

# Catalogues of Efficient Minimal Weakly Balanced RMDs in Circular Periods of Three Different Sizes

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**Abstract:** In this article, efficient minimal circular weakly balanced RMDs are constructed for the situations where these designs cannot be constructed through generators. Catalogues of these efficient designs are compiled in periods of three different sizes. In the proposed designs,  $v/2$  ordered pairs of distinct treatments do not appear as their preceded value while the remaining ones appear once, where  $v$  is number of treatments to be compared.

**Keywords:** Balanced RMDs; Efficiency of Separability; CBRMDs; CSBRMDs

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## 1 Introduction

In repeated measurements designs (RMDs) more than one measurements are observed on each subject. RMDs have application in many branches of scientific inquiry, therefore, choice of RMD must be made in a way that the treatments can be efficiently compared after allowing for the residual effects. Balanced RMDs (BRMDs) are called robust to residual effects. Experimenters always prefer the minimal BRMDs because these are most economical. If minimal BRMDs cannot be constructed then weakly BRMDs should be used to balance the residual effect.

- RMD is minimal balanced if each treatment is immediately preceded once by all other treatments (excluding itself).
- RMD is minimal weakly balanced if each treatment is immediately preceded once by most of the treatments (excluding itself) but not preceded with a few of remaining ones.
- Periods are considered as circular if periods are (i) divided in circular form, or (ii) not divided in circular form but pre period is used in which treatments applied are same as in the last period.

[1] constructed balanced RMDs for some cases. [2] introduced the idea of a circular BRMDs. [3] generated some classes of BRMDs. He also constructed extra balanced (strongly balanced) RMDs. [4] and [5] generated CSBRMDs and also discussed their optimality. [6] proposed balanced and strongly BRMDs in period of different sizes. [7] constructed (i) RMDs in periods of equal and different sizes through method of cyclic shifts for  $3 \leq v \leq 10$ ,  $3 \leq p \leq 10$ , (ii) Strongly BRMD for  $3 \leq v$ ,  $p \leq 10$ , and (iii) combinatorial balanced designs for two unequal period sizes. [8] constructed BRMDs when  $v$  is odd. Using cyclic shifts, [9] and [10] proposed CSBRMDs and CBRMDs which are balanced for first order as well as second order. These designs are constructed for  $p < v$ . [11] constructed SBRMDs in circular periods using integer programming. [12] presented some infinite series for CBRMDs in equal period sizes for different cases. [13] also presented construction procedures of CBRMDs for some cases. [14] developed some generators to obtain MCSBRMDs

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for  $p_1 < p_2 < p_3$  for different cases.

[15] presented some minimal circular strongly partially BRMDs (MCSPBRMDs) in periods of equal sizes using Rule I.[16] constructed some MCSPBRMDs in periods of two different sizes.[17] presented some minimal circular weakly BRMDs (MCWBRMDs) in periods of equal sizes. [18] presented the catalogues of MCWBRMDs in two different period sizes. [19] presented MCWBRMDs for  $p_1$  even, in periods of two and three different sizes. MCWBRMDs have not been constructed for  $p_1 = r$  (odd)  $> 3$ ,  $p_2 = s$  (integer) and  $p_3 = u$  (integer). In this article, therefore, efficient MCWBRMDs in which  $v/2$  ordered pairs of distinct treatments do not appear as their preceded value while the remaining ones appear once, are constructed for  $p_1$  odd and  $p_3$  very small. Catalogues of the proposed designs are also compiled. Method of cyclic shifts (Rule I) is used to construct efficient MCWBRMDs-I in periods of three different sizes. This method is explained in Section 2. Model and efficiency measures are given in Section 3. Catalogues of efficient MCWBRMDs are presented in Section 4 for  $v \leq 70, 5 \leq p_1$  (odd)  $\leq 11, 4 \leq p_2 \leq 10$  and  $p_3 = 3$  and 4.

## 2 Method of Cyclic Shifts

Method of cyclic shifts introduced by [20] is used to construct several types of designs. Its Rule I is explained here only for the construction of MCBRMDs and MCWBRMDs in the periods of three different sizes.

