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# The Inventory Models for Deteriorating Items in the Discounted Cash-Flows Approach Under Conditional Trade Credit and Cash Discount in a Supply Chain System

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**Abstract:** Yang [20] studies a deteriorating inventory model in which the supplier simultaneously offers the retailer either a conditional delay in payments or a cash discount. While considering the same inventory problem as that of Yang [20] in a systematic manner, the main purpose of this paper is twofold: First of all, since Yang [20] does not seem to have defined his inventory problem precisely, this paper aims at remodeling this inventory problem from different viewpoints in order to derive formulations for $APV(T)$, that is, the present value of all future cash flows, so that researchers and practitioners can understand the situation much more easily. Secondly, since the processes of proofs of Lemmas 1 and 2, and also of Theorems 1, 2 and 3 in Yang's paper [20], have some shortcomings from the mathematical viewpoints, this paper applies rigorous analytic methods of mathematics to explore the functional behaviors of $APV(T)$, not only to remove all of the shortcomings in Yang's work [20], but also to present the complete rigorous mathematical proofs for the results asserted by Yang [20].

**Keywords:** Inventory models; Conditional trade credit and cash discount; Discounted cash-flows (DCF) approach; Permissible delays in payments; Mathematical solution procedures; Deteriorating items and supply chain system.
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## 1 Introduction and Motivation

Any given inventory problem consists of two parts: The modeling part and the mathematical solution procedure part. Basically, both parts are equally important. In recent years, marketing researchers and practitioners have recognized the phenomenon that the supplier offers a permissible delay to the retailer if the outstanding amount is paid within the permitted fixed settlement period, known as the *trade credit period*. During the trade credit period, the retailer can accumulate revenues by selling items and earning interests. As a result, with no incentive for making early payments and earning interest through the accumulated revenue received during the credit period, the retailer postpones payment up to the last moment of the permissible period allowed by the supplier. Therefore, offering trade credit leads to delayed cash inflow and increases the risk of cash flow shortage and bad debt. From the viewpoints of suppliers, they always hope to be able to find a trade credit policy to increase sales and decrease the risk of cash flow shortage and bad debt. In reality, on the operations management side, a supplier is always willing to provide the retailer either a cash discount or a permissible delay in payments.

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Stokes [17] indicates that the trade credit represents one of the most flexible sources of short-term financing available to firms principally because it arises spontaneously with the firm's purchases. The decision to offer trade credit and the determination of the firm's terms of sale are important managerial considerations. In addition, the purchasing firm's decision to take advantage of a cash discount or not and the motivations behind such a decision are also important. Given the increasing saliency of a sales promotion tool, Arcelus *et al.* [1] analyze the advantages and disadvantages of the two most common payment reduction schemes: A cash discount and a permissible delay in the payment of the merchandise. A cash discount can encourage the customer to pay cash on delivery and reduce the default risk. A permissible delay in payments is considered as a type of price reduction and it can attract new customers and thereby increase sales. In reality, on the operations management side, a supplier is always willing to provide the retailer either a cash discount or a permissible delay in payments. On the other hand, the discounted cash-flow (DCF) model, which is is an economic model studied by the classical financial mathematical tools, describes some very general ways to characterize the expression of its present value (see, *e.g.*, Brigham [2], Sharpe *et al.* [14], Yao *et al.* [21]; see also [5] and [6]). In practice, the DCF model has become very popular in valuation because it is most consistent with the goal of the long-term value creation, and because it may capture all the elements that affect the value of the company in a comprehensive yet straightforward manner. It is also widely applied in many fields such as project management, insurance, and financial management. Some proponents of the DCF model approach have even suggested that the DCF model can provide a more sophisticated and reliable picture of a company's value than the accounting approach (see Copeland *et al.* [8]).

A number of published papers on the subject of investigation here assume that the supplier offers the retailer fully permissible delay in payments independent of the order quantity. Recently, Huang [10] considered the case of a conditionally permissible delay to assume that the supplier only offers the retailer fully permissible delay in payments if the retailer orders more than a predetermined quantity. Yang [20] further adopts the concept of Huang [10] to consider the inventory models for deteriorating items in discount cash flows analysis under alternatives of conditionally permissible delay in payments and cash discount (see also such closely-related recent investigations as [4,6,7,9,11,12,13,15,18]). This paper aims at presenting a rather systematic study of the same inventory problem as the one considered by Yang [20]. The main purpose of this paper is twofold: First of all, since Yang [20] does not seem to have defined his inventory problem precisely, this paper begins by remodeling this inventory problem from different viewpoints in order to derive formulations for $APV(T)$, that is, the present value of all future cash flows, so that researchers and practitioners can understand the situation much more easily. Secondly, since the processes of proofs of Lemmas 1 and 2, and also of Theorems 1, 2 and 3 in Yang's paper [20], have some shortcomings from the mathematical viewpoints, this paper applies rigorous analytic methods of mathematics to explore the functional behaviors of $APV(T)$, not only to remove all of the shortcomings in Yang's work [20], but also to present the complete rigorous mathematical proofs for the results asserted by Yang [20].

