Journal of Statistics Applications & Probability
An International Journal

"http://dx.doi.org/10.18576/jsap/120112

Analysis of Generalized Inverted Exponential Distribution under Adaptive Type-I Progressive Hybrid Censored Competing Risks Data

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Received: 19 Feb. 2022, Revised: 21 Apr. 2022, Accepted: 16 May 2022
Published online: 1 Jan. 2023

Abstract: The estimation of the unknown parameters of generalized inverted exponential distribution under adaptive type-I progressive hybrid censored scheme (AT-I PHCS) with competing risks data will be discussed. The reason why AT-I PHCS has exceeded other failure censored types; Time censored types enable analysts to accomplish their trials and experiments in a shorter time and with higher efficiency. In this regards, we obtain the maximum likelihood estimation of the parameters and the asymptotic confidence intervals for the unknown parameters. Further, Bayes estimates of the parameters which obtained based on squared error and LINEX loss functions under the assumptions of independent gamma priors of the scale parameters. For Bayesian estimation, we take advantage of Markov Chain Monte Carlo techniques to derive Bayesian estimators and the credible intervals. Finally, two data sets with Monte Carlo simulation study and a real data set are analyzed for illustrative purposes.

Keywords: Generalized inverted exponential distribution, Adaptive type-I progressive hybrid censored scheme, Competing risks, Maximum likelihood estimation, Bayesian estimation, Markov Chain Monte Carlo, Squared error and LINEX loss functions

1 Introduction

In the life-testing and reliability studies, both type-I and type-II censoring schemes are widely used. They can be described as follows: consider n identical unites are placed in the test, in type-I censoring schemes, the experiment continues up to a predetermined time T. However, in type-II censoring schemes, the experiment is terminated when a predetermined number of failures $r < n$ occurs. The combination of type-I and type-II censoring schemes is known as the hybrid censoring scheme which was first introduced by [1] in the context of life-testing experiments. In type-I hybrid censoring scheme, the life test experiment is terminated at a random time $T^* = \min(\mathrm{X}_{(r)}, T)$. [2] proposed a new hybrid censoring scheme called a type-II hybrid censoring scheme in which the experiment would be terminated at the random time $T^* = \max(X_{(r)}, T)$. These schemes do not allow to remove the units from the experiment at any time point other than the terminal point. To deal with this problem, a more general censoring scheme called progressive type-II censoring is used.

The progressive type-II censoring scheme can be conducted as follows: consider n identical units are put in a lifetime test and r is a predetermined number of units to be failed. At the time of the first failure $X_{(1)}$, R_1 units are randomly removed from the remaining $n-1$ surviving units. Then at the second failure time $X_{(2)}$, R_2 units of the remaining $n-2-R_1$ units are randomly removed. And so on, at the time of the rth failure $X_{(r)}$ all the remaining units are removed, that is, $R_r = n - r - \sum_{i=1}^{r-1} R_i$. The progressively censoring scheme $R_1, R_2, \ldots, R_r$ are fixed and predetermined prior to the study.

Combining progressive type-II and hybrid type-I censoring schemes, [3] introduced type-I progressive hybrid censoring scheme. If the rth failure occurs before the time point T, the experiment stops at the time point $X_{(r)}$, suppose the rth failure does not occur before the time point T and only D failures occur before the time point T, then at the time point T the experiment terminates and all the remaining units are removed. [4] proposed a type-II progressive hybrid censoring scheme, the experiment is terminated at random time $T^* = \max(X_{(r)}, T)$. Adaptive progressive hybrid censoring scheme, proposed by [5], called AT-I PHCS, which assures the termination of the life testing experiment at a fixed time T, and results in a higher efficiency in estimations.

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The AT-I PHCS can be described as: suppose n identical units placed under testing with progressive censoring scheme $R_i,\ i=1,\ldots,r,\ 1\le r\le n$, and the experiment is terminated at a prefixed time T, where $T\in(0,\infty)$. At the time $X_{(1)}$, R_1 of the remaining units are randomly removed, at the time $X_{(2)}$, R_2 of the remaining units are randomly removed and so on. Let the number of failures that occur before time T be D. If the rth failure $X_{(r)}$ occur before time T, the process will not stop but continue to observe failures without any further withdrawals until reach time T. Then, all remaining units $R_D^*=n-D-\sum_{i=1}^{D}R_i$ are removed at T and the experiment is terminated. The progressive censoring scheme become $R_1,R_2,\ldots,R_r,R_{r+1},\ldots,R_D$, where $R_r=R_{r+1}=\cdots=R_D=0$. Otherwise, when $X_{(r)}>T$ the process will have a progressive censoring scheme as $R_1,R_2,\ldots,R_D$. This censoring ensures terminate the experiment at a predetermined time in any case of number of failures, and it is useful when the time is the main goal in the experiment. But in an adaptive type-II progressive hybrid censoring (introduced by [6]), no units will be removed when the experimental time passes time T.

In reliability analysis, the failure of units at the same time may be attributable to more than one cause. These causes are competing for the failure of the experimental unit. In the statistical literature, this problem is known as the competing risks model. The causes of failure in the competing risks data analysis can be assumed to be dependent or independent where the data consists of a failure time and the associated cause of failure. [7] analyzed the competing risks data under AT-I PHCS with the assumption of exponential distribution. The exponential distribution has a constant failure rate, so it has serious limitations in modeling lifetime data. In addition, [8] investigated the inference for Weibull distribution under AT-I PHCS in the presence of competing risks data. [9] investigated E-Bayesian estimation for the Weibull distribution under AT-I PHCS with competing risks data. [10] investigated estimations of competing lifetime data from inverse Weibull distribution under adaptive progressively hybrid censored. [11] introduced generalized inverted exponential distribution (GIED) with two parameter is capable of modeling diverse shapes of failure rates and thus various shapes of aging criteria.

This paper goal is to analyze the competing risk model under AT-I PHCS using GIED. We assume the lifetime under the competing risks have independent GIED with common shape parameter. We derive the maximum likelihood estimators (MLEs) and the approximate confidence intervals (ACIs); also, obtained Bayes estimators under squared error (SE) and linear exponential (LINEX) loss functions and Bayes credible intervals (BCIs) using gamma priors based on Markov Chain Monte Carlo (MCMC) techniques, Bayes estimates based on squared error loss function under the assumption of independent gamma priors were produced. In terms of minimum mean square errors, the simulation results show that Bayes estimates applying the importance sampling technique better than the other estimates. We consider a simulation study and a real data set and see how the different models work in the practical situation.

The rest of this paper is organized as follows. In Section 2, we introduce the model description and the notation used throughout this paper. The maximum likelihood estimation of the unknown parameters is established in Section 3. In Section 4, we discuss Bayes estimators under SE and LINEX loss function using MCMC techniques. In Section 5, an simulation study is given to study the effectiveness of the proposed estimation of the unknown parameters. In Section 6, analysis of real data set is presented. Finally the conclusion is given in Section 7.

2 Model description

Consider a life time experiment with $n\in N$ identical units, where its lifetimes are described by independent and identically distributed (i.i.d) random variables $X_1,X_2,\ldots,X_n$. Without loss of generality, assume that there are only two causes of failure. We have $X_i=\min\{X_{1i},X_{2i}\}$ for $i=1,\ldots,n$, where $X_{ki},(k=1,2)$ denotes the latent failure time of the ith unit under the kth cause of failure. We assume that the latent failure times X_{1i} and X_{2i} are independent and the pairs (X_{1i},X_{2i}) are i.i.d. Assume that the failure times follow the GIED with the same shape parameter (α) and different scale parameters $(\lambda_k,\ k=1,2)$. The probability density function and cumulative distribution function of GIED are given by

$$f_k(x;\alpha,\lambda_k)=\frac{\alpha\lambda_k}{x^2}e^{-\lambda_k/x}\left(1-e^{-\lambda_k/x}\right)^{\alpha-1},\ x>0\,\alpha,\lambda_k>0,\ k=1,2,$$

$$F_k(x;\alpha,\lambda_k)=1-\left(1-e^{-\lambda_k/x}\right)^{\alpha},\ x>0,\ \alpha,\lambda_k>0,\ k=1,2,$$

and the corresponding survival function $\bar{F}_k$ and the failure rate function h_k are given by

$$\bar{F}_k=\left(1-e^{-\lambda_k/x}\right)^{\alpha}\quad\text{and}\quad h_k=\frac{\alpha\lambda_k}{x^2\left(e^{-\lambda_k/x}-1\right)},\ x>0,\ \alpha,\lambda_k>0,\ k=1,2. \tag{1}$$

The hazard rate function of GIED can be increasing, or decreasing but not constant depending on the value of the shape parameter, it has a unimodal and right skewed density function and in many situations, the GIED provides a better fit than gamma and Weibull. It has many applications in several areas of life testing such as horse racing, supermarket queues, sea currents and wind speeds and it has a great ability to synthesize different forms of failure rates.

In the presence of competing risks, the data from an AT-I PHCS is as follows:

$$(X_{(1)},\delta_1,R_1),\ldots,(X_{(r-1)},\delta_r,R_{r-1}),(X_{(r)},\delta_r,0),\ldots,(X_{(D)},\delta_D,0),(T,R_D^*),$$

where δ_i is the indicator denoting the cause of failure, D denote the number of failures before time T and R_D^* is the number of remaining units left at the time point T with $R_r = R_{r+1} = \cdots = R_D = 0$. Here, $\delta_i = k,\ k = 1,2$ means the unit i has failed due to cause k. Let

$$I_1(\delta_i=1)=\begin{cases}1, & \delta_i=1\\ 0, & \text{else}\end{cases} \quad \text{and} \quad I_2(\delta_i=2)=\begin{cases}1, & \delta_i=2\\ 0, & \text{else}\end{cases}$$

then the random variables $D_1 = \sum_{i=1}^{D} I_1(\delta_i = 1)$ and $D_2 = \sum_{i=1}^{D} I_2(\delta_i = 2)$ describe the number of failures due to the first and the second cause of failures, respectively. Both D_1 and D_2 follow binomial distributions with sample size D.

For a given censoring scheme $R_1,R_2,\ldots,R_{r-1},0,\ldots,0,R_D^*$, the likelihood function of the observed data $(X_{(1)},\delta_1),\ldots,(T,R_D^*)$ is given by

$$L = c_D\prod_{i=1}^{D}\left\{\left[f_1(x_{(i)})\bar{F}_2(x_{(i)})\right]^{I(\delta_i=1)}\left[f_2(x_{(i)})\bar{F}_1(x_{(i)})\right]^{I(\delta_i=2)}\left[\bar{F}_1(x_{(i)})\bar{F}_2(x_{(i)})\right]^{R_i}\right\}[\bar{F}_1(T)\bar{F}_2(T)]^{R_D^*},$$

where $c_D = \prod_{i=1}^{D} r-i+1-\sum_{D=1}^{r} R_D$.

Applying the identity $f_k = h_k\overline{F}_k$, we can write the likelihood function as follows

$$L = c_D\prod_{i=1}^{D}\left\{\left[h_1(x_{(i)})\right]^{I(\delta_i=1)}\left[h_2(x_{(i)})\right]^{I(\delta_i=2)}\left[\bar{F}_1(x_{(i)})\bar{F}_2(x_{(i)})\right]^{1+R_i}\right\}[\bar{F}_1(T)\bar{F}_2(T)]^{R_D^*}, \tag{2}$$

where $D = D_1 + D_2$ and $D > 0$.