- How a design is obtained from given set(s) of shifts using Rule I, is described here with the help of an example. Following arrays are obtained for  $v = 20$  and  $p = 8$  from  $S = [2, 12, 3, 4, 5, 6, 7]$ .

Get arrays from  $S = [2, 12, 3, 4, 5, 6, 7]$  using  $v$  subjects. Consider  $0, 1, \dots, v - 1$  respectively as the values of first period. Add 2 (mod 20) in these values to obtain the elements of second period, Similarly add 12, 3, 4, 5, 6 and 7 in the values of 2nd, 3rd, ..., 7th period to obtain the values of 3rd, 4th, ..., 8th periods respectively, see Table 1.

**Table 1:** Arrays obtained from  $S = [2, 12, 3, 4, 5, 6, 7]$

Periods	1	2	3	4	5	6	7	8	9	10
1	0	1	2	3	4	5	6	7	8	9
2	2	3	4	5	6	7	8	9	10	11
3	14	15	16	17	18	19	0	1	2	3
4	17	18	19	0	1	2	3	4	5	6
5	1	2	3	4	5	6	7	8	9	10
6	6	7	8	9	10	11	12	13	14	15
7	12	13	14	15	16	17	18	19	0	1
8	19	0	1	2	3	4	5	6	7	8

Periods	11	12	13	14	15	16	17	18	19	20
1	10	11	12	13	14	15	16	17	18	19
2	12	13	14	15	16	17	18	19	0	1
3	4	5	6	7	8	9	10	11	12	13
4	7	8	9	10	11	12	13	14	15	16
5	11	12	13	14	15	16	17	18	19	0
6	16	17	18	19	0	1	2	3	4	5
7	2	3	4	5	6	7	8	9	10	11
8	9	10	11	12	13	14	15	16	17	18

- **Construction of MCBRMDs and MCWBRMDs**

If  $[q_{11}, q_{12}, \dots, q_{1(p_1-1)}], [q_{21}, q_{22}, \dots, q_{2(p_2-1)}]$  and  $[q_{31}, q_{32}, \dots, q_{3(p_3-1)}]$  are the sets with  $1 \leq q_{ij} \leq v - 1$  and each of  $1, 2, \dots, v - 1$  appears once in  $S^*$ , where  $S^*$  contains (i) each element of all  $S_j$ , and (ii)  $v - [\text{sum of all elements (mod } v)]$  in each  $S_j$  then it will provide MCBRMD in  $p_1, p_2$  and  $p_3$ . If most of  $1, 2, \dots, v - 1$  appear once but a few do not appear then it will be MCWBRMD.

**Example 2.1:**  $S_1 = [2, 4, 6, 13], S_2 = [1, 8, 10], S_3 = [5, 11]$  provide MCWBRMD for  $v = 14, p_1 = 5, p_2 = 4, p_3 = 3$ .

**Proof:** Hence  $S^* = [2, 4, 6, 13, 3, 1, 8, 10, 9, 5, 11, 12]$ . Here 1, 2, ..., 13 appear once except 7 which does not appear. Therefore, it is MCWBRMD which is given below.

**Table 2:** Arrays obtained from  $S_1 = [2, 4, 6, 13]$

p	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
2	2	3	4	5	6	7	8	9	10	11	12	13	0	1
3	6	7	8	9	10	11	12	13	0	1	2	3	4	5
4	12	13	0	1	2	3	4	5	6	7	8	9	10	11
5	11	12	13	0	1	2	3	4	5	6	7	8	9	10

**Table 3:** Arrays obtained from  $S_2 = [1, 8, 10]$

p	15	16	17	18	19	20	21	22	23	24	25	26	27	28
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
2	1	2	3	4	5	6	7	8	9	10	11	12	13	0
3	9	10	11	12	13	0	1	2	3	4	5	6	7	8
4	5	6	7	8	9	10	11	12	13	0	1	2	3	4

**Table 4:** Arrays obtained from  $S_3 = [5, 11]$

p	29	30	31	32	33	34	35	36	37	38	39	40	41	42
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
2	5	6	7	8	9	10	11	12	13	0	1	2	3	4
3	2	3	4	5	6	7	8	9	10	11	12	13	0	1

MCWBRMD is obtained for  $v = 20, p_1 = 8, p_2 = 6, p_3 = 4$  using Table 1, 2 and 3 together.