We have chosen to organize our presentation here as follows. In Section 2, we lay out the notations, conventions and assumptions which are used in our investigation. Section 3 contains a detailed mathematical formulation of the inventory problem which we investigate in this paper rather systematically. It is here, in Section 3, that we point out (and deal appropriately with) some of the shortcomings in Yang's paper [20]. The relevant functional behaviors of $APV_2(T), APV_{11}(T)$ and $APV_{12}(T)$, that is, the present values of all future cash flows, are considered in Section 4. In Section 5, we briefly describe some *further* shortcomings of the solution procedures of Yang [20] in addition to those considered in section 3. Our main Theorem for the optimal cycle time $T^*$ of $APV(T)$ is then stated and proved, using rigorous mathematical techniques, in Section 6. Finally, in the concluding section (Section 7), we list a number of remarks and observations which pertain to the various arguments that are put forth in this paper.

## 2 Notations, Conventions and Assumptions

The following notations, conventions and assumptions are used throughout this paper.

**Notations Used:**

$D$ the demand rate.
$A$ the ordering cost per order.
$W$ the quantity for which the fully delayed payment is permitted per order.
$T_W$ the time interval in which $W$ units are depleted to zero.
$c$ the purchasing cost per unit.
$h$ the out-of-pocket inventory holding cost as a proportion of the value of the item per unit time.
$M_1$ the fixed period of cash discount in settling accounts.
$M_2$ the fixed period of permissible delay in settling accounts (with $M_2 > M_1$).
$\alpha$ the fraction of the delay payments permitted by the supplier when the order quantity is less than the quantity for which which the fully delayed payment is permitted $(0 \leqq \alpha \leqq 1)$.
$\delta$ the cash discount rate $(0 \leqq \delta < 1)$.

$r$ the opportunity cost (that is, the continuous discounting rate) per unit time.
$\theta$ the deterioration rate $(0 \leqq \theta < 1)$.
$T$ the replenishment cycle time.
$Q$ the order quantity.
$I(t)$ the level of inventory at time $t$ $(0 \leqq t \leqq T)$.
$PV(T)$ the present value of cash flows for the first replenishment cycle.
$APV(T)$ the present value of all future cash flows.
$T^*$ the optimal replenishment cycle time.

**Assumptions Made:**

1. The demand for the item is constant with time.
2. Shortages are not allowed.
3. Replenishment is instantaneous and the lead time is zero.
4. Time horizon is infinite.
5. The distribution of time for deterioration of the items follows the *exponential distribution* with constant deteriorating rate $\theta$ (see, for details, [16]). There is no replacement or repair of the deteriorated units during the period under consideration.
6. In the case of a conditionally permissible delay, if the order quantity $Q$ is greater than or equal to $W$, then the fully delayed payment is permitted. Otherwise, the partially delayed payment is permitted. This means that, if $Q < W$, then the retailer must pay the supplier the partial payment of $(1-\alpha)cQ$ when the order is filled and pay the rest on the last time of the credit period.
7. If $Q < W$, then the retailer has two payment policies to be chosen: (1) Cash discount and (2) Partially delayed payment. Furthermore, if $Q \geqq W$, then the retailer has two payment policies to be chosen: (1) Cash discount and (2) Fully delayed payment.
8. If the retailer chooses the cash discount, then the account should be settled at time $M_2$.

**Conventions Applied:**

Based upon the above notations and assumptions, there are three policies to be defined as follows.

Policy 1. The retailer adopts the cash discount to settle the account and makes the payment at time $M_1$.
Policy 2. The retailer adopts the partially delayed payment to settle the account. Then the retailer must pay the supplier the partial payment of $(1-\alpha)cQ$ when the order is filled and pay the rest at time $M_2$.
Policy 3. The retailer adopts the fully delayed payment to settle the account and makes payment at time $M_2$.

**Remark.** We remark in passing that Assumption 6 in Yang [20] only allows Policies 1 and 2 to be chosen. In our present investigation, however, we adopt all of the above three policies to remodel the inventory problem considered by Yang [20].