3 Maximum likelihood estimation

From (1) and (2), the likelihood function ignoring the normalized constant can be written as follows

$$\begin{aligned}L(\lambda_1,\lambda_2,\alpha) \propto{}& \alpha^D\lambda_1^{D_1}\lambda_2^{D_2}\prod_{i=1}^{D}x_{(i)}^{-2D}e^{-\frac{1}{x_{(i)}}(D_1\lambda_1+D_2\lambda_2)}\left(1-e^{-\lambda_1/x_{(i)}}\right)^{\alpha(1+R_i)-D_1}\\ &\times\left(1-e^{-\lambda_2/x_{(i)}}\right)^{\alpha(1+R_i)-D_2}\left(1-e^{-\lambda_1/T}\right)^{\alpha R_D^*}\left(1-e^{-\lambda_2/T}\right)^{\alpha R_D^*}.\end{aligned} \tag{3}$$

The log-likelihood $\ln L(\lambda_1,\lambda_2,\alpha)$ corresponding to equation (3), is given by

$$\begin{aligned}\ln L(\lambda_1,\lambda_2,\alpha) \propto{}& D\ln\alpha + D_1\ln\lambda_1 + D_2\ln\lambda_2 - 2D\sum_{i=1}^{D}\ln x_{(i)} - (D_1\lambda_1 + D_2\lambda_2)\sum_{i=1}^{D}x_{(i)}^{-1}\\ &+(\alpha(1+R_i)-D_1)\sum_{i=1}^{D}\ln(1-e^{-\lambda_1/x_{(i)}}) + (\alpha(1+R_i)-D_2)\sum_{i=1}^{D}\ln(1-e^{-\lambda_2/x_{(i)}})\\ &+\alpha R_D^*\ln(1-e^{-\lambda_1/T}) + \alpha R_D^*\ln(1-e^{-\lambda_2/T}).\end{aligned} \tag{4}$$

The first order derivations of equation (4), with respect to λ_1, λ_2 and α, are given respectively by

$$\frac{\partial\ln L(\lambda_1,\lambda_2,\alpha)}{\partial\lambda_1} = \frac{D_1}{\lambda_1} - D_1\sum_{i=1}^{D}x_{(i)}^{-1} + (\alpha(1+R_i)-D_1)\sum_{i=1}^{D}\frac{x_{(i)}^{-1}e^{-\lambda_1/x_{(i)}}}{(1-e^{-\lambda_1/x_{(i)}})} + \frac{T^{-1}\alpha R_D^* e^{-\lambda_1/T}}{(1-e^{-\lambda_1/T})},$$

$$\frac{\partial\ln L(\lambda_1,\lambda_2,\alpha)}{\partial\lambda_2} = \frac{D_2}{\lambda_2} - D_2\sum_{i=1}^{D}x_{(i)}^{-1} + (\alpha(1+R_i)-D_2)\sum_{i=1}^{D}\frac{x_{(i)}^{-1}e^{-\lambda_2/x_{(i)}}}{(1-e^{-\lambda_2/x_{(i)}})} + \frac{T^{-1}\alpha R_D^* e^{-\lambda_2/T}}{(1-e^{-\lambda_2/T})},$$

and

$$\frac{\partial \ln L(\lambda_1,\lambda_2,\alpha)}{\partial \alpha} = \frac{D}{\alpha} + (1+R_i)\sum_{i=1}^{D}\ln(1-e^{-\lambda_1/x_{(i)}}) + (1+R_i)\sum_{i=1}^{D}\ln(1-e^{-\lambda_2/x_{(i)}}) + R_D^*\ln(1-e^{-\lambda_1/T}) + R_D^*\ln(1-e^{-\lambda_2/T}). \quad (5)$$

Assuming the shape parameter α is known and equating the first derivatives in equation (5) to zero, one can obtain the maximum likelihood estimates of the unknown parameters λ_1 and λ_2. As it seems, the system of non-linear equations has no closed form solution in λ_1 and λ_2, so a numerical method technique is required for computing the maximum likelihood estimates of those parameters.

To construct confidence intervals for the unknown parameters, we need to compute the asymptotic variance-covariance matrix that is obtained by inverting Fisher information matrix $I(\lambda_1,\lambda_2,\alpha)$, in which elements are negatives of expected values of the second partial derivatives of $\ln L(\lambda_1,\lambda_2,\alpha)$. The elements of the sample information matrix will be

$$-\frac{\partial^2 \ln L(\lambda_1,\lambda_2,\alpha)}{\partial \lambda_1^2} = \frac{D_1}{\lambda_1^2} - (\alpha(1+R_i)-D_1)\sum_{i=1}^{D}\frac{x_{(i)}^{-2}e^{-\lambda_1/x_{(i)}}(1-e^{-\lambda_1/x_{(i)}}) + x_{(i)}^{-2}e^{-2\lambda_1/x_{(i)}}}{(1-e^{-\lambda_1/x_{(i)}})^2} - \frac{T^{-2}\alpha R_D^* e^{-\lambda_1/T}(1-e^{-\lambda_1/T}) + T^{-2}\alpha R_D^* e^{-2\lambda_1/T}}{(1-e^{-\lambda_1/T})^2},$$

$$-\frac{\partial^2 \ln L(\lambda_1,\lambda_2,\alpha)}{\partial \lambda_2^2} = \frac{D_2}{\lambda_2^2} - (\alpha(1+R_i)-D_2)\sum_{i=1}^{D}\frac{x_{(i)}^{-2}e^{-\lambda_2/x_{(i)}}(1-e^{-\lambda_2/x_{(i)}}) + x_{(i)}^{-2}e^{-2\lambda_2/x_{(i)}}}{(1-e^{-\lambda_2/x_{(i)}})^2} - \frac{T^{-2}\alpha R_D^* e^{-\lambda_2/T}(1-e^{-\lambda_2/T}) + T^{-2}\alpha R_D^* e^{-2\lambda_2/T}}{(1-e^{-\lambda_2/T})^2},$$

$$-\frac{\partial^2 \ln L(\lambda_1,\lambda_2,\alpha)}{\partial \alpha^2} = \frac{D}{\alpha^2}, \quad -\frac{\partial^2 \ln L(\lambda_1,\lambda_2,\alpha)}{\partial \lambda_1 \partial \lambda_2} = 0,$$

$$-\frac{\partial^2 \ln L(\lambda_1,\lambda_2,\alpha)}{\partial \lambda_1 \partial \alpha} = -(1+R_i)\sum_{i=1}^{D}\frac{x_{(i)}^{-1}e^{-\lambda_1/x_{(i)}}}{(1-e^{-\lambda_1/x_{(i)}})} - \frac{T^{-1}R_D^* e^{-\lambda_1/T}}{(1-e^{-\lambda_1/T})},$$

and

$$-\frac{\partial^2 \ln L(\lambda_1,\lambda_2,\alpha)}{\partial \lambda_2 \partial \alpha} = -(1+R_i)\sum_{i=1}^{D}\frac{x_{(i)}^{-1}e^{-\lambda_2/x_{(i)}}}{(1-e^{-\lambda_2/x_{(i)}})} - \frac{T^{-1}R_D^* e^{-\lambda_2/T}}{(1-e^{-\lambda_2/T})}.$$

Under some regularity conditions, $(\hat{\lambda}_1,\hat{\lambda}_2,\alpha)$ is approximately normal with mean $(\lambda_1,\lambda_2,\alpha)$ and covariance matrix $I_0^{-1}(\hat{\lambda}_1,\hat{\lambda}_2,\alpha)$. Practically, we estimate $I^{-1}(\lambda_1,\lambda_2,\alpha)$ by $I_0^{-1}(\hat{\lambda}_1,\hat{\lambda}_2,\alpha)$, then

$$I_0^{-1}(\hat{\lambda}_1,\hat{\lambda}_2,\alpha) \cong \begin{bmatrix} -\frac{\partial^2 \ln L(\lambda_1,\lambda_2,\alpha)}{\partial \lambda_1^2} & -\frac{\partial^2 \ln L(\lambda_1,\lambda_2,\alpha)}{\partial \lambda_1 \partial \lambda_2} & -\frac{\partial^2 \ln L(\lambda_1,\lambda_2,\alpha)}{\partial \lambda_1 \partial \alpha} \\ -\frac{\partial^2 \ln L(\lambda_1,\lambda_2,\alpha)}{\partial \lambda_1 \partial \lambda_2} & -\frac{\partial^2 \ln L(\lambda_1,\lambda_2,\alpha)}{\partial \lambda_2^2} & -\frac{\partial^2 \ln L(\lambda_1,\lambda_2,\alpha)}{\partial \lambda_2 \partial \alpha} \\ -\frac{\partial^2 \ln L(\lambda_1,\lambda_2,\alpha)}{\partial \lambda_1 \partial \alpha} & -\frac{\partial^2 \ln L(\lambda_1,\lambda_2,\alpha)}{\partial \lambda_2 \partial \alpha} & -\frac{\partial^2 \ln L(\lambda_1,\lambda_2,\alpha)}{\partial \alpha^2} \end{bmatrix}^{-1}_{(\hat{\lambda}_1,\hat{\lambda}_2,\alpha)}$$

Using the independence of the latent failure times X_{1i}, X_{2i}, $i = 1,\ldots,n$, we can obtain the relative risk rate due to a particular cause (say, cause 1) as follows

$$\psi_1 = P(X_{1i} \le X_{2i}) = \int_0^\infty f_1(x).\bar{F}_2(x)dx = \alpha\lambda_1\int_0^\infty x^{-2}e^{-\lambda_1/x}(1-e^{-\lambda_1/x})^{\alpha-1}(1-e^{-\lambda_2/x})^{\alpha}dx, \quad (6)$$

and

$$\psi_2 = P(X_{2i} \le X_{1i}) = \int_0^\infty f_2(x).\bar{F}_1(x)dx = \alpha\lambda_2 \int_0^\infty x^{-2} e^{-\lambda_2/x} \left(1 - e^{-\lambda_2/x}\right)^{\alpha-1} \left(1 - e^{-\lambda_1/x}\right)^{\alpha} dx. \tag{7}$$

As the integral in the right side of equations (6) and (7) have no analytical solution, so a numerical technique may be used for solving these integrals. The MLE of the relative risk rates of ψ_1 and ψ_2 may be obtained by replacing the estimates of λ_1 and λ_2 in equations (6) and (7).

Now, assuming α unknown ACI for λ_1, λ_2 and α can be obtained as follows

$$\hat{\lambda}_k \pm Z_{\gamma/2}\sqrt{\text{var}(\hat{\lambda}_k)},\ k = 1,2, \quad \text{and} \quad \hat{\alpha} \pm Z_{\gamma/2}\sqrt{\text{var}(\hat{\alpha})},$$

where $Z_{\gamma/2}$ is the $100(1-\gamma/2)\%$ standard normal percentile.

4 Bayesian estimation

In this section, we describe how to obtain the Bayes estimates of the unknown parameters when the shape parameter of the GIED is known under AT-I PHCS based on competing risks data and also when the shape parameter is unknown.

4.1 Shape parameter α is known

We consider the Bayesian estimation under the assumption that the random variables $\lambda_k,\ k = 1,2$, have an independent gamma prior distribution. Assuming that $\lambda_k \sim \text{Gamma}(a_k, b_k),\ k = 1,2$, the joint prior density of λ_1 and λ_2 can be written as

$$\pi(\lambda_k) \propto \lambda_k^{a_k-1} e^{-\lambda_k b_k},\ a_k, b_k > 0,\ k = 1,2. \tag{8}$$

Combining (3) and (8), the joint posterior density of λ_1 and λ_2 given the data is

$$\pi(\lambda_1,\lambda_2|\alpha,x) \propto \frac{\lambda_1^{D_1+a_1-1}\lambda_2^{D_2+a_2-1}}{A_1A_2\sum_{g=0}^{\infty}\rho_g\sum_{h=0}^{\infty}\eta_h\sum_{w=0}^{\infty}\upsilon_w\sum_{s=0}^{\infty}\varepsilon_s} e^{-\lambda_1(D_1/x_{(i)}+b_1)-\lambda_2(D_2/x_{(i)}+b_2)} \left(1 - e^{-\lambda_1/x_{(i)}}\right)^{\alpha(1+R_i)-D_1}$$
$$\times \left(1 - e^{-\lambda_2/x_{(i)}}\right)^{\alpha(1+R_i)-D_2} \left(1 - e^{-\lambda_1/T}\right)^{\alpha R_D^*} \left(1 - e^{-\lambda_2/T}\right)^{\alpha R_D^*}, \tag{9}$$

where
$\rho_g = \frac{(-1)^g\Gamma(\alpha(1+R_i)-D_1+1)}{g!\Gamma(\alpha(1+R_i)-D_1-g+1)}$, $\eta_h = \frac{(-1)^h\Gamma(\alpha R_D^*+1)}{h!\Gamma(\alpha R_D^*-h+1)}$, $\upsilon_w = \frac{(-1)^w\Gamma(\alpha(1+R_i)-D_2+1)}{w!\Gamma(\alpha(1+R_i)-D_2-w+1)}$, $\varepsilon_s = \frac{(-1)^s\Gamma(\alpha R_D^*+1)}{s!\Gamma(\alpha R_D^*-s+1)}$,
$A_1 = \frac{\Gamma(D_1+a_1)}{\left(\left(\frac{D_1}{x_{(i)}}+b_1\right)+\frac{g}{x_{(i)}}+\frac{h}{T}\right)^{(D_1+a_1)}}$ and $A_2 = \frac{\Gamma(D_2+a_2)}{\left(\left(\frac{D_2}{x_{(i)}}+b_2\right)+\frac{w}{x_{(i)}}+\frac{s}{T}\right)^{(D_2+a_2)}}$