### 3 Efficiency Measures

#### 3.1 Efficiency of Residual Effects

Commonly used model for circular RMDs is, see [21].

$$Y = \mu E + D\delta + R\rho + Uv + PII + \varepsilon \tag{1}$$

Canonical efficiency factors are non-zero Eigen values of information matrix  $C^*$ , and efficiency factor for both direct and carryover effect can be regarded as the harmonic mean of non-zero Eigen values of their respective information matrix, see [22] and [23]. Design will be suitable to estimate residual effects when value of  $E_r$  is high.

#### 3.2 Efficiency of Separability

[24] developed following efficiency of Separability ( $E_s$ ) for BRMDs.

$$E_s = [1 - 1/(v\sqrt{v-1})] * 100\%$$

### 4 Construction of efficient MCWBRMDs for $p_1, p_2$ and $p_3$

In this Section, efficient MCWBRMDs are constructed for  $p_1$  odd and  $p_3$  very small using the following sets of shifts with  $p_1 = r > 3, p_2 = s, p_3 = u$ .

$$S_j = [q_{j1}, q_{j2}, \dots, q_{j(r-1)}]; \quad j = 1, 2, \dots, i.$$

$$S_{i+1} = [q_{(i+1)1}, q_{(i+1)2}, \dots, q_{(i+1)(s-1)}]$$

$$S_{i+2} = [q_{(i+2)1}, q_{(i+2)2}, \dots, q_{(i+2)(u-1)}]$$

Each of  $1, 2, \dots, v-1$  appears once in  $S^*$  but  $v/2$  does not appear.

Where  $S^*$  contains (i) Each element of all  $S_j$ , and (ii)  $v - [\text{Sum of all elements (mod } v) \text{ in each } S_j]$ .