## 3 Mathematical Formulation of the Inventory Problem

In this section, we present a detailed mathematical formulation of the inventory problem which we investigate here rather systematically. Indeed, since the inventory level decreases owing to demand as well as deterioration, the change of inventory level can be represented by the following differential equation:

$$\frac{d}{dt}\{I(t)\} + \theta I(t) = -D \qquad (0 < t < T) \tag{1}$$

with the boundary condition given by

$$I(T) = 0.$$

The explicit solution of (1) is easily found as follows:

$$I(t) = \frac{D}{\theta}\left(e^{\theta(T-t)} - 1\right) \qquad (0 \leqq t \leqq T). \tag{2}$$

The order quantity $Q$ for each cycle is given by

$$Q = I(0) = \frac{D}{\theta}\left(e^{\theta T} - 1\right). \tag{3}$$

The time interval in which $W$ units are depleted to zero due to both demand and deterioration can be obtained from the equation (3) as follows:

$$T_W = \frac{1}{\theta}\ln\left(\frac{\theta}{D}W + 1\right). \tag{4}$$

By assumption, if $Q \geqq W$ (that is, if $T \geqq T_W$), then fully delayed payment is permitted. Otherwise, the partially delayed payment is permitted. Hence, if $Q < W$ (that is, if $T < T_W$), then the retailer must pay the supplier the partial payment of $(1-\alpha)cQ$ when the order is filled at time 0 and pay the rest on the last day of the credit period $M_2$. The following two cases will occur then.

**Case I.** Let $0 < T < T_W$.

Under Case I, the retailer has two payment policies to be chosen: (1) Cash discount (Policy 1) and (2) Partially delayed payment (Policy 2).

(A) If the cash discount policy (Policy 1) is chosen, then the supplier offers a cash discount if the retailer pays for purchases within a certain period $M_1$. At the beginning of each replenishment cycle, there will be cash outflows of the ordering cost $A$. Following the same arguments of Case

2 in Yang [20], we have the present value of cash outflows given by

$$\begin{aligned} PV_2(T) &= A + \frac{c(1-\delta)De^{-rM_1}}{\theta}(e^{\theta T}-1) \\ &\quad + hc(1-\delta)D\int_0^T \frac{e^{-rt}}{\theta}\left(e^{\theta(T-t)}-1\right)dt \\ &= A + \frac{c(1-\delta)D}{\theta}(e^{\theta T}-1)e^{-rM_1} \\ &\quad + hc(1-\delta)\frac{D}{\theta}\left(\frac{e^{\theta T}}{\theta+r}-\frac{1}{r}+\frac{\theta}{r(\theta+r)}e^{-rT}\right). \end{aligned} \tag{5}$$

Then, according to Yang [20], the present value of all future cash outflows for Policy 1 is given by

$$\begin{aligned} APV_2(T) &= \sum_{n=0}^{\infty} PV_2(T)e^{-nrT} = PV_2(T)\sum_{n=0}^{\infty} e^{-nrT} = \frac{PV_2(T)}{1-e^{-rT}} \\ &= \frac{1}{1-e^{-rT}}\left\{A + \frac{c(1-\delta)D}{\theta}\left[\left(e^{\theta T}-1\right)e^{-rM_1}\right.\right. \\ &\quad \left.\left. + h\left(\frac{e^{\theta T}}{\theta+r}-\frac{1}{r}+\frac{\theta e^{-rT}}{r(\theta+r)}\right)\right]\right\}. \end{aligned} \tag{6}$$

(B) If the partially delayed payment is chosen, then the retailer must pay the supplier the partial payment of $(1-\alpha)cQ$ when the order is filled and thereafter pay the rest on the end of the credit period. By the same arguments for Cases 1 and 2 as in Yang [20], we have the present value of cash flows for the first cycle as follows:

$$\begin{aligned} PV_{12}(T) &= A + \frac{c(1-\alpha)D}{\theta}(e^{\theta T}-1) + \frac{c\alpha De^{-rM_2}}{\theta}(e^{\theta T}-1) \\ &\quad + \frac{hcD}{\theta}\left(\frac{e^{\theta T}}{\theta+r}-\frac{1}{r}+\frac{\theta e^{-rT}}{r(\theta+r)}\right). \end{aligned} \tag{7}$$

Thus the present value of all future cash flows for Policy 2 is given by

$$\begin{aligned} APV_{12}(T) &= \sum_{n=0}^{\infty} PV_{12}(T)e^{-nrT} = PV_{12}(T)\sum_{n=0}^{\infty} e^{-nrT} = \frac{PV_{12}(T)}{1-e^{-rT}} \\ &= \frac{1}{1-e^{-rT}}\left\{A + \frac{c(1-\delta)D}{\theta}(e^{\theta T}-1) + \frac{c\alpha De^{-rM_2}}{\theta}(e^{\theta T}-1)\right. \\ &\quad \left. + hc\frac{D}{\theta}\left(\frac{e^{\theta T}}{\theta+r}-\frac{1}{r}+\frac{\theta e^{-rT}}{r(\theta+r)}\right)\right\}. \end{aligned} \tag{8}$$

From the equations (6) and (8), we have the present value of all future cash flows for Case I as follows:

$$APV_I(T) = \begin{cases} APV_2(T) & (0 < T < T_W \text{ and Policy 1 is adopted}) \quad (9a) \\ APV_{12}(T) & (0 < T < T_W \text{ and Policy 2 is adopted}). \quad (9b) \end{cases}$$

**Case II.** Let $T_W \leqq T$.

Under Case II, if the retailer orders more than or equal to a predetermined quantity $W$, then the retailer has two payment policies to be chosen: (1) Cash discount (Policy 1) and (2) Fully delayed payment (Policy 3).