The marginal posterior of λ_1 and λ_2 are given as follows

$$\pi_1(\lambda_1|\alpha,x) = \frac{1}{A_1\sum_{g=0}^{\infty}\rho_g\sum_{h=0}^{\infty}\eta_h} \lambda_1^{D_1+a_1-1} e^{-\lambda_1(D_1/x_{(i)}+b_1)} \left(1 - e^{-\lambda_1/x_{(i)}}\right)^{\alpha(1+R_i)-D_1} \left(1 - e^{-\lambda_1/T}\right)^{\alpha R_D^*},$$

and

$$\pi_2(\lambda_2|\alpha,x) = \frac{1}{A_2\sum_{w=0}^{\infty}\upsilon_w\sum_{s=0}^{\infty}\varepsilon_s} \lambda_2^{D_2+a_2-1} e^{-\lambda_2(D_2/x_{(i)}+b_2)} \left(1 - e^{-\lambda_2/x_{(i)}}\right)^{\alpha(1+R_i)-D_2} \left(1 - e^{-\lambda_2/T}\right)^{\alpha R_D^*}.$$

In Bayesian estimation, two types of loss functions are considered. The first is the squared error loss function. The second, the LINEX loss function. Under squared error loss function, Bayes estimator of λ_1 and λ_2 is the posterior mean which obtained as follows

$$\tilde{\lambda}_{SE1} = \frac{D_1 + a_1}{\left(\left(\frac{D_1}{x_{(i)}} + b_1\right) + \frac{g}{x_{(i)}} + \frac{h}{T}\right)} \quad \text{and} \quad \tilde{\lambda}_{SE2} = \frac{D_2 + a_2}{\left(\left(\frac{D_2}{x_{(i)}} + b_2\right) + \frac{w}{x_{(i)}} + \frac{s}{T}\right)},$$

respectively.

For the non-informative priors $a_1 = a_2 = b_1 = b_2 = 0$, the Bayes estimators coincide with the corresponding MLEs. Because the posterior distributions of λ_1 and λ_2 are gamma, the credible intervals of λ_1 and λ_2 can be obtained using

$$Z_1 = 2\lambda_1((\frac{D_1}{x_{(i)}} + b_1) + \frac{g}{x_{(i)}} + \frac{h}{T}) \quad \text{and} \quad Z_2 = 2\lambda_2((\frac{D_2}{x_{(i)}} + b_2) + \frac{w}{x_{(i)}} + \frac{s}{T}),$$

which follows χ^2 distributions with $(2(D_1 + a_1))$ and $(2(D_2 + a_2))$ degrees of freedom, respectively. Therefore, $100(1-\gamma)\%$ credible intervals for λ_1 and λ_2 are

$$\left(\frac{\chi^2_{(2(D_1+a_1)),1-\gamma/2}}{2((\frac{D_1}{x_{(i)}} + b_1) + \frac{g}{x_{(i)}} + \frac{h}{T})}, \frac{\chi^2_{(2(D_1+a_1)),\gamma/2}}{2((\frac{D_1}{x_{(i)}} + b_1) + \frac{g}{x_{(i)}} + \frac{h}{T})} \right),$$

and

$$\left(\frac{\chi^2_{(2(D_2+a_2)),1-\gamma/2}}{2(\frac{D_2}{x_{(i)}} + b_2) + \frac{w}{x_{(i)}} + \frac{s}{T}}, \frac{\chi^2_{(2(D_2+a_2)),\gamma/2}}{2(\frac{D_2}{x_{(i)}} + b_2) + \frac{w}{x_{(i)}} + \frac{s}{T}} \right),$$

respectively, where $(D_1 + a_1) > 0$ and $(D_2 + a_2) > 0$. Note that if $(2(D_1 + a_1))$ and $(2(D_2 + a_2))$ are not integer values, then gamma distribution can be used to construct the credible intervals.

The LINEX loss function with parameters d and c is given by

$$\ell(\tilde{\theta}_{Lin}, \theta) = d\left\{e^{c(\tilde{\theta}_{Lin},\theta)} - c(\tilde{\theta}_{Lin}, \theta) - 1\right\}, \tag{10}$$

where d and c are constants. The sign and magnitude of c represent the direction and degree of symmetry, respectively. From (10) the Bayes estimator $\tilde{\theta}$ of θ is given by

$$\tilde{\theta}_{Lin} = -\frac{1}{c}\ln E(e^{-c\theta}),\ c \neq 0, \tag{11}$$

for c closed to zero, the LINEX loss is approximately SE loss.

Under LINEX loss function (10), the Bayes estimator of λ_1 and λ_2 can be obtained as follows

$$\tilde{\lambda}_{Lin1} = -\frac{D_1 + a_1}{c}\ln\left(\frac{(\frac{D_1}{x_{(i)}} + b_1) + \frac{g}{x_{(i)}} + \frac{h}{T}}{(\frac{D_1}{x_{(i)}} + b_1) + c + \frac{g}{x_{(i)}} + \frac{h}{T}}\right),$$

and

$$\tilde{\lambda}_{Lin2} = -\frac{D_2 + a_2}{c}\ln\left(\frac{(\frac{D_2}{x_{(i)}} + b_2) + \frac{w}{x_{(i)}} + \frac{s}{T}}{(\frac{D_2}{x_{(i)}} + b_2) + c + \frac{w}{x_{(i)}} + \frac{s}{T}}\right),\ c \neq 0,$$

where $E(e^{-c\lambda_1}) = \left(\frac{(\frac{D_1}{x_{(i)}}+b_1)+\frac{g}{x_{(i)}}+\frac{h}{T}}{(\frac{D_1}{x_{(i)}}+b_1)+c+\frac{g}{x_{(i)}}+\frac{h}{T}}\right)^{(D_1+a_1)}$ and $E(e^{-c\lambda_2}) = \left(\frac{(\frac{D_2}{x_{(i)}}+b_2)+\frac{w}{x_{(i)}}+\frac{s}{T}}{(\frac{D_2}{x_{(i)}}+b_2)+c+\frac{w}{x_{(i)}}+\frac{s}{T}}\right)^{(D_2+a_2)}$.

4.2 Shape parameter α is unknown

Bayes estimates of the unknown parameters of the GIED when the shape and scale parameters are unknown under AT-I PHCS based on competing risks data are obtained. Bayesian estimation under the assumption that the random variables λ_1, λ_2 and α, have an independent gamma prior distribution. Assuming that λ_1 and λ_2 have gamma priors with joint prior given in (8), and $\alpha \sim Gamma(p,q)$ where the joint prior density of α can be written as

$$\pi(\alpha) \propto \alpha^{p-1}e^{-\alpha q},\ p,\ q > 0.$$

Hence, the joint prior density of unknown parameters λ_1, λ_2 and α can be written as

$$\pi(\lambda_1,\lambda_2,\alpha) \propto \lambda_1^{a_1-1}\lambda_2^{a_2-1}\alpha^{p-1}e^{-(\lambda_1 b_1+\lambda_2 b_2+\alpha q)},\ a_k,\ b_k,\ p,q>0. \tag{12}$$

Combining equation (3) with equation (12) then

$$L(\lambda_1,\lambda_2,\alpha)\times\pi(\lambda_1,\lambda_2,\alpha) \propto \alpha^{D+p-1}\lambda_1^{D_1+a_1-1}\lambda_2^{D_2+a_2-1}\prod_{i=1}^{D}x_{(i)}^{-2D}e^{-\alpha q-\lambda_1(b_1+D_1/x_{(i)})-\lambda_2(b_2+D_2/x_{(i)})}$$
$$\times\left(1-e^{-\lambda_1/x_{(i)}}\right)^{\alpha(1+R_i)-D_1}\left(1-e^{-\lambda_2/x_{(i)}}\right)^{\alpha(1+R_i)-D_2}$$
$$\times\left(1-e^{-\lambda_1/T}\right)^{\alpha R_D^*}\left(1-e^{-\lambda_2/T}\right)^{\alpha R_D^*},$$

The joint posterior density of λ_1, λ_2 and α, given the data is

$$\pi(\lambda_1,\lambda_2,\alpha|x) = \frac{L(\lambda_1,\lambda_2,\alpha)\pi(\lambda_1,\lambda_2,\alpha)}{\int_0^\infty\int_0^\infty\int_0^\infty L(\lambda_1,\lambda_2,\alpha)\pi(\lambda_1,\lambda_2,\alpha)d\lambda_1 d\lambda_2 d\alpha}. \tag{13}$$

Therefore, the Bayes estimates of any function of λ_1, λ_2 and α, say $\delta(\lambda_1,\lambda_2,\alpha)$ is

$$\tilde{\delta}(\lambda_1,\lambda_2,\alpha) = E(\lambda_1,\lambda_2,\alpha|\underline{x}) = \int_0^\infty\int_0^\infty\int_0^\infty \delta(\lambda_1,\lambda_2,\alpha)L(\lambda_1,\lambda_2,\alpha).\pi(\lambda_1,\lambda_2,\alpha)d\lambda_1 d\lambda_2 d\alpha, \tag{14}$$

Normally, the ratio of three integrals given by Equation (14) cannot be obtained in a closed form. In this case, one may utilize the MCMC technique to generate samples from the posterior distributions, after that, compute the Bayes estimators of the individual parameters.

A broad diversity of MCMC schemes is accessible, and any researcher may find difficulty in selecting one of them. A vital sub-class of MCMC methods is Gibbs sampling and more general Metropolis-Hastings (M-H) introduced by [12] within Gibbs samplers. The benefit of employing the MCMC method over the maximum likelihood method can be revealed as one may always attain a sound interval estimate of the parameters by constructing the probability intervals depending on empirical posterior distribution. The aforementioned may not frequently be feasible in maximum likelihood estimation. The method of MCMC can be used to generate samples from the posterior density function (13) and in turn to compute the Bayes estimates of the unknown parameters and compute the corresponding BCIs. To generate samples from (13), we need the posterior probability density functions of λ_1, λ_2 and α which can be written as

$$\lambda_1\,|\lambda_2,\alpha,x \sim Gamma\left(D_1+a_1,\left(\frac{D_1}{x_{(i)}}+b_1\right)+\frac{g}{x_{(i)}}+\frac{h}{T}\right), \tag{15}$$

$$\lambda_2\,|\lambda_1,\alpha,x \sim Gamma\left(D_2+a_2,\left(\frac{D_2}{x_{(i)}}+b_2\right)+\frac{w}{x_{(i)}}+\frac{s}{T}\right), \tag{16}$$

and

$$\alpha\,|\lambda_1,\lambda_2,x \propto \alpha^{D+p-1}\prod_{i=1}^{D}x_{(i)}^{-2D}e^{-\alpha q}\left(1-e^{-\lambda_1/x_{(i)}}\right)^{\alpha(1+R_i)-D_1}\left(1-e^{-\lambda_2/x_{(i)}}\right)^{\alpha(1+R_i)-D_2}$$
$$\times\left(1-e^{-\lambda_1/T}\right)^{\alpha R_D^*}\left(1-e^{-\lambda_2/T}\right)^{\alpha R_D^*}. \tag{17}$$

It is rather obvious that both Equations (15) and (16) are gamma distributed, consequently, samples of λ_1 and λ_2 may be created without difficulty by employing any of the gamma generating procedures. The posterior of α in Equations (17) is not known. Thus, to derive from this distributions, one may employ the Metropolis-Hastings method with normal proposal distribution. For more information concerning the application of M-H, readers may refer to [13].