**Procedure 4.1:** Efficient MCWBRMDs for  $v = p_1 + p_2 + p_3 + 2; i$  odd,  $p_1$  odd,  $p_2$  even and  $p_3 = 3$

$v$	$p_1$	$p_2$	$p_3$	sets	Er	Es
14	5	4	3	[13,2,4,6]+[1,8,10]+[11,5]	0.80	0.89
24	5	4	3	[1,2,4,14]+[6,7,9,18]+[16,17,11,15]+ [23,10,20]+[21,5]	0.80	0.94
34	5	4	3	[6,2,8,5]+[1,7,12,10]+[23,31,24,15]+ [16,26,20,21]+[11,14,25,30]+[27,29,18]+[3,33]	0.80	0.96
44	5	4	3	[1,12,13,4]+[6,11,8,9]+[7,2,3,17]+ [16,18,29,20]+[26,24,21,38]+[28,35,43,31]+ [32,33,36,41]+[37,30,40]+[19,27]	0.80	0.97
54	5	4	3	[1,17,18,5]+[52,20,11,16]+[10,4,25,8]+ [23,2,14,12]+[41,35,24,19]+[44,29,30,31]+ [32,34,21,36]+[37,45,49,47]+[42,33,40,48]+[51,46,50]+[26,22]	0.80	0.97
64	5	4	3	[1,8,13,39]+[27,18,61,15]+[12,58,5,10]+ [6,25,4,20]+[21,2,35,37]+[36,22,26,28]+ [31,23,47,57]+[19,44,45,46]+[42,29,50,41]+ [24,49,54,17]+[52,40,55,56]+[62,59,60]+[14,63]	0.80	0.98
16	7	4	3	[1,2,9,13,6,7]+[5,12,11]+[3,15]	0.83	0.91
30	7	4	3	[1,2,3,4,6,9]+[7,8,12,11,13,14]+[26,18,19,21,27,22]+ [23,28,29]+[20,16]	0.85	0.95
44	7	4	3	[2,1,21,42,12,7]+[8,9,10,11,13,14]+[16,15,5,18,20,19]+ [6,24,27,28,31,35]+[29,30,32,33,34,36]+[37,17,40]+ [41,43]	0.85	0.97
58	7	4	3	[1,2,39,56,10,3]+[8,48,15,12,13,14]+[38,16,28,19,20,21]+ [9,22,23,26,18,36]+[27,51,54,7,31,45]+[37,11,25,41,42,43]+ [44,46,47,24,49,50]+[35,53,34]+[55,57]	0.85	0.98
18	9	4	3	[10,7,8,12,5,3,6,1]+[11,13,14]+[15,17]	0.86	0.92
36	9	4	3	[1,2,3,14,6,7,8,9]+[4,12,13,15,17,16,27,19]+ [20,11,35,31,30,25,26,28]+[29,24,32]+[33,5]	0.88	0.96
54	9	4	3	[10,2,3,11,6,7,8,9]+[1,5,13,16,17,18,21,34]+ [22,23,25,24,26,19,28,20]+[30,32,31,33,14,35,12,36]+ [38,39,41,42,43,44,45,46]+[15,49,50]+[51,53]	0.88	0.97
20	11	4	3	[11,12,1,2,3,6,7,8,9,17]+[13,15,18]+[5,19]	0.87	0.93
42	11	4	3	[1,2,4,6,12,38,8,9,10,11]+[5,14,15,17,37,18,19,22,23,27]+ [24,7,26,20,30,28,31,32,33,34]+[35,36,39]+[3,41]	0.90	0.97
64	11	4	3	[50,12,30,4,6,7,8,29,36,11]+[17,18,19,20,21,22,39,13,14,15]+ [23,24,25,26,27,28,2,3,33,34]+[9,10,35,38,40,41,42,43,44,45]+ [46,47,48,49,1,51,52,53,55,56]+[57,16,60]+[61,5]	0.90	0.98
18	7	6	3	[7,6,3,5,2,1]+[8,10,11,13,14]+[15,17]	0.74	0.92
32	7	6	3	[1,2,3,5,6,11]+[8,9,12,13,14,30]+[25,18,19,20,27,22]+ [23,24,15,28,17]+[26,31]	0.86	0.95
46	7	6	3	[1,2,21,4,5,7]+[8,9,11,12,13,14]+[15,17,18,19,10,43]+ [22,20,24,27,26,29]+[37,31,32,33,35,28]+[30,39,40,41,42]+[3,45]	0.86	0.97
60	7	6	3	[18,7,42,39,12,13]+[8,9,11,5,14,3]+[15,17,1,38,6,27]+ [22,34,24,25,21,28]+[32,54,31,23,35,36]+[37,19,40,41,2,43]+ [44,45,46,47,48,20]+[52,53,33,55,56]+[57,59]	0.86	0.98
20	9	6	3	[13,2,12,4,6,7,8,9]+[1,3,14,15,16]+[17,18]	0.87	0.93