(A) If the cash discount policy is chosen, by following the same arguments as those given in Case I(A), we have the present value of all future cash outflows for Policy 1 as follows:

$$\begin{aligned} APV_2(T) &= \frac{1}{1-e^{-rT}}\left\{A + \frac{c(1-\delta)D}{\theta}\left[\left(e^{\theta T}-1\right)e^{-rM_1}\right.\right. \\ &\quad \left.\left. + h\left(\frac{e^{\theta T}}{\theta+r}-\frac{1}{r}+\frac{\theta e^{-rT}}{r(\theta+r)}\right)\right]\right\}. \end{aligned} \tag{10}$$

(B) If the fully delayed payment policy is chosen, the retailer pays the supplier full purchase cost on the last day of the credit period $M_2$ in order to enjoy the permissible delay in payments. At the beginning of each replenishment cycle, there will be cash outflows of the ordering cost $A$. Following the same arguments used in Case 1-1 in Yang [20], we have the present value of cash outflows for the first cycles as follows:

$$\begin{aligned} PV_{11}(T) &= A + \frac{cDe^{-rM_2}}{\theta}(e^{\theta T}-1) + hcD\int_0^T \frac{e^{-rt}}{\theta}\left(e^{\theta(T-t)}-1\right)dt \\ &= A + \frac{cD}{\theta}(e^{\theta T}-1)e^{-rM_2} \\ &\quad + \frac{hcD}{\theta}\left(\frac{e^{\theta T}}{\theta+r}-\frac{1}{r}+\frac{\theta}{r(\theta+r)}e^{-rT}\right). \end{aligned} \tag{11}$$

Yang [20] shows that the present value of all future cash outflows for Policy 3 is given by

$$\begin{aligned} APV_{11}(T) &= \sum_{n=0}^{\infty} PV_{11}(T)e^{-nrT} = PV_{11}(T)\sum_{n=0}^{\infty} e^{-nrT} = \frac{PV_{11}(T)}{1-e^{-rT}} \\ &= \frac{1}{1-e^{-rT}}\left\{A + \frac{cD}{\theta}\left(e^{\theta T}-1\right)e^{-rM_2}\right. \\ &\quad \left. + \frac{hcD}{\theta}\left(\frac{e^{\theta T}}{\theta+r}-\frac{1}{r}+\frac{\theta e^{-rT}}{r(\theta+r)}\right)\right\}. \end{aligned} \tag{12}$$

From equations (10) and (12), we have the present value of all future cash flows for Case II as follows:

$$APV_{II}(T) = \begin{cases} APV_2(T) & (T_W \leqq T \text{ and Policy 1 is adopted}) \quad (13a) \\ APV_{11}(T) & (T_W \leqq T \text{ and Policy 3 is adopted}). \quad (13b) \end{cases}$$

By combining Cases I and II, we have the present value of all future cash flows as follows:

$$APV(T) = \begin{cases} APV_I(T) & (0 < T < T_W) \quad (14a) \\ APV_{II}(T) & (T_W \leqq T). \quad (14b) \end{cases}$$

Yang [20] does not give the formulation for $APV(T)$. This means that he does not define $APV(T)$ *prescisely*. However, the equations (14a) and (14b) in this paper not only give the formulation for $APV(T)$, but also correctly

define $APV(T)$ in order to overcome the shortcomings in the paper by Yang [20] (see also Section 5 for other shortcomings in in Tang's paper [20]). Our problem now is to determine the optimal replenishment cycle time $T^*$ which minimizes the present value of all future cash flows $APV(T)$. We thus find from the equations (14a) and (14b) that

$$APV(T^*) = \min \left\{ \begin{array}{l} \min\{APV_I(T) : 0 < T < T_W\} \\ \\ \min\{APV_{II}(T) : T_W \leqq T\} \end{array} \right\}. \tag{15}$$

Let $T_I^*$, $T_{II}^*$, $t_{2I}^*$, $t_{12}^*$, $t_{2II}^*$ and $t_{11}^*$ denote the minimum points of $APV_I(T)$ on $(0,T_W)$, $APV_{II}(T)$ on $[T_W,\infty)$, $APV_2(T)$ on $(0,T_W)$, $APV_{12}(T)$ on $(0,T_W)$, $APV_2(T)$ on $[T_W,\infty)$ and $APV_{11}(T)$ on $[T_W,\infty)$, respectively. Then, by means of the equations (9a), (9b), (13a), (13b), (14a) and (14b), we get

$$APV(T^*) = \min\{APV_I(T_I^*),\, APV_{II}(T_{II}^*)\}, \tag{16}$$

$$APV_I(T_I^*) = \min\{APV_2(t_{2I}^*),\, APV_{12}(t_{12}^*)\} \tag{17}$$

and

$$APV_{II}(T_{II}^*) = \min\{APV_2(t_{2II}^*),\, APV_{11}(t_{11}^*)\}. \tag{18}$$