To run the Gibbs within M-H algorithm, we started with the MLEs. We then drew samples from various full conditionals, in turn, using the most recent values of all other conditioning variables unless some systematic pattern of convergence was achieved. The algorithm Gibbs sampling can be described as follows:

Step 1: Initial $(\lambda_1,\lambda_2,\alpha)=(\lambda_1^{(0)},\lambda_2^{(0)},\alpha^{(0)})$ and set $t=1$.
Step 2: Generate $\lambda_1^{(t)}$ from (15).
Step 3: Generate $\lambda_2^{(t)}$ from (16).
Step 4: Generate $\alpha^{(t)}$ from (17).
Step 5: Set $t=t+1$.
Step 6: Repeat steps 2-5, M times, and obtain $(\lambda_1^{(i)},\lambda_2^{(i)},\alpha^{(i)}),\ i=1,\ldots,M$.
Step 7: Under SE loss function, obtain the Bayes estimates of λ_1, λ_2 and α as
$\tilde{\lambda}_{SE1}=\sum_{i=1}^{M}\lambda_1^{(i)}/M$, $\tilde{\lambda}_{SE2}=\sum_{i=1}^{M}\lambda_2^{(i)}/M$ and $\tilde{\alpha}_{SE}=\sum_{i=1}^{M}\alpha^{(i)}/M$.
Step 8: To obtain the BCIs of λ_1, λ_2 and α order $\lambda_1^{(i)}$, $\lambda_2^{(i)}$ and $\alpha^{(i)}$ as $(\lambda_1^{[1]},\lambda_1^{[2]},\ldots,\lambda_1^{[M]})$, $(\lambda_2^{[1]},\lambda_2^{[2]},\ldots,\lambda_2^{[M]})$ and $(\alpha^{[1]},\alpha^{[2]},\ldots,\alpha^{[M]})$. Then, the $100(1-2\gamma)\%$ symmetric BCIs of λ_1, λ_2 and α become $(\lambda_1^{[\gamma M]},\lambda_1^{[(1-\gamma)M]})$, $(\lambda_2^{[\gamma M]},\lambda_2^{[(1-\gamma)M]})$ and $(\alpha^{[\gamma M]},\alpha^{[(1-\gamma)M]})$.
Step 9: Under LINEX loss function, obtain the Bayes estimates of λ_1, λ_2 and α as
$\tilde{\lambda}_{Lin1}=-\frac{1}{c}\ln\left(\sum_{i=1}^{M}e^{-c\lambda_1^{(i)}}/M\right)$, $\tilde{\lambda}_{Lin2}=-\frac{1}{c}\ln\left(\sum_{i=1}^{M}e^{-c\lambda_2^{(i)}}/M\right)$ and $\tilde{\alpha}_{Lin}=-\frac{1}{c}\ln\left(\sum_{i=1}^{M}e^{-c\alpha^{(i)}}/M\right)$.

5 Simulation study

To study the effectiveness of the proposed estimators of the unknown parameters α, λ_1 and λ_2, a large number 1000 of AT-I PHCS samples based on choices of some different values of sample size $n=30$ and 50 for each distinct time $T=1$ and 2 are generated from the GIED when the true values of parameters $(\alpha,\lambda_1,\lambda_2)$ are taken as $(0.5,0.2,0.3)$ and $(1.0,0.3,0.5)$ respectively. The test is terminated when the number of failed subjects achieves or exceeds a certain value r, where the percentages of failure information $(r/n)100\%$ are considered as 40 and 80%. In each setting, the proposed point and interval estimators are evaluated. Three different censoring schemes are considered as

$$\text{Scheme-I}: R_1=n-r,\quad R_i=0\quad \text{for}\quad i\neq 1,$$

$$\text{Scheme-II}: \begin{cases} R_{\frac{r}{2}}=n-r,\ R_i=0 & \text{for } i\neq \frac{r}{2} \text{ if } r \text{ is even},\\ R_{\frac{(r+1)}{2}}=n-r,\ R_i=0 & \text{for } i\neq \frac{(r+1)}{2} \text{ if } r \text{ is odd},\end{cases}$$

and

$$\text{Scheme-III}: R_r=n-r,\quad R_i=0\quad \text{for}\quad i\neq r.$$

In Bayes paradigm, the choice of the hyper-parameters is the main issue, when an informative prior of the density parameter is taken into account. If the proper prior information, i.e., $a_i=b_i=0$ for $i=1,2,3$ is available, the joint posterior distribution (9) of α, λ_1 and λ_2 are proportional to the likelihood function (3). Thus, to see the effects of the priors on the Bayes estimators and the associated BCIs, two different sets of hyper-parameters for each set of α , λ_1 and λ_2 are used, namely; prior (1): $(a_1,a_2,a_3)=(2.5,1.0,1.5)$, $b_i=5,\ i=1,2,3$, and prior (2): $(a_1,a_2,a_3)=(5,2,3)$, $b_i=10,\ i=1,2,3$, when $(\alpha,\lambda_1,\lambda_2)=(0.5,0.2,0.3)$, similarly, prior (1): $(a_1,a_2,a_3)=(5,1.5,2.5)$, $b_i=5,\ i=1,2,3$, and prior (2): $(a_1,a_2,a_3)=(10,3,5)$, $b_i=10,\ i=1,2,3$, when $(\alpha,\lambda_1,\lambda_2)=(1.0,0.3,0.5)$. It should be noted that, if one does not have prior information on the unknown parameters of interest, it is better to use the MLEs rather than the Bayes estimators because the Bayes estimators are computationally more expensive. It is clear that the values of hyper-parameters of the unknown parameters α, λ_1 and λ_2 are chosen in such way that the prior meaning become the expected value of the corresponding parameter, for detail, see [14].

To develop the Bayesian computations, using a hybrid strategy combining Gibbs within M-H proposed in Section 4, 12,000 MCMC samples are generated and discard the first 2,000 values as 'burn-in'. Hence, based on 10,000 MCMC samples, the average Bayes MCMC estimates and corresponding 95% BCIs are computed using square error and LINEX (with $c=\pm 2$) functions. Using different initial guesses of α, λ_1 and λ_2, the convergence of the MCMC sequences for their stationary distributions has been checked. Here, the corresponding MLEs of unknown parameters are used as initial values to run the MCMC sampler, consequently, one can observe that the Markov chains reached the stationary condition very quickly.

Comparison between different point estimates that is made based on their estimated root mean squared errors (RMSEs) and relative absolute biases (RABs) values. Also, the performances of 95% two-sided ACI/BCI estimates are compared by using their average confidence lengths (ACLs).

For each test, the average estimates with their RMSEs and RABs of classical and Bayesian estimates of the parametric function (say φ) are calculated using the following formulas, respectively, as

$$\bar{\hat{\varphi}}_\tau = \frac{1}{S}\sum_{j=1}^{S}\hat{\varphi}_\tau^{(j)}, \tau = 1,2,3, \text{ RMSE}(\hat{\varphi}_\tau) = \sqrt{\frac{1}{S}\sum_{j=1}^{S}\left(\hat{\varphi}_\tau^{(j)} - \varphi_\tau\right)^2}, \ \tau = 1,2,3, \text{ RAB}(\hat{\varphi}_\tau) = \frac{1}{S}\sum_{j=1}^{S}\frac{\left|\hat{\varphi}_\tau^{(j)} - \varphi_\tau\right|}{\varphi_\tau}, \ \tau = 1,2,3,$$

and

$$\text{ACL}_{\varphi_\tau}(1-\gamma)\% = \frac{1}{S}\sum_{j=1}^{S}\left(U_{\hat{\varphi}_\tau^{(j)}}(1-\gamma)\% - L_{\hat{\varphi}_\tau^{(j)}}(1-\gamma)\%\right), \ \tau = 1,2,3,$$

where $\hat{\varphi}_\tau^{(j)}$ denotes the obtained estimate at the jth sample of the unknown parameter φ_τ, $L_{\hat{\varphi}_\tau^{(j)}}(\cdot)$ and $U_{\hat{\varphi}_\tau^{(j)}}(\cdot)$ denote the lower and upper estimated bounds, respectively, of $(1-\gamma)\%$ asymptotic (or credible) interval of φ_τ, S is number of generated sequence data, $\varphi_1 = \alpha$, $\varphi_2 = \lambda_1$ and $\varphi_3 = \lambda_2$.

The average estimates of α, λ_1 and λ_2 with their RMSEs and RABs of MLEs and Bayes estimators are calculated and reported in Tables 1-7. In these Tables, the average estimates with associated RMSEs and RABs for each test are tabulated in the first, second and third rows, respectively. In addition, the corresponding ACLs of 95% ACIs/BCIs of α, λ_1 and λ_2 are listed in Tables 9 and 10, respectively. All necessary computational algorithms are coded in R statistical programming language software version 4.0.4 using three useful statistical packages; namely: 'coda' package which used for carrying out the computations of MCMC Bayes estimators proposed by [15], 'maxLik' package which uses Newton-Raphson method of maximization in the computations, proposed by [16], and 'GoFKernel' package proposed by [17] which generate a sequence of random numbers. These packages were recently recommended by [18, 19].

From the simulation results shown in Tables 1-7, we can make the following observations. In general, it can be seen that the classical and Bayes estimates of the unknown parameters α, λ_1 and λ_2 are good in terms of minimum RMSEs and RABs. As n (or the failure proportion r/n) increases RMSEs and RABs of all estimates of α, λ_1 and λ_2 decrease as expected. So, to get better estimation results, one may tend to increase the effective sample size. Similarly, as T increases, the corresponding RMSEs and RABs of α, λ_1 and λ_2 decrease. Also, Bayes MCMC estimates using gamma informative prior are better as they include prior information than MLEs in terms of RMSEs and RABs. Further, when T increases, the average number of observed failures, D, are also increases and when the failure percentage r/n is high, the MLEs of the shape parameter α have performed better than Bayes MCMC estimates in respect of their RMSEs and RABs.

Moreover, when actual value of the parameters α, λ_1 and λ_2 increases, it can be seen that the corresponding RMSEs and RABs associated with MLEs and Bayes estimators increase. In addition, for each set of parameter values, Bayes estimators based on prior 2 is better than prior 1 in terms of minimum RMSEs, RABs and ACLs for all estimates. This is because the fact that the prior 2 variance is lower than prior 1 variance, and both are more informative than the improper prior. In most cases, in respect of minimum RMSEs and RABs, the performance of the Bayes estimators relative to LINEX loss function have perform better than square error loss function, this is due to the use of square error loss function gives an equal weight to underestimation and overestimation due to its symmetrical nature. Also RMSEs and RABs of LINEX loss function of α, λ_1 and λ_2 are overestimates for $c < 0$ and when $c > 0$ are underestimates.

Moreover, comparing the censoring scheme (I) and censoring scheme (III), it is clear that the RMSEs and RABs of the MLEs and Bayes estimators for the unknown parameters α, λ_1 and λ_2 are greater for the censoring scheme (III) than censoring scheme (I). This because the expected duration of the experiments for censoring scheme (I), where the remaining $n-r$ units are withdrawn in first stage, is greater than censoring scheme (III), where the remaining $n-r$ units are withdrawn from the life-test at the time of observed rth failure. Also, the average number of observed failures D close the effective sample size n in censoring scheme (III).

From Tables 9-10, regarding interval estimates, the ACLs of ACIs/BCIs of α, λ_1 and λ_2 tend to decrease as n increases as expected. For each case, due to gamma prior information about unknown parameters, the BCIs perform better than the asymptotic intervals. Further, it is clear that the ACLs of BCIs under prior 2 become even better than prior 1 that is due to the fact that the variance of prior 2 is lower than the variance of prior 1. Furthermore, when the true values of α, λ_1 and λ_2 increase, the ACLs for ACIs/BCIs of α and λ_2 increase, while the ACLs of λ_1 narrow down under asymptotic intervals than BCIs. It has been also noted that the ACLs of 95% ACIs/BCIs of α, λ_1 and λ_2 are narrows down for censoring scheme (I) than other competing schemes. Therefore, it is expected that the data obtained by using censoring scheme (I) will contain more information about the unknown parameters α, λ_1 and λ_2 than the data obtained by using censoring scheme (III). In most cases, when T increases, the ACLs of α increase while that associated with λ_1 and λ_2 tends to decrease. This may not be very surprising, because when T increases some additional information is gathered.

Finally, to study the behavior of different estimators of the parameters α, λ_1 and λ_2, Bayesian (point and interval) inference using the hybrid strategy combining the Gibbs sampler within M-H algorithm is recommended.

Table 1: Average estimated values, MSEs and RABs of the MLEs and Bayes estimators based on AT-I PHCS when ($n = 30$) for ($T = 1,2$) and ($r = 12,24$) for $(\alpha, \lambda_1, \lambda_2) = (0.5, 0.2, 0.3)$.