38	9	6	3	[27,14,34,5,4,7,8,9]+[10,11,12,13,16,2,17,18]+ [20,21,22,24,25,26,28,1]+[29,31,32,33,35]+[3,37]	0.88	0.96
56	9	6	3	[1,2,3,4,16,6,8,9]+[50,12,13,14,15,5,17,18]+ [19,20,21,23,11,55,26,27]+[49,29,32,53,34,35,36,37]+ [52,40,41,42,43,44,45,46]+[47,48,30,10,38]+[54,25]	0.88	0.97
22	11	6	3	[1,2,3,5,6,7,8,9,12,20]+[13,10,16,17,18]+[19,21]	0.89	0.93
44	11	6	3	[1,2,3,4,43,6,7,8,26,11]+[12,14,15,16,17,18,19,20,9,23]+ [24,25,27,10,28,35,33,34,31,32]+[30,37,38,39,40]+[42,5]	0.90	0.97
66	11	6	3	[45,30,63,4,5,6,8,9,10,11]+[34,13,15,28,27,18,19,20,21,55]+ [23,24,25,26,17,16,29,2,31,12]+[36,37,38,39,40,41,42,43,44,1]+ [46,47,48,49,50,53,54,52,22,56]+[57,32,60,61,62]+[64,65]	0.90	0.98
22	9	8	3	[1,2,3,5,20,8,7,14]+[10,13,12,15,9,16,18]+[19,21]	0.89	0.93
40	9	8	3	[2,1,3,4,6,7,10,8]+[9,11,28,14,35,16,17,18]+ [21,22,19,23,24,27,25,13]+[29,31,32,33,34,15,36]+[37,5]	0.89	0.96
58	9	8	3	[2,48,3,24,6,7,9,18]+[11,10,13,14,15,17,16,8]+ [19,20,21,28,23,43,26,27]+[22,30,32,33,34,35,36,37]+ [38,39,49,41,44,4,45,46]+[47,1,50,51,52,53,54]+[55,5]	0.89	0.98
24	11	8	3	[4,3,2,7,1,9,8,11,18,23]+[13,15,16,6,17,19,20]+[21,5]	0.90	0.94
46	11	8	3	[1,2,3,4,6,8,10,11,18,22]+[12,14,15,16,17,19,20,21,41,42]+ [25,24,26,27,28,29,30,32,37,33]+[35,36,34,38,39,40,9]+[44,5]	0.90	0.97
68	11	8	3	[1,2,3,4,67,6,7,8,9,42]+[13,14,15,16,17,18,19,20,21,22]+ [23,24,25,26,28,27,12,31,33,32]+[35,37,38,39,40,41,10,43,44,45]+ [46,47,48,49,50,51,52,53,62,56]+[57,59,60,61,54,63,64]+[65,5]	0.91	0.98
26	11	10	3	[19,2,3,4,25,6,7,8,10,11]+[15,16,12,17,1,18,21,22,20]+[23,5]	0.91	0.94
48	11	10	3	[45,33,1,5,6,7,8,9,17,11]+[12,14,15,16,10,18,19,20,21,34]+ [23,22,32,31,30,28,29,25,26,38]+[35,37,27,39,40,41,42,43,44]+ [3,47]	0.91	0.97
70	11	10	3	[25,2,3,4,69,6,7,8,65,12]+[11,14,16,15,17,18,19,20,23,21]+ [22,24,44,46,27,28,29,34,32,33]+[30,37,38,39,40,41,43,1,42,26]+ [45,47,49,48,50,51,53,54,55,56]+[57,58,59,60,61,64,63,10,66]+ [67,5]	0.91	0.98

**Procedure 4.2:** Efficient MCWBRMDs for  $v = p_1 + p_2 + p_3 + 2$ ;  $i$  even,  $p_1, p_2$  odd and  $p_3 = 3$ .

$v$	$p_1$	$p_2$	$p_3$	sets	Er	Es
24	7	5	3	[1,7,4,14,20,23]+[8,11,9,13,6,15]+[16,18,19,2]+[21,5]	0.85	0.94
38	7	5	3	[2,1,18,4,6,8]+[7,9,11,12,13,14]+[15,17,3,20,22,21]+ [23,24,25,34,28,29]+[30,32,33,26]+[35,5]	0.85	0.96
52	7	5	3	[2,1,33,4,6,7]+[28,3,11,19,13,21]+[15,17,18,12,20,14]+ [42,23,24,47,27,29]+[30,31,10,34,35,36]+[37,38,40,22,41,43]+ [44,45,46,48]+[49,5]	0.85	0.97
66	7	5	3	[1,2,3,45,65,10]+[24,56,11,18,13,14]+[27,17,12,19,20,21]+ [46,39,36,25,15,28]+[32,31,29,34,35,7]+[37,23,40,41,42,43]+ [44,4,22,47,49,50]+[51,53,54,55,8,57]+[58,60,61,26]+[63,64]	0.85	0.98
28	9	5	3	[24,2,3,4,20,6,7,10]+[12,11,13,9,15,16,27,19]+[17,21,23,1]+[25,5]	0.87	0.95
46	9	5	3	[1,42,17,3,45,6,8,9]+[11,10,35,13,19,16,4,18]+ [15,21,22,20,24,25,2,28]+[29,30,32,33,34,14,36,37]+ [38,40,41,26]+[43,5]	0.88	0.97
64	9	5	3	[1,3,4,63,6,60,8,9]+[13,12,10,20,15,16,17,18]+ [14,19,21,29,22,24,11,27]+[28,30,31,2,34,35,36,37]+ [33,39,40,41,42,44,45,46]+[47,49,50,51,52,53,43,55]+ [56,58,59,26]+[61,5]	0.88	0.98
32	11	5	3	[1,2,3,31,6,7,12,9,10,11]+[14,8,15,18,17,19,20,21,24,23]+ [22,26,27,28]+[29,5]	0.89	0.95
54	11	5	3	[12,3,4,53,6,7,8,9,33,11]+[2,14,15,17,1,18,19,20,21,22]+ [23,24,25,28,29,38,31,34,10,50]+[30,35,39,36,40,41,42,43,44,45]+ [46,47,48,26]+[52,5]	0.90	0.97
30	9	7	3	[12,2,13,6,29,7,8,9]+[10,1,3,25,16,17,19,18]+[20,22,23,24,14,26]+ [27,5]	0.86	0.95