## 4 Functional Behaviors of $APV_2(T)$, $APV_{11}(T)$ and $APV_{12}(T)$

For convenience, we treat $APV_2(T)$, $APV_{11}(T)$ and $APV_{12}(T)$ defined on the semi-infinite interval $(0,\infty)$. Then the equations (6), (8) and (12) yield the first-order derivatives of $APV_2(T)$, $APV_{11}(T)$ and $APV_{12}(T)$ with respect to $T$ as follows:

$$APV_2'(T) = \frac{F_2(T)e^{-rT}}{(1-e^{-rT})^2}, \tag{19}$$

$$APV_{11}'(T) = \frac{F_{11}(T)e^{-rT}}{(1-e^{-rT})^2} \tag{20}$$

and

$$APV_{12}'(T) = \frac{F_{12}(T)e^{-rT}}{(1-e^{-rT})^2}, \tag{21}$$

where

$$F_2(T) = \left(c(1-\delta)De^{-rM_1} + \frac{hc(1-\delta)D}{\theta+r}\right)\left(e^{(\theta+r)T} - \frac{(\theta+r)e^{\theta T}}{\theta} + \frac{r}{\theta}\right) - Ar, \tag{22}$$

$$F_{11}(T) = \left(cDe^{-rM_2} + \frac{hcD}{\theta+r}\right)\left(e^{(\theta+r)T} - \frac{(\theta+r)e^{\theta T}}{\theta} + \frac{r}{\theta}\right) - Ar \tag{23}$$

and

$$F_{12}(T) = \left(c(1-\alpha)D + c\alpha De^{-rM_2} + \frac{hcD}{\theta+r}\right)\left(e^{(\theta+r)T} - \frac{(\theta+r)e^{\theta T}}{\theta} + \frac{r}{\theta}\right) - Ar. \tag{24}$$

Equations (22), (23) and (24) show that

$$F_2'(T) = (\theta+r)\left(c(1-\delta)De^{-rM_1} + \frac{hc(1-\delta)D}{\theta+r}\right)\left(e^{(\theta+r)T} - e^{\theta T}\right) > 0, \tag{25}$$

$$F_{11}'(T) = (\theta+r)\left(cDe^{-rM_2} + \frac{hcD}{\theta+r}\right)\left(e^{(\theta+r)T} - e^{\theta T}\right) > 0 \tag{26}$$

and

$$F_{12}'(T) = (\theta+r)\left(c(1-\alpha)D + c\alpha De^{-rM_2} + \frac{hcD}{\theta+r}\right)\left(e^{(\theta+r)T} - e^{\theta T}\right) > 0. \tag{27}$$

Thus, clearly, the equations (25), (26) and (27) imply that $F_i(T)$ is increasing on $T>0$ for $i=2,11,12$. Furthermore, since

$$F_i(0) = -Ar < 0 \quad \text{and} \quad \lim_{T\to\infty} F_i(T) = \infty \quad (i=2,11,12),$$

by the *Intermediate Value Theorem* (see, for example, Varberg *et al.* [19]), we conclude that there exist $T_2$, $T_{11}$ and $T_{12}$in the *open* interval $(0,\infty)$ such that

$$F_i(T_i) = 0 \qquad (i=2,11,12). \tag{28}$$

Since $F_i(T)$ is increasing on $T>0$, the equation (28) shows (for all $i=2,11,12$) that

$$F_i(T) \begin{cases} <0 & (0<T<T_i) \quad (29a) \\ =0 & (T=T_i) \quad (29b) \\ >0 & (T>T_i). \quad (29c) \end{cases}$$

Therefore, according to the equations (19), (20) and (21), we find (for all $i=2,11,12$) that

$$APV_i'(T) \begin{cases} <0 & (0<T<T_i) \quad (30a) \\ =0 & (T=T_i) \quad (30b) \\ >0 & (T>T_i). \quad (30c) \end{cases}$$

So, finally, we have shown that (for all $i=2,11,12$) the function $APV_i(T)$ is decreasing on $(0,T_i]$ and increasing on $[T_i,\infty)$, and also that $T_i$ is the minimum point of $APV_i(T)$ on $T>0$. Furthermore, the equations (30a), (30b) and (30c) characterize all functional behaviors of $APV_{11}(T)$, $APV_{12}(T)$ and $APV_2(T)$ on $T>0$.

## 5 Further Shortcomings of the Solution Procedures of Yang [20]

In this section, we present (and deal appropriately with) *further* shortcomings in Yang's paper [20] in addition, of course, to the shortcomings which we have already indicated and remedied in Section 3.