T	n	r	Scheme	Parameter	MLE	SE		LINEX				D
$c \rightarrow$								-2		+2		
Prior $\rightarrow$						1	2	1	2	1	2	
1	30	12	I	α	0.1353	0.1432	0.1747	0.1444	0.1764	0.1421	0.1732	9.5690
					0.3720	0.3584	0.3277	0.3556	0.3236	0.3579	0.3268	
					0.7295	0.7136	0.6506	0.7112	0.6473	0.7158	0.6536	
				λ_1	0.0194	0.1924	0.2049	0.1924	0.2049	0.1924	0.2049	
					0.1837	0.0086	0.0062	0.0076	0.0049	0.0076	0.0049	
					0.9031	0.0380	0.0248	0.0379	0.0247	0.0381	0.0245	
				λ_2	0.1559	0.2459	0.3225	0.2462	0.3226	0.2456	0.3225	
					0.1695	0.0569	0.0236	0.0538	0.0226	0.0544	0.0225	
					0.4803	0.1804	0.0751	0.1794	0.0753	0.1814	0.0749	
			II	α	0.1445	0.2145	0.2414	0.2170	0.2442	0.2122	0.2389	10.384
					0.3627	0.2896	0.2637	0.2830	0.2558	0.2878	0.2611	
					0.7110	0.5709	0.5173	0.5659	0.5116	0.5756	0.5222	
				λ_1	0.0206	0.2028	0.1996	0.1996	0.2029	0.1996	0.2029	
					0.1827	0.0048	0.0031	0.0029	0.0004	0.0029	0.0004	
					0.8969	0.0188	0.0132	0.0144	0.0020	0.0143	0.0021	
				λ_2	0.1648	0.2727	0.3137	0.2732	0.3139	0.2722	0.3134	
					0.1619	0.0353	0.0206	0.0278	0.0139	0.0278	0.0134	
					0.4508	0.0920	0.0545	0.0892	0.0463	0.0926	0.0448	
			III	α	0.2026	0.2997	0.3218	0.2997	0.3218	0.2997	0.3218	12.000
					0.2990	0.2003	0.1782	0.2003	0.1782	0.2003	0.1782	
					0.5949	0.4006	0.3563	0.4005	0.3563	0.4006	0.3563	
				λ_1	0.2064	0.2002	0.2007	0.2002	0.2008	0.2002	0.2007	
					0.0676	0.0049	0.0047	0.0008	0.0002	0.0007	0.0002	
					0.2419	0.0190	0.0190	0.0038	0.0010	0.0036	0.0008	
				λ_2	0.2367	0.2812	0.3008	0.2813	0.3010	0.2811	0.3007	
					0.1265	0.0217	0.0123	0.0187	0.0010	0.0189	0.0007	
					0.3443	0.0626	0.0346	0.0622	0.0032	0.0630	0.0011	
		24	I	α	0.4667	0.3469	0.3568	0.3496	0.3596	0.3442	0.3541	18.767
					0.0381	0.1618	0.1525	0.1504	0.1404	0.1558	0.1459	
					0.0666	0.3068	0.2880	0.3007	0.2808	0.3116	0.2918	
				λ_1	0.0290	0.1952	0.1980	0.1952	0.1980	0.1952	0.1980	
					0.1772	0.0060	0.0049	0.0048	0.0020	0.0048	0.0020	
					0.8548	0.0245	0.0192	0.0241	0.0098	0.0242	0.0100	
				λ_2	0.2662	0.2777	0.2827	0.2779	0.2829	0.2775	0.2825	
					0.0889	0.0267	0.0222	0.0221	0.0171	0.0225	0.0175	
					0.1126	0.0745	0.0629	0.0737	0.0569	0.0751	0.0582	
			II	α	0.4715	0.4007	0.4013	0.4043	0.4045	0.3973	0.3983	19.556
					0.0318	0.1156	0.1133	0.0956	0.0955	0.1026	0.1017	
					0.0571	0.2078	0.2038	0.1913	0.1910	0.2054	0.2034	
				λ_1	0.0297	0.2045	0.1952	0.1952	0.2045	0.1952	0.2045	
					0.1769	0.0055	0.0052	0.0048	0.0045	0.0048	0.0045	
					0.8516	0.0229	0.0240	0.0239	0.0227	0.0239	0.0226	
				λ_2	0.2685	0.2907	0.2922	0.2907	0.2922	0.2906	0.2922	
					0.0878	0.0100	0.0085	0.0093	0.0078	0.0094	0.0078	
					0.1051	0.0311	0.0260	0.0311	0.0259	0.0312	0.0259	
			III	α	0.4972	0.5155	0.5314	0.5259	0.5398	0.5062	0.5237	22.092
					0.0259	0.1003	0.0948	0.0398	0.0259	0.0237	0.0062	
					0.0182	0.1545	0.1437	0.0795	0.0518	0.0475	0.0124	
				λ_1	0.0387	0.1954	0.2016	0.1954	0.2016	0.1954	0.2016	
					0.1764	0.0051	0.0033	0.0046	0.0016	0.0046	0.0016	
					0.8506	0.0229	0.0131	0.0229	0.0080	0.0230	0.0080	
				λ_2	0.2855	0.2912	0.3065	0.2912	0.3065	0.2912	0.3065	
					0.0765	0.0001	0.0075	0.0088	0.0065	0.0088	0.0065	
					0.0696	0.0091	0.0217	0.0294	0.0216	0.0295	0.0215	

Table 2: *

Continue Table 1.

T	n	r	Scheme	Parameter	MLE	SE		LINEX				D
c →								-2		+2		
Prior →						1	2	1	2	1	2	
2	30	12	I	α	0.1633	0.1720	0.1854	0.1742	0.1874	0.1700	0.1834	10.671
					0.3449	0.3312	0.3178	0.3258	0.3126	0.3300	0.3166	
					0.6735	0.6559	0.6293	0.6515	0.6251	0.6600	0.6331	
				λ_1	0.0201	0.1888	0.1956	0.1889	0.1956	0.1888	0.1956	
					0.1831	0.0120	0.0061	0.0111	0.0043	0.0112	0.0044	
					0.8997	0.0558	0.0247	0.0557	0.0217	0.0559	0.0219	
				λ_2	0.1582	0.3259	0.2887	0.3260	0.2887	0.3258	0.2886	
					0.1692	0.0285	0.0133	0.0260	0.0113	0.0258	0.0114	
					0.4726	0.0868	0.0396	0.0868	0.0376	0.0858	0.0379	
			II	α	0.3512	0.3672	0.3697	0.3752	0.3757	0.3600	0.3642	11.172
					0.1649	0.1588	0.1506	0.1247	0.1243	0.1400	0.1358	
					0.2975	0.2870	0.2728	0.2494	0.2487	0.2799	0.2715	
				λ_1	0.0302	0.2089	0.2047	0.2089	0.2048	0.2088	0.2047	
					0.1766	0.0122	0.0075	0.0089	0.0048	0.0088	0.0047	
					0.8488	0.0532	0.0297	0.0447	0.0238	0.0440	0.0235	
				λ_2	0.2083	0.3148	0.2939	0.3148	0.2939	0.3147	0.2938	
					0.1420	0.0164	0.0079	0.0148	0.0061	0.0147	0.0062	
					0.3056	0.0493	0.0217	0.0495	0.0203	0.0492	0.0205	
			III	α	0.4509	0.4890	0.4876	0.4948	0.4922	0.4832	0.4830	12.000
					0.0572	0.0768	0.0691	0.0078	0.0052	0.0170	0.0168	
					0.0983	0.1246	0.1114	0.0155	0.0104	0.0340	0.0335	
				λ_1	0.0324	0.1906	0.2065	0.1907	0.2066	0.1906	0.2065	
					0.1757	0.0119	0.0091	0.0093	0.0066	0.0094	0.0065	
					0.8483	0.0469	0.0364	0.0467	0.0328	0.0472	0.0324	
				λ_2	0.2384	0.2904	0.2927	0.2904	0.2927	0.2904	0.2927	
					0.1246	0.0107	0.0079	0.0096	0.0073	0.0096	0.0073	
					0.2054	0.0323	0.0244	0.0319	0.0243	0.0321	0.0244	
		24	I	α	0.4748	0.3845	0.4102	0.3884	0.4145	0.3807	0.4062	21.321
					0.0274	0.1312	0.1105	0.1116	0.0855	0.1193	0.0938	
					0.0504	0.2386	0.1953	0.2231	0.1709	0.2386	0.1876	
				λ_1	0.0297	0.1860	0.2051	0.1861	0.2051	0.1859	0.2051	
					0.1764	0.0181	0.0054	0.0139	0.0051	0.0141	0.0051	
					0.8513	0.0710	0.0253	0.0693	0.0254	0.0706	0.0253	
				λ_2	0.2710	0.2855	0.2908	0.2856	0.2909	0.2855	0.2908	
					0.0837	0.0157	0.0114	0.0144	0.0091	0.0145	0.0092	
					0.0967	0.0482	0.0325	0.0480	0.0305	0.0483	0.0308	
			II	α	0.4771	0.4467	0.4533	0.4523	0.4580	0.4415	0.4489	21.667
					0.0247	0.0907	0.0819	0.0477	0.0420	0.0585	0.0511	
					0.0459	0.1526	0.1374	0.0954	0.0840	0.1170	0.1022	
				λ_1	0.0300	0.1887	0.1926	0.1887	0.1926	0.1887	0.1926	
					0.1761	0.0124	0.0076	0.0113	0.0074	0.0113	0.0074	
					0.8498	0.0565	0.0370	0.0563	0.0370	0.0565	0.0370	
				λ_2	0.2737	0.3086	0.2981	0.3086	0.2981	0.3086	0.2981	
					0.0807	0.0096	0.0045	0.0086	0.0019	0.0086	0.0019	
					0.0877	0.0298	0.0117	0.0287	0.0370	0.0286	0.0065	
			III	α	0.4948	0.4633	0.5059	0.4694	0.5127	0.4576	0.4992	23.841
					0.0150	0.0850	0.0823	0.0306	0.0127	0.0424	0.0008	
					0.0137	0.1402	0.1281	0.0613	0.0253	0.0848	0.0017	
				λ_1	0.0367	0.1942	0.1918	0.1943	0.1919	0.1942	0.1918	
					0.1731	0.0088	0.0087	0.0081	0.0057	0.0082	0.0058	
					0.8167	0.0408	0.0350	0.0407	0.0286	0.0409	0.0290	
				λ_2	0.2839	0.2941	0.2973	0.2941	0.2973	0.2941	0.2973	
					0.0774	0.0064	0.0043	0.0059	0.0027	0.0059	0.0027	
					0.0638	0.0196	0.0098	0.0195	0.0090	0.0196	0.0091	

Table 3: Average estimated values, MSEs and RABs of the MLEs and Bayes estimators based on AT-I PHCS for when ($n = 50$) for ($T = 1, 2$) and ($r = 20, 40$) for $(\alpha, \lambda_1, \lambda_2) = (0.5, 0.2, 0.3)$.