48	9	7	3	[1,44,47,6,4,7,8,25]+[10,12,13,14,15,16,35,18]+ [19,21,22,23,9,26,27,28]+[31,30,32,39,34,20,36,37]+ [38,33,40,41,43,3]+[46,5]	0.89	0.97
66	9	7	3	[1,42,53,4,65,6,8,12]+[10,9,13,14,15,16,17,27]+ [18,19,21,20,22,23,24,25]+[29,31,32,2,34,35,36,37]+ [38,39,57,54,30,43,44,46]+[47,49,50,51,52,3,41,55]+ [56,40,58,59,61,62]+[64,5]	0.89	0.98
34	11	7	3	[1,2,3,15,18,33,7,9,29,11]+[12,14,4,6,16,19,20,21,22,23]+ [24,25,26,10,27,30]+[31,5]	0.90	0.96
56	11	7	3	[33,47,3,4,55,36,7,8,9,12]+[11,14,15,17,16,18,19,20,23,2]+ [25,21,26,27,24,29,30,31,32,34]+[35,37,38,39,22,41,42,43,44,45]+ [46,40,48,49,51,52]+[53,5]	0.90	0.97
36	11	9	3	[1,2,4,33,6,5,7,9,14,17]+[12,8,13,16,15,19,20,21,22,23]+ [24,25,26,27,28,30,31,32]+[3,35]	0.91	0.96
58	11	9	3	[1,2,3,4,57,6,20,8,54,45]+[12,14,15,16,17,18,19,7,21,22]+ [23,24,25,26,27,28,9,30,33,34]+[35,36,38,39,40,41,42,43,44,11]+ [46,47,48,49,50,51,53,10]+[56,5]	0.91	0.98