Yang [20] has made the following assertions:

(C1) Since the equation (A3) in Yang [20] holds true, $T_{11}$ is the minimum point of $APV_{11}(T)$ on $T \geqq T_W$.

(C2) Since the equation (B3) in Yang [20] holds true, $T_2$ is the minimum point $APV_2(T)$ on $T > 0$.

We now recall a well-known result (Theorem B in [19, p. 164]) as follows.

**Theorem B (Second Derivative Test).** *Let $f'(x)$ and $f''(x)$ exist at every point in an open interval $(a,b)$ containing the point $x = c$. Suppose also that $f'(c) = 0$.*
(i) *If $f''(c) < 0$, then $f(c)$ is a local maximum value of $f$.*
(ii) *If $f''(c) > 0$, then $f(c)$ is a local minimum value of $f$.*

Theorem B(ii) explains clearly that the equations (A3) and (B3) in Yang's paper [20] are only able to imply that $T_{11}$ and $T_2$ are local minimum points of $APV_{11}(T)$ and $APV_2(T)$, respectively. Many examples given by Varberg *et al.* [19] show that Theorem B can not draw a conclusion about maxima or minima *without* more information in general. Therefore, the processes of arguments of both assertions (C1) and (C2) by Yang [20] are doubtful. Thus, in general, it is unacceptable from the viewpoint of mathematics that, although the results of Lemmas 1 and 2, and Theorems 1, 2 and 3, in Yang's paper [20] are correct, the processes of arguments in their proofs have some shortcomings. Equations (30a), (30b)and (30c) in this paper reveal the correct ways of discussions about the minimum points $T_{11}$, $T_{12}$ and $T_2$ of $APV_{11}(T)$, $APV_{12}(T)$ and $APV_2(T)$, respectively.

## 6 Theorem for the Optimal Cycle Time $T^*$ of $APV(T)$

Let

$$\Delta_{11} = F_{11}(T_W) = \left(cDe^{-rM_2} + \frac{hcD}{\theta + r}\right) \cdot \left\{\left(\frac{\theta W}{D} + 1\right)\left[\left(\frac{\theta W}{D} + 1\right)^{r/\theta} - \frac{\theta + r}{\theta}\right] + \frac{r}{\theta}\right\} - Ar, \tag{31}$$

$$\Delta_{12} = F_{12}(T_W) = \left(c(1-\alpha)D + c\alpha De^{-rM_2} + \frac{hcD}{\theta + r}\right) \cdot \left\{\left(\frac{\theta W}{D} + 1\right)\left[\left(\frac{\theta W}{D} + 1\right)^{r/\theta} - \frac{\theta + r}{\theta}\right] + \frac{r}{\theta}\right\} - Ar \tag{32}$$

and

$$\Delta_2 = F_2(T_W) = \left(c(1-\delta)De^{-rM_1} + \frac{hc(1-\delta)D}{\theta + r}\right) \cdot \left\{\left(\frac{\theta W}{D} + 1\right)\left[\left(\frac{\theta W}{D} + 1\right)^{r/\theta} - \frac{\theta + r}{\theta}\right] + \frac{r}{\theta}\right\} - Ar. \tag{33}$$

Equations (31) and (32) imply that

$$\Delta_{12} \geqq \Delta_{11}, \tag{34}$$

which leads us easily to the following results.

**Lemma.** *Each of the following assertions hold true*:

(A) *If $\Delta_{11} > 0$, then the function $APV_{11}(T)$ is increasing on $T \geqq T_W$ and $t_{11}^* = T_W$.*
(B) *If $\Delta_{11} \leqq 0$, then the function $APV_{11}(T)$ is decreasing on $[T_W, T_{11}]$ and increasing on $[T_{11}, \infty)$. Furthermore, $t_{11}^* = T_{11}$.*
(C) *If $\Delta_{12} > 0$, then the function $APV_{12}(T)$ is decreasing on $(0, T_{12}]$ and increasing on $[T_{12}, T_W)$. Furthermore, $t_{12}^* = T_{12}$.*
(D) *If $\Delta_{12} \leqq 0$, then the function $APV_{12}(T)$ is decreasing on $(0, T_W)$.*
(E) *If $\Delta_2 > 0$, then*

$$0 < T_2 < T_W \qquad \text{and} \qquad t_{2I}^* = T_2.$$

(F) *If $\Delta_2 \leqq 0$, then*

$$T_2 \geqq T_W \qquad \text{and} \qquad t_{2II}^* = T_2.$$

**Theorem.** *Each of the following assertions hold true*:

(A) *If $\Delta_{11} > 0$, then*

$$APV(T^*) = \min\{APV_{11}(T_W), APV_{12}(T_{12}), APV_2(T_2)\}. \tag{35}$$

(B) *If $\Delta_{11} \leqq 0 < \Delta_{12}$, then*

$$APV(T^*) = \min\{APV_{11}(T_{11}), APV_{12}(T_{12}), APV_2(T_2)\}. \tag{36}$$

(C) *If $\Delta_{12} \leqq 0$, then*

$$APV(T^*) = \min\{APV_{11}(T_{11}), APV_2(T_2)\} \tag{37}$$

*Proof.* Our demonstration of the above Theorem is being outlined as follows.

