/N$, and the rate other of convergence, R^N, for the θ-methods as well as our extended step-one methods. All examples confirm the theoretical studies introduced in this paper, mainly the third order of convergence of our extended one-step methods.

7 Conclusion and perspective

We have introduced a new numerical methods of third order for solving delay differential equations. The P-stability region has been investigated for different values of parameter $\beta_1 \in (-\infty, 2]$. We showed that the larger P-stability region occurs for $\beta = 0$. Moreover, the methods are L-stable for solving ODEs for the case corresponding to $\beta = 0$. The effectiveness of our methods is clearly indicated with the numerical results. Our research perspective is to extend the current study for integral-deferential equations.

Table 2: Comparison of Extended one-step method with parameter β set to equal 1 with the θ-methods with θ set to equal 0.5 for Example 2.

θ-methods

	$y_1(x)$		$y_2(x)$	
N	E^N	R^N	E^N	R^N
10	1.12E-02		8.93E-03	
20	2.80E-03	2.00	2.23E-03	2.00
40	7.00E-04	2.00	5.57E-04	2.00
80	1.75E-04	2.00	1.39E-04	2.00
160	4.37E-05	2.00	3.48E-05	2.00

Extended one-step methods

	$y_1(x)$		$y_2(x)$	
N	E^N	R^N	E^N	R^N
10	4.00E-06		3.49E-07	
20	5.00E-07	3.00	4.33E-08	3.01
40	6.26E-08	3.00	5.39E-09	3.01
80	7.80E-09	3.00	6.73E-10	3.00
160	9.74E-10	3.00	8.40E-11	3.00

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Fatma Ibrahim recived her Ph.D at Department of Mathematics, Augsburg university. Her main focus are numerical techniques for optimal control problems and delay differential equations (DDES)

Abdelhay A. Salama is a Professor at Department of Mathematics, Faculty of Science, Assiut University. His main focus are numerical techniques for partial differential equations (PDEs), ordinary differential equation (ODE) and delay differential equations(DDES)

Abderrahim Ouazzib received the PhD degree (Dr. rer. nat.) in Mathematics at Dortmund University. His research interests are in the areas of applied mathematics and industrial mathematics including the mathematical methods and models for complex fluids and numerical methods for nonlinear fluids. He has published research articles in reputed international journals of mathematical and engineering sciences.

Stefan Turek is a Professor at TU Dortmund and Dean of Department of Mathematics, TU Dortmund. He helds the chair of Applied Mathematics and Numerics (LS3) at the Department of Mathematics of the TU Dortmund. His main focus are numerical techniques for partial differential equations (PDEs), high performance computing and scientific computing with respect to engineering sciences (fluid and structural mechanics). Prof. Dr. Stefan Turek main research topics include finite element discretizations, enhanced and adapted to the special characteristics of convection-diffusion equations and saddle point problems like the Navier-Stokes equations. He is also interested in the design and implementation of fast multigrid and domain decomposition solvers which combine and supersede his ouwn solver variant ScaRC (Scalable Recursive Clustering). Exceeding these mathematical aspects he focus on (numerically and implementationally) efficient FEM software, in particular by pursuing hardware-oriented approaches. While maintaining and steadily improving the legacy FEM software packages FEAT2D/3D and FEATFLOW, he is currently working on the successor packages as a part of the FEAST project which will yield a high performance FEM toolbox. His software is used worldwide to solve complex problems in the field of fluid mechanics that have an industrial background. Models are usually based on variants of incompressible Navier-Stokes equations with extensions like non-linear viscosity (granular flow, non-Newtonian flow, viscoelasticity), fluid-structure-interaction, multiphase flows with chemical reactions, and free boundary value problems emerging in solidification processes. Typically, some of them are put to use in industrial projects as well.