**Procedure 4.3:** Efficient MCWBRMDs for  $v = p_1 + p_2 + p_3 + 2$ ;  $i$  odd,  $p_1, p_2$  odd and  $p_3 = 4$

$v$	$p_1$	$p_2$	$p_3$	sets	$Er$	$Es$
18	7	5	4	[1,2,3,5,6,15]+[8,11,10,13]+[14,7,17]	0.86	0.92
32	7	5	4	[27,10,4,5,7,9]+[30,6,15,12,13,17]+[11,18,19,23,21,22]+ [20,25,26,1]+[28,29,31]	0.86	0.95
46	7	5	4	[2,4,5,6,30,44]+[10,9,11,12,13,17]+[14,15,18,19,3,7]+ [22,27,24,37,21,28]+[31,33,32,34,35,36]+[38,39,40,41]+[43,8,45]	0.86	0.97
60	7	5	4	[49,28,24,5,9,7]+[6,10,11,22,15,14]+[16,17,18,34,1,21]+ [23,4,25,26,27,3]+[31,32,33,19,35,36]+[39,38,40,41,2,43]+ [45,46,47,48,20,50]+[52,53,29,55]+[56,57,59]	0.86	0.98
20	9	5	4	[1,2,3,4,5,16,7,14]+[12,11,13,15]+[6,17,18]	0.87	0.93
38	9	5	4	[27,14,3,4,5,7,1,9]+[10,12,13,2,15,16,17,18]+ [21,22,24,23,25,36,29,28]+[26,30,32,33]+[35,8,37]	0.88	0.96
56	9	5	4	[51,27,3,4,5,6,54,11]+[9,12,1,14,15,16,17,18]+ [20,21,22,23,24,25,26,44]+[13,29,31,32,33,34,35,37]+ [40,39,41,42,43,2,45,46]+[47,49,50,30]+[52,53,55]	0.88	0.97
22	11	5	4	[1,2,4,3,5,15,8,12,7,21]+[13,14,6,17]+[18,19,9]	0.89	0.93
44	11	5	4	[1,2,3,5,16,40,7,8,10,11]+[12,14,15,4,17,18,19,21,20,23]+ [24,25,26,27,28,9,39,32,33,34]+[36,37,38,30]+[41,42,43]	0.90	0.97
66	11	5	4	[12,61,3,4,5,6,13,64,10,11]+[45,14,15,16,17,19,18,53,21,22]+ [7,23,25,26,27,28,29,32,2,31]+[36,37,38,39,40,41,42,43,44,1]+ [46,47,48,49,50,52,51,20,54,56]+[58,59,60,30]+[63,8,65]	0.90	0.98
22	9	7	4	[10,2,4,3,5,6,7,21]+[1,13,14,15,16,17]+[19,20,9]	0.89	0.93
40	9	7	4	[1,2,3,4,5,12,38,9]+[34,7,14,13,15,16,17,18]+ [22,19,23,24,25,11,27,28]+[29,30,31,32,33,35]+[37,8,39]	0.89	0.96
58	9	7	4	[20,12,3,4,25,38,7,9]+[11,13,2,14,15,16,17,18]+ [1,21,28,23,5,24,26,27]+[30,31,32,33,34,35,36,37]+ [6,39,41,40,42,44,47,46]+[48,49,50,51,52,53]+[55,8,57]	0.89	0.98
24	11	7	4	[1,4,19,2,5,6,7,22,9,11]+[15,14,16,17,18,3]+[21,8,23]	0.82	0.94
46	11	7	4	[1,2,40,28,3,6,7,44,13,11]+[10,14,15,16,17,18,19,20,21,22]+ [25,24,26,27,5,9,30,31,33,34]+[36,37,38,39,4,41]+[43,8,45]	0.90	0.97
68	11	7	4	[23,2,3,30,5,6,7,66,9,11]+[1,14,15,16,17,19,18,49,21,22]+ [13,25,24,26,27,28,29,4,32,33]+[36,37,38,39,40,41,10,43,44,45]+ [58,47,48,50,20,51,52,53,55,56]+[46,59,60,61,62,63]+[65,8,67]	0.91	0.98
26	11	9	4	[14,1,4,8,7,10,11,6,15,25]+[12,5,16,18,17,19,20,21]+[23,24,9]	0.91	0.94
48	11	9	4	[26,17,3,4,5,6,7,46,9,11]+[13,14,15,16,2,38,19,20,21,22]+ [25,18,1,27,28,31,29,32,33,34]+[36,23,39,37,40,41,42,43]+[45,8,47]	0.91	0.97
70	11	9	4	[22,2,3,5,4,65,67,12,10,11]+[13,14,16,15,17,18,19,23,21,46]+ [20,24,25,28,1,29,30,31,32,33]+[37,36,38,39,40,44,41,26,42,43]+ [47,45,49,48,51,50,53,52,54,56]+[58,59,60,61,62,6,63,64]+[7,68,69]	0.91	0.98