T	n	r	Scheme	Parameter	MLE	SE		LINEX				D
$c \rightarrow$								-2		+2		
Prior $\rightarrow$						1	2	1	2	1	2	
1	50	20	I	α	0.1450	0.1730	0.1812	0.1758	0.1837	0.1704	0.1787	15.808
					0.3617	0.3311	0.3227	0.3242	0.3163	0.3296	0.3213	
					0.7099	0.6539	0.6377	0.6484	0.6326	0.6592	0.6426	
				λ_1	0.0146	0.2055	0.1911	0.2055	0.1911	0.2054	0.1911	
					0.1872	0.0105	0.0093	0.0089	0.0055	0.0089	0.0054	
					0.9272	0.0420	0.0445	0.0444	0.0277	0.0445	0.0269	
				λ_2	0.1667	0.2759	0.2880	0.2760	0.2880	0.2758	0.2879	
					0.1550	0.0259	0.0131	0.0240	0.0120	0.0242	0.0121	
					0.4443	0.0804	0.0401	0.0801	0.0400	0.0807	0.0402	
			II	α	0.1522	0.2477	0.2749	0.2527	0.2811	0.2430	0.2692	17.307
					0.3545	0.2617	0.2380	0.2473	0.2189	0.2570	0.2308	
					0.6956	0.5061	0.4524	0.4946	0.4378	0.5139	0.4617	
				λ_1	0.0156	0.1950	0.1977	0.1950	0.1977	0.1950	0.1977	
					0.1863	0.0076	0.0038	0.0050	0.0023	0.0050	0.0023	
					0.9222	0.0324	0.0150	0.0249	0.0113	0.0252	0.0114	
				λ_2	0.1729	0.2891	0.3062	0.2892	0.3062	0.2891	0.3062	
					0.1494	0.0119	0.0072	0.0108	0.0062	0.0109	0.0062	
					0.4238	0.0362	0.0210	0.0361	0.0208	0.0363	0.0207	
			III	α	0.1962	0.3224	0.3432	0.3298	0.3505	0.3155	0.3364	20.000
					0.3045	0.1966	0.1778	0.1702	0.1495	0.1845	0.1636	
					0.6077	0.3624	0.3229	0.3404	0.2989	0.3689	0.3271	
				λ_1	0.0312	0.1937	0.2015	0.1937	0.2015	0.1937	0.2015	
					0.1751	0.0066	0.0034	0.0063	0.0015	0.0063	0.0015	
					0.8637	0.0313	0.0151	0.0313	0.0074	0.0313	0.0073	
				λ_2	0.2143	0.2955	0.2944	0.2956	0.2944	0.2594	0.2944	
					0.1033	0.0111	0.0065	0.0056	0.0044	0.0046	0.0015	
					0.3111	0.0325	0.0187	0.0186	0.0147	0.0153	0.0187	
		40	I	α	0.4744	0.4142	0.4270	0.4230	0.4354	0.4061	0.4191	31.313
					0.0280	0.1256	0.1159	0.0770	0.0646	0.0939	0.0809	
					0.0511	0.2140	0.1969	0.1540	0.1293	0.1878	0.1617	
				λ_1	0.0256	0.1915	0.1966	0.1915	0.1966	0.1914	0.1966	
					0.1776	0.0105	0.0050	0.0085	0.0034	0.0085	0.0034	
					0.8720	0.0425	0.0195	0.0424	0.0170	0.0427	0.0172	
				λ_2	0.2837	0.2875	0.2940	0.2875	0.2940	0.2874	0.2940	
					0.0598	0.0140	0.0064	0.0125	0.0060	0.0126	0.0060	
					0.0544	0.0417	0.0200	0.0416	0.0199	0.0418	0.0200	
			II	α	0.4776	0.4787	0.4858	0.4885	0.4949	0.4696	0.4770	32.738
					0.0240	0.0994	0.0956	0.0115	0.0051	0.0304	0.0230	
					0.0447	0.1613	0.1552	0.0230	0.0103	0.0608	0.0460	
				λ_1	0.0258	0.1960	0.1990	0.1961	0.1990	0.1960	0.1989	
					0.1772	0.0062	0.0024	0.0039	0.0010	0.0040	0.0010	
					0.8711	0.0233	0.0084	0.0197	0.0051	0.0199	0.0501	
				λ_2	0.2851	0.2908	0.3022	0.2907	0.3022	0.2907	0.3022	
					0.0578	0.0111	0.0040	0.0093	0.0022	0.0093	0.0022	
					0.0496	0.0310	0.0109	0.0308	0.0074	0.0311	0.0073	
			III	α	0.4989	0.5336	0.5416	0.5638	0.5640	0.5107	0.5233	37.064
					0.0223	0.1650	0.1479	0.0640	0.0638	0.0233	0.0107	
					0.0115	0.2349	0.2165	0.1281	0.1277	0.0465	0.0214	
				λ_1	0.1956	0.1954	0.2023	0.1954	0.2023	0.1954	0.2023	
					0.0476	0.0053	0.0025	0.0046	0.0023	0.0046	0.0023	
					0.1817	0.0232	0.0115	0.0231	0.0105	0.0232	0.0115	
				λ_2	0.2963	0.2939	0.3028	0.2939	0.3028	0.2939	0.3028	
					0.0401	0.0074	0.0030	0.0061	0.0028	0.0061	0.0028	
					0.0205	0.0206	0.0093	0.0202	0.0093	0.0203	0.0093	

Table 4: *

Continue Table 2.

T	n	r	Scheme	Parameter	MLE	SE		LINEX				D
$c \rightarrow$								-2		+2		
Prior $\rightarrow$						1	2	1	2	1	2	
2	50	20	I	α	0.1771	0.1921	0.1997	0.1953	0.2028	0.1891	0.1967	17.737
					0.3302	0.3129	0.3053	0.3047	0.2972	0.3109	0.3033	
					0.6458	0.6158	0.6007	0.6094	0.5945	0.6219	0.6066	
				λ_1	0.0142	0.2084	0.2070	0.2085	0.2070	0.2084	0.2070	
					0.1875	0.0095	0.0075	0.0085	0.0070	0.0084	0.0070	
					0.9292	0.0441	0.0351	0.0423	0.0352	0.0421	0.0351	
				λ_2	0.1699	0.3088	0.2922	0.3088	0.2923	0.3087	0.2922	
					0.1530	0.0111	0.0088	0.0088	0.0077	0.0087	0.0078	
					0.4337	0.0335	0.0259	0.0294	0.0258	0.0291	0.0259	
			II	α	0.3760	0.4990	0.5028	0.5199	0.5211	0.4809	0.4870	18.677
					0.1370	0.1388	0.1300	0.0211	0.0199	0.0190	0.0130	
					0.2479	0.2156	0.2002	0.0422	0.0397	0.0381	0.0259	
				λ_1	0.0243	0.1939	0.2010	0.1939	0.2010	0.1939	0.2009	
					0.1801	0.0075	0.0029	0.0061	0.0010	0.0061	0.0010	
					0.8783	0.0309	0.0124	0.0306	0.0050	0.0307	0.0049	
				λ_2	0.2307	0.3044	0.2997	0.3044	0.2997	0.3044	0.2997	
					0.1167	0.0052	0.0020	0.0044	0.0003	0.0044	0.0003	
					0.2309	0.0147	0.0053	0.0147	0.0009	0.0147	0.0009	
			III	α	0.4676	0.4856	0.4949	0.5083	0.5007	0.4719	0.4823	20.000
					0.0358	0.1205	0.1140	0.0083	0.0007	0.0281	0.0177	
					0.0648	0.1909	0.1834	0.0165	0.0014	0.0561	0.0354	
				λ_1	0.0282	0.1937	0.1996	0.1937	0.1996	0.1937	0.1996	
					0.1774	0.0068	0.0019	0.0063	0.0004	0.0063	0.0004	
					0.8589	0.0315	0.0076	0.0315	0.0020	0.0316	0.0020	
				λ_2	0.2681	0.2974	0.2979	0.2974	0.2979	0.2973	0.2979	
					0.0877	0.0043	0.0024	0.0026	0.0021	0.0027	0.0021	
					0.1064	0.0117	0.0069	0.0088	0.0069	0.0088	0.0069	
		40	I	α	0.4797	0.4597	0.4636	0.4699	0.4731	0.4502	0.4548	35.514
					0.0215	0.1072	0.1024	0.0301	0.0269	0.0498	0.0452	
					0.0407	0.1739	0.1661	0.0602	0.0538	0.0997	0.0905	
				λ_1	0.0251	0.1943	0.1950	0.1943	0.1951	0.1943	0.1950	
					0.1772	0.0062	0.0069	0.0057	0.0049	0.0057	0.0050	
					0.8747	0.0286	0.0278	0.0284	0.0246	0.0285	0.0249	
				λ_2	0.2885	0.2935	0.2959	0.2935	0.2959	0.2934	0.2959	
					0.0501	0.0074	0.0046	0.0065	0.0041	0.0066	0.0041	
					0.0382	0.0219	0.0137	0.0218	0.0136	0.0218	0.0137	
			II	α	0.4811	0.4745	0.4905	0.4861	0.4999	0.4643	0.4816	36.179
					0.0200	0.1072	0.0961	0.0139	0.0001	0.0357	0.0184	
					0.0379	0.1708	0.1554	0.0278	0.0002	0.0715	0.0369	
				λ_1	0.0250	0.1956	0.1978	0.1956	0.1978	0.1955	0.1978	
					0.1771	0.0047	0.0026	0.0044	0.0022	0.0045	0.0022	
					0.8749	0.1708	0.0116	0.0222	0.0108	0.0223	0.0108	
				λ_2	0.2901	0.2943	0.2985	0.2943	0.2985	0.2943	0.2985	
					0.0463	0.0059	0.0016	0.0057	0.0015	0.0057	0.0015	
					0.0332	0.0191	0.0050	0.0191	0.0050	0.0191	0.0050	
			III	α	0.4963	0.4746	0.4790	0.4852	0.4889	0.4648	0.4698	39.945
					0.0068	0.1040	0.1001	0.0148	0.0110	0.0352	0.0302	
					0.0082	0.1685	0.1623	0.0296	0.0221	0.0704	0.0604	
				λ_1	0.0283	0.1966	0.1985	0.1966	0.1985	0.1966	0.1985	
					0.1746	0.0042	0.0018	0.0034	0.0015	0.0034	0.0015	
					0.8600	0.0171	0.0079	0.0171	0.0074	0.0172	0.0074	
				λ_2	0.2932	0.2961	0.2982	0.2961	0.2982	0.2961	0.2982	
					0.0433	0.0045	0.0019	0.0039	0.0018	0.0039	0.0018	
					0.0227	0.0130	0.0059	0.0130	0.0059	0.0130	0.0059	

Table 5: Average estimated values, MSEs and RABs of the MLEs and Bayes estimators based on AT-I PHCS for when ($n = 30$) for ($T = 1,2$) and ($r = 12,24$) for $(\alpha,\lambda_1,\lambda_2) = (1.0,0.3,0.5)$.

T	n	r	Scheme	Parameter	MLE	SE		LINEX				D
$c \rightarrow$								-2		+2		
Prior $\rightarrow$						1	2	1	2	1	2	
1	30	12	I	α	0.1528	0.3379	0.3992	0.3617	0.4227	0.3196	0.3798	10.797
					0.8517	0.6775	0.6182	0.6383	0.5774	0.6804	0.6201	
					0.8472	0.6624	0.6009	0.6382	0.5773	0.6804	0.6202	
				λ_1	0.0150	0.5022	0.4895	0.5046	0.4949	0.4997	0.4842	
					0.2861	0.2082	0.2031	0.2046	0.1949	0.1997	0.1842	
					0.9499	0.6740	0.6316	0.6819	0.6496	0.6656	0.6140	
				λ_2	0.2934	0.6631	0.6882	0.6749	0.6892	0.6529	0.6872	
					0.2304	0.1940	0.1908	0.1892	0.1749	0.1872	0.1529	
					0.4133	0.3262	0.3764	0.3784	0.3498	0.3745	0.3057	
			II	α	0.1618	0.4696	0.5014	0.5275	0.5345	0.4332	0.4748	11.259
					0.8425	0.5704	0.5274	0.4725	0.4655	0.5667	0.5252	
					0.8382	0.5402	0.5012	0.4725	0.4654	0.5668	0.5252	
				λ_1	0.0156	0.4743	0.4626	0.4765	0.4637	0.4722	0.4615	
					0.2854	0.1804	0.1660	0.1765	0.1637	0.1721	0.1615	
					0.9479	0.5811	0.5421	0.5883	0.5458	0.5739	0.5383	
				λ_2	0.3157	0.6548	0.6369	0.6579	0.6382	0.6515	0.6356	
					0.2110	0.1647	0.1415	0.1578	0.1381	0.1515	0.1356	
					0.3686	0.3095	0.2739	0.3157	0.2764	0.3030	0.2713	
			III	α	0.5348	0.4823	0.6272	0.5062	0.6526	0.4621	0.6043	12.000
					0.5153	0.5383	0.4038	0.4938	0.3474	0.5379	0.3957	
					0.4652	0.5185	0.3755	0.4938	0.3474	0.5378	0.3957	
				λ_1	0.0164	0.4486	0.4325	0.4506	0.4335	0.4466	0.4315	
					0.2853	0.1553	0.1362	0.1506	0.1335	0.1465	0.1315	
					0.9453	0.4955	0.4417	0.5021	0.4450	0.4886	0.4384	
				λ_2	0.3735	0.3849	0.5865	0.3883	0.5923	0.3816	0.5801	
					0.1766	0.1287	0.1165	0.1117	0.0923	0.1184	0.0801	
					0.2529	0.2302	0.2115	0.2234	0.1846	0.2367	0.1603	
		24	I	α	0.9250	0.8822	0.9454	0.9474	0.9999	0.8325	0.9002	21.508
					0.0885	0.2637	0.2281	0.0526	0.0001	0.1675	0.0997	
					0.0750	0.2147	0.1832	0.0526	0.0001	0.1675	0.0998	
				λ_1	0.0314	0.2454	0.2702	0.2459	0.2707	0.2449	0.2697	
					0.2711	0.0589	0.0374	0.0541	0.0293	0.0551	0.0303	
					0.8954	0.1821	0.0992	0.1804	0.0975	0.1837	0.1009	
				λ_2	0.4750	0.4636	0.4823	0.4639	0.4824	0.4633	0.4821	
					0.2706	0.0401	0.0209	0.0361	0.0176	0.0367	0.0179	
					0.8944	0.0728	0.0374	0.0723	0.0352	0.0734	0.0357	
			II	α	0.9309	0.9769	1.0187	1.0498	1.0817	0.9180	0.9674	22.032
					0.0835	0.2564	0.2384	0.0816	0.0497	0.0819	0.0326	
					0.0691	0.2074	0.1867	0.0817	0.0498	0.0820	0.0326	
				λ_1	0.0317	0.2584	0.2720	0.2591	0.2726	0.2576	0.2715	
					0.2706	0.0498	0.0364	0.0409	0.0274	0.0424	0.0285	
					0.8944	0.1391	0.0953	0.1363	0.0914	0.1412	0.0950	
				λ_2	0.4734	0.4667	0.4800	0.4669	0.4805	0.4665	0.4795	
					0.0893	0.0362	0.0297	0.0331	0.0195	0.0335	0.0205	
					0.0532	0.0666	0.0483	0.0661	0.0391	0.0669	0.0410	
			III	α	0.9599	0.9455	0.9738	1.0062	1.0171	0.8957	0.9363	23.592
					0.0447	0.2399	0.2019	0.0170	0.0061	0.1042	0.0637	
					0.0400	0.1925	0.1617	0.0171	0.0062	0.1043	0.0637	
				λ_1	0.0355	0.2627	0.2665	0.2635	0.2667	0.2620	0.2664	
					0.2680	0.0462	0.0358	0.0365	0.0333	0.0380	0.0336	
					0.8832	0.1262	0.1115	0.1218	0.1110	0.1268	0.1120	
				λ_2	0.4831	0.4720	0.4813	0.4723	0.4814	0.4718	0.4812	
					0.0797	0.0325	0.0205	0.0277	0.0186	0.0282	0.0188	
					0.0337	0.0571	0.0374	0.0553	0.0373	0.0564	0.0375	

Table 6: *

Continue Table 3.