(A) If $\Delta_{11} > 0$, there are the following two cases that would occur:

(a1) If $\Delta_2 > 0$, then we find from the equations (16), (17) and (18) and the assertions (A), (C) and (E) of the Lemma that

$$APV(T^*) = \min\{APV_{11}(T_W),\ APV_{12}(T_{12}),\ APV_2(T_2)\}; \tag{38}$$

(a2) If $\Delta_2 \leqq 0$, then we find from the equations (16), (17) and (18) and the assertions (A), (C) and (F) of the Lemma that

$$APV(T^*) = \min\{APV_{11}(T_W),\ APV_{12}(T_{12}),\ APV_2(T_2)\}. \tag{39}$$

Thus, by combining (a1) and (a2), we see that the assertion (A) of the Theorem holds true.

(B) If $\Delta_{11} \leqq 0 < \Delta_{12}$, there are the following two cases that would occur:

(b1) If $\Delta_2 > 0$, then we find from the equations (16), (17) and (18) and the assertions (A), (C) and (E) of the Lemma that

$$APV(T^*) = \min\{APV_{11}(T_{11}),\ APV_{12}(T_{12}),\ APV_2(T_2)\}. \tag{40}$$

(b2) If $\Delta_2 \leqq 0$, then we find from the equations (16), (17) and (18) and the assertions (B), (C) and (F) of the Lemma that

$$APV(T^*) = \min\{APV_{11}(T_{11}),\ APV_{12}(T_{12}),\ APV_2(T_2)\}. \tag{41}$$

Thus, by combining (b1) and (b2), we see that Theorem (B) holds true.

(C) If $\Delta_{12} \leqq 0$, there are the following two cases which would occur:

(c1) If $\Delta_2 > 0$, then we find from the equations (16), (17) and (18) and the assertions (B), (D) and (E) of the Lemma, together with the following inequality:

$$APV_{12}(T_W) \geqq APV_{11}(T_W), \tag{42}$$

that

$$APV(T^*) = \min\{APV_{11}(T_{11}),\ APV_2(T_2)\}. \tag{43}$$

(c2) If $\Delta_2 \leqq 0$, then we find from the equations (16), (17) and (18) and the assertions (B), (D) and (F) of the Lemma, together with the following inequality:

$$APV_{12}(T_W) \geqq APV_{11}(T_W), \tag{44}$$

that

$$APV(T^*) = \min\{APV_{11}(T_{11}),\ APV_2(T_2)\}. \tag{45}$$

Thus, by combining (c1) and (c2), we see that Theorem (C) holds true.

Finally, by incorporating (A), (B) and (C) above, we complete the proof of the Theorem.

The Lemma and and the Theorem, which we have presented in this section, not only remove all shortcomings in Yang's paper [20], but also give the complete rigorous proofs for Lemmas 1 and 2, and Theorems 1, 2 and 3, in Yang's paper [20].

## 7 Concluding Remarks and Observations

The inventory problem consists of two parts: (1) the modeling part and (2) the solution procedure part. The modeling part can provide insight into the solvability of the inventory problem and the solution procedure part involves the implementation of the inventory model. Basically, both the modeling part and the solution procedure part of the inventory problem are equally important.

**(A) About the Modeling Part.** The $APV(T)$ appears in the Notation section of Yang's paper [20], but Yang [20] does not have any formulation to define it explicitly. Equations (14a) and (14b) in this paper give the precise definition of $APV(T)$ in order to overcome the shortcomings of Yang's paper [20].

**(B) About the Solution Procedure Part.** Theorem B (Second Derivative Test) in [19, p. 164] reveals why equations (A3) and (B3) in Yang's paper [20] can not assure that $T_{11}$ and $T_2$ are the minimum points of $APV_{11}(T)$ and $APV_2(T)$, respectively. The validity of the arguments in the proofs of Lemmas 1 and 2, and Theorems 1, 2 and 3, in Yang's paper [20] is, therefore, doubtful from the mathematical viewpoints. Our Equations (30a), (30b) and (30c) correctly demonstrate that $T_{11}$, $T_{12}$ and $T_2$ are the minimum points of $APV_{11}(T)$, $APV_{12}(T)$ and $APV_2(T)$, respectively, thereby overcoming the corresponding shortcomings in Yang's paper [20].