**Procedure 4.4:** Efficient MCWBRMDs for  $v = p_1 + p_2 + p_3 + 2$ ;  $i$  odd,  $p_1, p_2$  even and  $p_3 = 4$

$v$	$p_1$	$p_2$	$p_3$	sets	Er	Es
26	7	6	4	[25,2,4,5,6,7]+[8,1,11,17,12,15]+[16,18,19,20,21]+[23,24,9]	0.78	0.94
40	7	6	4	[35,2,8,3,19,7]+[4,28,10,12,13,14]+[16,17,18,11,21,22]+ [24,23,25,26,5,30]+[31,32,33,34,1]+[37,38,9]	0.86	0.96
54	7	6	4	[23,9,4,10,50,7]+[20,3,11,12,13,14]+[17,16,18,19,2,21]+ [22,8,25,24,28,29]+[31,32,33,34,1,36]+[38,39,30,41,42,43]+ [44,45,46,47,40]+[51,52,53]	0.86	0.97
68	7	6	4	[48,43,3,23,25,56]+[9,10,11,12,13,15]+[16,29,17,19,20,21]+ [22,4,24,5,27,28]+[7,32,33,31,35,36]+[38,39,63,41,42,2]+ [44,45,46,37,1,50]+[51,53,54,55,18,57]+[59,60,61,62,40]+[65,8,67]	0.86	0.98
30	9	6	4	[2,1,3,4,5,7,9,8]+[11,13,14,12,16,17,23,19]+[26,22,18,10,24]+[6,27,29]	0.88	0.95
48	9	6	4	[32,13,14,5,44,7,8,9]+[11,2,3,15,4,16,17,18]+[19,1,22,23,21,26,28,25]+ [31,30,20,33,34,38,36,37]+[35,39,40,41,42]+[6,46,47]	0.88	0.97
66	9	6	4	[54,2,3,4,5,11,48,9]+[8,12,13,15,14,16,17,18]+ [10,21,20,22,23,24,26,27]+[29,30,32,31,34,7,36,37]+ [35,39,61,41,43,42,44,45]+[38,49,50,51,52,53,1,55]+ [56,57,58,59,40]+[6,64,65]	0.89	0.98
34	11	6	4	[1,2,3,4,19,28,7,8,10,11]+[12,15,18,14,16,30,20,21,22,23]+ [24,25,5,26,29]+[6,32,33]	0.90	0.96
56	11	6	4	[1,29,3,4,5,23,7,8,12,11]+[52,14,15,16,17,18,19,20,21,22]+ [13,23,25,26,27,2,30,31,36,34]+[37,33,38,39,40,41,42,43,44,45]+ [35,47,48,49,50]+[53,54,55]	0.90	0.95
38	13	6	4	[1,33,3,4,5,34,8,7,10,11,9,13]+[16,12,18,15,20,17,22,25,21,27,24,26]+ [29,30,31,32,2]+[6,35,36]	0.91	0.96
64	13	6	4	[1,14,3,5,60,7,8,9,10,47,11,13]+[2,15,17,18,19,28,21,22,23,24,25,26]+ [27,30,31,20,33,34,35,36,38,37,39,59]+ [41,43,45,44,46,12,48,49,50,52,51,53]+[54,56,57,58,40]+[6,62,63]	0.92	0.98
32	9	8	4	[2,3,4,5,28,7,26,9]+[11,15,1,14,13,17,18,19]+ [10,21,23,24,22,8,27]+[6,30,31]	0.89	0.95
50	9	8	4	[1,42,29,3,5,46,7,8]+[31,22,13,14,15,16,17,18]+ [20,19,21,12,23,24,27,28]+[30,11,32,33,34,35,36,37]+ [38,39,40,41,10,43,44]+[47,48,49]	0.89	0.97
68	9	8	4	[13,2,54,42,5,6,7,66]+[11,31,1,14,15,19,17,18]+ [20,16,21,22,23,24,26,27]+[29,30,33,12,35,32,37,36]+ [38,39,40,4,41,43,44,45]+[48,49,50,51,52,53,3,55]+ [56,57,58,59,60,61,62]+[65,8,67]	0.89	0.98
36	11	8	4	[17,2,3,5,4,7,32,8,9,11]+[13,14,16,15,1,19,24,21,22,23]+ [20,25,26,27,28,29,30]+[6,34,35]	0.91	0.96
58	11	8	4	[35,50,26,17,5,6,7,56,9,11]+[13,14,15,18,4,16,19,20,21,22]+ [23,24,3,25,28,27,30,32,31,34]+[1,37,38,39,40,42,41,43,44,45]+ [46,47,48,49,2,51,52]+[55,8,57]	0.91	0.98
38	11	10	4	[1,2,3,4,5,6,7,9,36,29]+[13,15,17,14,18,16,20,21,22,23]+ [24,25,26,28,27,10,30,32,33]+[35,8,37]	0.91	0.96
60	11	10	4	[1,22,35,4,5,6,7,8,59,10]+[14,3,13,17,16,18,20,46,19,2]+ [11,24,26,25,27,29,28,31,33,34]+[36,37,39,38,40,41,42,43,45,44]+ [21,47,48,49,50,51,52,53,54]+[57,58,9]	0.91	0.98

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