T	n	r	Scheme	Parameter	MLE	SE		LINEX				D
$c \rightarrow$								-2		+2		
Prior $\rightarrow$						1	2	1	2	1	2	
2	30	12	I	α	0.1876	0.3474	0.3763	0.3728	0.3942	0.3270	0.3614	11.640
					0.8176	0.6697	0.6366	0.6271	0.6058	0.6730	0.6385	
					0.8124	0.6528	0.6239	0.6272	0.6058	0.6730	0.6386	
				λ_1	0.0138	0.1975	0.1985	0.2013	0.2020	0.1935	0.1952	
					0.2872	0.1201	0.1171	0.0987	0.0980	0.1065	0.1048	
					0.9539	0.3416	0.3484	0.3290	0.3268	0.3551	0.3495	
				λ_2	0.2962	0.5922	0.4009	0.6038	0.4058	0.5815	0.3961	
					0.2282	0.1403	0.1211	0.1038	0.0942	0.1039	0.0815	
					0.4077	0.2094	0.2034	0.2077	0.1885	0.2078	0.1630	
			II	α	0.4235	0.4582	0.4989	0.5024	0.5308	0.4273	0.4740	11.798
					0.6021	0.5742	0.5382	0.4975	0.4692	0.5727	0.5260	
					0.5765	0.5479	0.5038	0.4976	0.4692	0.5727	0.5260	
				λ_1	0.0184	0.2164	0.2288	0.2203	0.2307	0.2127	0.2269	
					0.2836	0.1039	0.0834	0.0797	0.0693	0.0873	0.0731	
					0.9386	0.3063	0.2401	0.2657	0.2310	0.2910	0.2436	
				λ_2	0.3497	0.5520	0.4000	0.5623	0.4021	0.5417	0.3978	
					0.1962	0.1142	0.1102	0.0979	0.0623	0.1022	0.0417	
					0.3006	0.1949	0.2004	0.1958	0.1246	0.2043	0.0834	
			III	α	0.5623	0.5029	0.6895	0.5257	0.7312	0.4826	0.6558	12.000
					0.4789	0.5183	0.3657	0.4742	0.2687	0.5174	0.3441	
					0.4377	0.4975	0.3309	0.4743	0.2688	0.5174	0.3442	
				λ_1	0.0206	0.3489	0.2643	0.3543	0.2664	0.3434	0.2623	
					0.2818	0.0888	0.0575	0.0543	0.0336	0.0434	0.0377	
					0.9314	0.2477	0.1696	0.1811	0.1121	0.1446	0.1257	
				λ_2	0.3691	0.5826	0.4416	0.5873	0.4453	0.5777	0.4377	
					0.1852	0.1078	0.0849	0.0873	0.0547	0.0777	0.0623	
					0.2618	0.1870	0.1322	0.1746	0.1039	0.1554	0.1245	
		24	I	α	0.9306	0.9004	0.9335	0.9667	0.9849	0.8482	0.8901	23.250
					0.0786	0.2615	0.2271	0.0333	0.0151	0.1518	0.1099	
					0.0694	0.2165	0.1854	0.0333	0.0151	0.1518	0.1099	
				λ_1	0.0291	0.2554	0.5668	0.2563	0.2671	0.2546	0.2664	
					0.2728	0.0532	0.0379	0.0437	0.0329	0.0454	0.0336	
					0.9028	0.1485	0.1108	0.1457	0.1097	0.1514	0.1119	
				λ_2	0.4750	0.4751	0.4921	0.4752	0.4924	0.4750	0.4919	
					0.0853	0.0267	0.0176	0.0248	0.0076	0.0250	0.0081	
					0.0499	0.0503	0.0259	0.0497	0.0152	0.0500	0.0162	
			II	α	0.9796	1.0106	1.0126	1.0877	1.0649	0.9521	0.9676	23.367
					0.0217	0.2577	0.2204	0.0876	0.0649	0.0478	0.0323	
					0.0204	0.1984	0.1725	0.0877	0.0649	0.0479	0.0324	
				λ_1	0.0361	0.2635	0.2651	0.2639	0.2652	0.2631	0.2651	
					0.2661	0.0414	0.0358	0.0361	0.0348	0.0369	0.0349	
					0.8795	0.1221	0.1162	0.1203	0.1160	0.1229	0.1164	
				λ_2	0.4929	0.4781	0.4999	0.4782	0.5000	0.4780	0.4998	
					0.0519	0.0247	0.0102	0.0218	0.0001	0.0220	0.0002	
					0.0143	0.0438	0.0174	0.0435	0.0001	0.0441	0.0004	
			III	α	0.9865	1.0607	1.0877	1.1421	1.1249	1.0410	1.0078	23.996
					0.0141	0.2483	0.2411	0.1421	0.1249	0.0409	0.0078	
					0.0135	0.1912	0.1853	0.1421	0.1249	0.0410	0.0078	
				λ_1	0.0364	0.2660	0.2716	0.2662	0.2718	0.2658	0.2715	
					0.2652	0.0364	0.0310	0.0338	0.0282	0.0342	0.0285	
					0.8787	0.1133	0.0945	0.1127	0.0940	0.1138	0.0950	
				λ_2	0.4977	0.4794	0.4980	0.4794	0.4981	0.4794	0.4978	
					0.0294	0.0214	0.0108	0.0206	0.0019	0.0206	0.0022	
					0.0046	0.0412	0.0176	0.0412	0.0039	0.0413	0.0043	

Table 7: Average estimated values, MSEs and RABs of the MLEs and Bayes estimators based on AT-I PHCS for when ($n = 50$) for ($T = 1,2$) and ($r = 20,40$) for $(\alpha,\lambda_1,\lambda_2) = (1.0,0.3,0.5)$.

T	n	r	Scheme	Parameter	MLE	SE		LINEX				D
c →								-2		+2		
Prior →						1	2	1	2	1	2	
1	50	20	I	α	0.4141	0.4562	0.5219	0.4732	0.5411	0.4408	0.5050	17.966
					0.6072	0.5585	0.4965	0.5267	0.4589	0.5591	0.4950	
					0.5859	0.5438	0.4784	0.5268	0.4589	0.5592	0.4949	
				λ_1	0.0143	0.2715	0.2749	0.2721	0.2756	0.2708	0.2742	
					0.2865	0.0385	0.0366	0.0279	0.0244	0.0292	0.0258	
					0.9525	0.0987	0.0970	0.0929	0.0813	0.0973	0.0860	
				λ_2	0.3474	0.4652	0.4787	0.4653	0.4788	0.4652	0.4786	
					0.1838	0.0357	0.0234	0.0347	0.0212	0.0348	0.0214	
					0.3052	0.0695	0.0426	0.0694	0.0424	0.0697	0.0428	
			II	α	0.4177	0.6811	0.7574	0.7172	0.7970	0.6511	0.7236	18.708
					0.6052	0.3667	0.3089	0.2828	0.2030	0.3489	0.2764	
					0.5823	0.3321	0.2707	0.2827	0.2029	0.3489	0.2764	
				λ_1	0.0158	0.2986	0.3045	0.2989	0.3047	0.2982	0.3044	
					0.2850	0.0187	0.0135	0.0011	0.0047	0.0018	0.0043	
					0.9472	0.0582	0.0373	0.0036	0.0156	0.0059	0.0145	
				λ_2	0.3599	0.4795	0.5128	0.4795	0.5129	0.4795	0.5127	
					0.1756	0.0212	0.0156	0.0205	0.0128	0.0205	0.0127	
					0.2802	0.0410	0.0259	0.0410	0.0258	0.0411	0.0255	
			III	α	0.4403	0.7467	0.8019	0.7836	0.8356	0.7147	0.7720	20.000
					0.5794	0.3137	0.2664	0.2164	0.1644	0.2853	0.2280	
					0.5597	0.2746	0.2282	0.2164	0.1644	0.2853	0.2280	
				λ_1	0.0156	0.2906	0.2970	0.2907	0.2971	0.2906	0.2969	
					0.2849	0.0113	0.0098	0.0093	0.0029	0.0094	0.0031	
					0.9480	0.0313	0.0261	0.0311	0.0097	0.0313	0.0103	
				λ_2	0.3749	0.4912	0.5027	0.4916	0.5030	0.4909	0.5024	
					0.1464	0.0205	0.0176	0.0084	0.0030	0.0091	0.0024	
					0.2501	0.0320	0.0302	0.0167	0.0060	0.0181	0.0048	
		40	I	α	0.9468	0.6862	0.7101	0.7059	0.7297	0.6682	0.6921	35.935
					0.0580	0.3425	0.3207	0.2940	0.2702	0.3318	0.3079	
					0.0532	0.3163	0.2942	0.2941	0.2703	0.3318	0.3079	
				λ_1	0.0354	0.2749	0.2777	0.2750	0.2778	0.2748	0.2775	
					0.2658	0.0269	0.0252	0.0250	0.0222	0.0252	0.0225	
					0.8819	0.0837	0.0744	0.0834	0.0740	0.0840	0.0749	
				λ_2	0.4923	0.4745	0.4789	0.4748	0.4789	0.4743	0.4789	
					0.0362	0.0299	0.0219	0.0252	0.0211	0.0257	0.0211	
					0.0155	0.0510	0.0422	0.0505	0.0421	0.0515	0.0423	
			II	α	0.9507	0.7675	0.8211	0.7910	0.8475	0.7462	0.7974	36.609
					0.0529	0.2765	0.2387	0.2089	0.1524	0.2538	0.2025	
					0.0493	0.2447	0.2065	0.2090	0.1525	0.2538	0.2026	
				λ_1	0.0366	0.2797	0.2880	0.2797	0.2881	0.2796	0.2879	
					0.2646	0.0214	0.0154	0.0203	0.0119	0.0204	0.0121	
					0.8781	0.0677	0.0413	0.0676	0.0398	0.0679	0.0404	
				λ_2	0.4946	0.4832	0.4901	0.4832	0.4902	0.4831	0.4900	
					0.0243	0.0180	0.0127	0.0168	0.0098	0.0169	0.0100	
					0.0109	0.0337	0.0198	0.0336	0.0197	0.0337	0.0199	
			III	α	0.9854	0.8884	0.9060	0.9223	0.9375	0.8580	0.8781	39.968
					0.0253	0.2110	0.1962	0.0776	0.0624	0.1420	0.1218	
					0.0168	0.1750	0.1648	0.0777	0.0625	0.1420	0.1219	
				λ_1	0.0423	0.2900	0.2910	0.2900	0.2910	0.2900	0.2910	
					0.2600	0.0111	0.0105	0.0100	0.0090	0.0100	0.0090	
					0.8613	0.0334	0.0301	0.0333	0.0299	0.0335	0.0301	
				λ_2	0.4993	0.4902	0.5101	0.4903	0.5101	0.4900	0.5101	
					0.0064	0.0162	0.0113	0.0097	0.0101	0.0100	0.101	
					0.0013	0.0220	0.0202	0.0194	0.0203	0.0200	0.0202	

Table 8: *

Continue Table 4.