Based upon all arguments in (A) and (B) above, we conclude that we have not only overcome all shortcomings in Yang's paper [20], but we have also presented the mathematically correct formulations and complete rigorous proofs for the assertions made by Yang [20].

## 8 Highlights

We have chosen to include, in this last section of our paper, some of the main highlights of the work presented here.

• We systematically model and investigate deteriorating inventory problems.

• We apply mathematical tools and techniques in dealing rigorously with the model.

• Our main assertions and results are stated and proved as Lemmas and Theorems.

• By our rigorous arguments, we have overcome all shortcomings in the literature.

• Corrections for the assertions made in the cited works are also presented.

## References


[1] F. J. Arcelus, N. H. Shah and G. Srinivasan, Retailer's response to special sales: Price discount vs. trade credit, *Omega*, **29**, 417–428 (2001).

[2] E. F. Brigham, *Fundamentals of Financial Management*, The Dryden Press, New York, (1992).

[3] C.-T. Chang, L.-Y. Ouyang and J.-T. Teng, An EOQ model for deteriorating items under supplier credits linked to order quantity, *Appl. Math. Modelling*, **27**, 983–996 (2003).

[4] K.-J. Chung and L. E. Cárdenas-Barrón, The simplified solution procedure for deteriorating items under stock-dependent demand and two-level trade credit in the supply chain management, *Appl. Math. Modelling*, **37**, 4653–4660 (2013).

[5] K.-J. Chung, K.-L. Hou and S.-P. Lan, The optimal production cycle time in an integrated production-inventory model for decaying raw materials, *Appl. Math. Modelling*, **33**, 1–10 (2009).

[6] K.-J. Chung, S.-D. Lin and H. M. Srivastava, The complete solution procedures for the mathematical analysis of some families of optimal inventory models with order-size dependent trade credit and deterministic and constant demand, *Appl. Math. Comput.*, **219**, 141–156 (2012).

[7] K.-J. Chung, S.-D. Lin and H. M. Srivastava, The inventory models under conditional trade credit in a supply chain system, *Appl. Math. Modelling*, **37**, 10036–10052 (2013).

[8] T. Copeland, T. Koller and J. Murrin, *Valuation*: *Measuring and Managing the Value of Companies*, Third edition, Wiley Series on Finance, John Wiley and Sons (McKinsey and Company Incorporated), New York, Chichester, Brisbane and Toronto, (2000).

[9] N. Esfandiari and M. Seifbarghy, Modeling a stochastic multi-objective supplier quota allocation problem with price-dependent ordering, *Appl. Math. Modelling*, **37**, 5790–5800 (2013).

[10] Y.-F. Huang, Economic order quantity under conditionally permissible delay in payments, *European J. Oper. Res.*, **176**, 911-924 (2007).

[11] J.-J. Liao, On an EPQ model for deteriorating items under permissible delay in payments, *Appl. Math. Modelling*, **31**, 393–403 (2007).

[12] K.-R. Lou and W.-C. Wang, A comprehensive extension of an integrated inventory model with ordering cost reduction and permissible delay in payments, *Appl. Math. Modelling*, **37**, 4709–4716 (2013).

[13] J. Min, Y.-W. Zhou and J. Zhao, An inventory model for deteriorating items under stock-dependent demand and two-level trade credit, *Appl. Math. Modelling*, **34**, 3273–3285 (2010).

[14] W. F. Sharpe, G. J. Alexander and J. V. Bailey, *Investment*, Prentice-Hall Incorporated, Englewood Cliffs, New Jersey, (1999).

[15] H. N. Soni, Optimal replenishment policies for deteriorating items with stock sensitive demand under two-level trade credit and limited capacity, *Appl. Math. Modelling*, **37**, 5887–5895 (2013).

[16] H. M. Srivastava and B. R. K. Kashyap, *Special Functions in Queuing Theory and Related Stochastic Processes*, Academic Press, New York and London, (1982).

[17] J. R. Stokes, Dynamic cash discounts when sales volume is stochastic, *Quart. Rev. Econ. Finance*, **45**, 144–160 (2005).

[18] J.-T. Teng, J. Chen and S. K. Goyal, A comprehensive note on: An inventory model under two levels of trade credit and limited storage space derived without derivatives, *Appl. Math. Modelling*, **33**, 4388–4396 (2009).

[19] D. Varberg, E. J. Purcell and S. E. Steven, *Calculus*, Ninth edition, Pearson Education Incorporated, Upper Saddle River, New Jersey, (2007).

[20] C.-T. Yang, The optimal order and payment policies for deteriorating items in discount cash and payment policies for deteriorating items in discount cash flows analysis under the alternatives of conditionally permissible delay in payments and cash discount, *Top*, **18**, 429–443 (2010).

[21] J.-S. Yao, M.-S. Chen and H.-W. Lin, Valuation by using a fuzzy discounted cash flow model, *Expert Systems Appl.*, **28**, 209–222 (2005).


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