T	n	r	Scheme	Parameter	MLE	SE		LINEX				D
$c \rightarrow$								-2		+2		
Prior $\rightarrow$						1	2	1	2	1	2	
2	50	20	I	α	0.4885	0.5783	0.6501	0.6132	0.6847	0.5519	0.6219	19.438
					0.5311	0.4557	0.3915	0.3867	0.3153	0.4481	0.3781	
					0.5115	0.4292	0.3584	0.3868	0.3153	0.4481	0.3781	
				λ_1	0.0144	0.2702	0.2851	0.2704	0.2853	0.2700	0.2848	
					0.2862	0.0333	0.0220	0.0296	0.0147	0.0300	0.0152	
					0.9519	0.0994	0.0584	0.0986	0.0489	0.1001	0.0507	
				λ_2	0.3614	0.4711	0.5110	0.4714	0.5111	0.4708	0.5109	
					0.1717	0.0331	0.0150	0.0286	0.0111	0.0292	0.0109	
					0.2773	0.0597	0.0245	0.0572	0.0223	0.0583	0.0219	
			II	α	0.7401	0.9635	0.9756	1.0545	1.0476	0.8951	0.9162	19.641
					0.2910	0.2820	0.2560	0.0544	0.0475	0.1049	0.0837	
					0.2599	0.2245	0.2055	0.0545	0.0476	0.1048	0.0838	
				λ_1	0.0213	0.2872	0.2896	0.2873	0.2897	0.2871	0.2895	
					0.2805	0.0156	0.0141	0.0127	0.0103	0.0128	0.0105	
					0.9290	0.0430	0.0398	0.0423	0.0342	0.0428	0.0349	
				λ_2	0.4249	0.4878	0.4889	0.4878	0.4889	0.4878	0.4889	
					0.1352	0.0132	0.0116	0.0122	0.0111	0.0122	0.0111	
					0.1501	0.0244	0.0222	0.0244	0.0222	0.0245	0.0223	
			III	α	0.7889	0.9717	0.9919	1.0279	1.0374	0.9230	0.9507	20.000
					0.2497	0.2295	0.2081	0.0278	0.0373	0.0769	0.0492	
					0.2111	0.1829	0.1651	0.0279	0.0374	0.0770	0.0493	
				λ_1	0.0209	0.2882	0.2933	0.2882	0.2933	0.2882	0.2933	
					0.2805	0.0123	0.0077	0.0118	0.0067	0.0118	0.0067	
					0.9304	0.0394	0.0224	0.0394	0.0222	0.0394	0.0223	
				λ_2	0.4389	0.4898	0.4910	0.4898	0.4910	0.4898	0.4909	
					0.1186	0.0112	0.0097	0.0102	0.0090	0.0102	0.0091	
					0.1223	0.0205	0.0181	0.0204	0.0180	0.0205	0.0181	
		40	I	α	0.9561	0.9522	0.9791	0.9943	1.0162	0.9159	0.9464	38.800
					0.0468	0.2030	0.1876	0.0056	0.0161	0.0840	0.0535	
					0.0439	0.1626	0.1492	0.0057	0.0162	0.0841	0.0536	
				λ_1	0.0399	0.2823	0.2865	0.2823	0.2865	0.2723	0.2865	
					0.2618	0.0181	0.0145	0.0177	0.0135	0.0177	0.0135	
					0.8671	0.0590	0.0450	0.0590	0.0449	0.0591	0.0451	
				λ_2	0.4947	0.4777	0.5126	0.4778	0.5126	0.4777	0.5125	
					0.0315	0.0229	0.0129	0.0222	0.0126	0.0223	0.0125	
					0.0106	0.0445	0.0251	0.0444	0.0251	0.0446	0.0251	
			II	α	0.9837	1.0658	1.0670	1.1217	1.1095	1.0179	1.0292	38.945
					0.0168	0.2364	0.2108	0.1216	0.1094	0.0178	0.0291	
					0.0163	0.1823	0.1628	0.1217	0.1095	0.0179	0.0292	
				λ_1	0.0406	0.2903	0.2899	0.2903	0.2900	0.2902	0.2899	
					0.2610	0.0123	0.0121	0.0100	0.0097	0.0101	0.0098	
					0.8645	0.0324	0.0336	0.0334	0.0322	0.0337	0.0326	
				λ_2	0.4987	0.4905	0.5051	0.4906	0.5051	0.4905	0.5050	
					0.0206	0.0105	0.0075	0.0094	0.0051	0.0095	0.0050	
					0.0026	0.0189	0.0129	0.0189	0.0102	0.0190	0.0101	
			III	α	0.9905	1.0954	1.0151	1.1451	1.0548	1.0515	0.9793	40.000
					0.0097	0.2361	0.1946	0.1450	0.0548	0.0514	0.0206	
					0.0095	0.1828	0.1559	0.1451	0.0548	0.0515	0.0207	
				λ_1	0.0438	0.2907	0.2968	0.2907	0.2969	0.2907	0.2969	
					0.2577	0.0101	0.0038	0.0093	0.0031	0.0093	0.0031	
					0.8539	0.0311	0.0105	0.0310	0.0104	0.0311	0.0104	
				λ_2	0.4998	0.4925	0.5026	0.4925	0.5026	0.4925	0.5026	
					0.0007	0.0083	0.0030	0.0075	0.0025	0.0075	0.0026	
					0.0003	0.0150	0.0051	0.0149	0.0051	0.0150	0.0051	

Table 9: The ACLs for 95% ACIs/BCIs of α, λ_1 and λ_2 based on AT- I PHCS under various censoring schemes when $(\alpha, \lambda_1, \lambda_2) = (0.5, 0.2, 0.3)$.

T	n	r	Scheme	α			λ_1			λ_2		
				ACI	BCI		ACI	BCI		ACI	BCI	
Prior →					1	2		1	2		1	2
1	30	12	I	0.2274	0.0132	0.0095	0.0156	0.0141	0.0105	0.1116	0.0385	0.0252
			II	0.2376	0.1480	0.1243	0.0154	0.0142	0.0137	0.1185	0.0608	0.0362
			III	0.2442	0.1841	0.1825	0.2138	0.0197	0.0163	0.2405	0.0694	0.0446
		24	I	0.4909	0.1990	0.1966	0.0157	0.0099	0.0077	0.1262	0.0109	0.0080
			II	0.4989	0.2263	0.2104	0.0160	0.0118	0.0081	0.1274	0.0137	0.0130
			III	0.5445	0.3707	0.3370	0.0393	0.0151	0.0125	0.2562	0.0425	0.0419
	50	20	I	0.1824	0.1928	0.1885	0.0071	0.0104	0.0081	0.0706	0.0164	0.0126
			II	0.1873	0.2746	0.2563	0.0079	0.0202	0.0107	0.0805	0.0300	0.0180
			III	0.1955	0.3191	0.3121	0.1560	0.0294	0.0108	0.1692	0.0328	0.0109
		40	I	0.2425	0.3570	0.3456	0.0058	0.0079	0.0041	0.1035	0.0133	0.0043
			II	0.2448	0.3714	0.3598	0.0062	0.0142	0.0072	0.1051	0.0203	0.0081
			III	0.4610	0.6024	0.5389	0.0241	0.0187	0.0135	0.2047	0.0202	0.0117
2	30	12	I	0.2605	0.1618	0.1608	0.0149	0.0157	0.0147	0.1099	0.0185	0.0110
			II	0.5255	0.3262	0.2716	0.0234	0.0284	0.0218	0.1565	0.0207	0.0184
			III	0.8660	0.2881	0.2597	0.0398	0.0285	0.0243	0.2106	0.0428	0.0271
		24	I	0.4690	0.2472	0.2340	0.0147	0.0177	0.0065	0.1160	0.0109	0.0087
			II	0.4804	0.2798	0.2411	0.0155	0.0346	0.0065	0.1203	0.0179	0.0144
			III	0.5713	0.3333	0.2896	0.0317	0.0216	0.0102	0.2432	0.0205	0.0193
	50	20	I	0.2037	0.2080	0.2061	0.0060	0.0092	0.0073	0.0707	0.0092	0.0076
			II	0.2718	0.5221	0.4850	0.0087	0.0138	0.0080	0.1060	0.0103	0.0043
			III	0.4581	0.4426	0.4344	0.0146	0.0188	0.0085	0.1409	0.0226	0.0161
		40	I	0.2273	0.3747	0.3582	0.0046	0.0048	0.0039	0.0966	0.0046	0.0025
			II	0.2304	0.3829	0.3586	0.0049	0.0074	0.0046	0.0986	0.0067	0.0026
			III	0.5281	0.3778	0.3714	0.0986	0.0145	0.0105	0.1986	0.0129	0.0076

Table 10: The ACLs for 95% ACIs/BCIs of α, λ_1 and λ_2 based on AT- I PHCS under various censoring schemes when $(\alpha,\lambda_1,\lambda_2) = (1.0,0.3,0.5)$.

T	n	r	Scheme	α			λ_1			λ_2		
				ACI	BCI		ACI	BCI		ACI	BCI	
Prior →					1	2		1	2		1	2
1	30	12	I	0.2485	0.5267	0.5131	0.0072	0.1597	0.1189	0.1662	0.1976	0.1196
			II	0.2665	0.7230	0.6133	0.0076	0.1746	0.1190	0.1870	0.2627	0.1943
			III	0.6608	0.5862	0.5442	0.0099	0.2124	0.1884	0.2522	0.3680	0.1260
		24	I	0.5063	0.8863	0.7370	0.0075	0.0724	0.0417	0.2359	0.0557	0.0275
			II	0.5138	0.8942	0.8245	0.0077	0.0806	0.0653	0.2402	0.0588	0.0409
			III	0.5705	0.9170	0.8522	0.0107	0.0813	0.0702	0.2504	0.0699	0.0519
	50	20	I	0.3708	0.5125	0.4815	0.0047	0.0389	0.0235	0.1414	0.0321	0.0211
			II	0.3816	0.6903	0.6756	0.0057	0.0537	0.0397	0.1649	0.0368	0.0313
			III	0.4519	0.7224	0.6746	0.0072	0.0757	0.0679	0.1696	0.0581	0.0534
		40	I	0.2902	0.5314	0.5300	0.0025	0.0191	0.0152	0.1595	0.0282	0.0209
			II	0.2956	0.5967	0.5546	0.0028	0.0325	0.0264	0.1641	0.0409	0.0144
			III	0.4403	0.6907	0.6502	0.0085	0.0356	0.0337	0.1929	0.0512	0.0223
2	30	12	I	0.3111	0.5333	0.4590	0.0077	0.2161	0.2106	0.1650	0.2510	0.2118
			II	0.5883	0.6895	0.6056	0.0132	0.2235	0.1545	0.2539	0.2890	0.1722
			III	0.7094	0.7442	0.5622	0.0162	0.2604	0.1826	0.2881	0.3658	0.2327
		24	I	0.4226	0.9072	0.8256	0.0038	0.0463	0.0274	0.1875	0.0356	0.0196
			II	0.8740	0.9696	0.8542	0.0172	0.0655	0.0393	0.2739	0.0372	0.0366
			III	1.3819	0.9288	0.8363	0.0330	0.0827	0.0606	0.2647	0.0506	0.0379
	50	20	I	0.3833	0.6484	0.6410	0.0025	0.0133	0.0114	0.1396	0.0150	0.0138
			II	0.4369	0.9792	0.9465	0.0077	0.0322	0.0318	0.2141	0.0194	0.0137
			III	0.4637	0.8501	0.8082	0.0157	0.0584	0.0526	0.2108	0.0587	0.0348
		40	I	0.2800	0.7366	0.7286	0.0040	0.0122	0.0075	0.1537	0.0109	0.0051
			II	0.4448	0.8567	0.7647	0.0051	0.0167	0.0127	0.1967	0.0189	0.0147
			III	0.6697	0.8095	0.7223	0.0045	0.0241	0.0217	0.2079	0.0232	0.0113

6 Real-life data analysis

To show how the proposed methodology can be applied in real phenomenon, we shall analyze one real data set which was originally reported in [20] and later by [21]. The data set contains 58 electrodes (segments cut from bars) and was put on a high-stress voltage endurance life-test. The failures were attributed to one of two (causes) modes, the first cause called Mode E; is the insulation defect due to a processing problem which tends to occur early in life and the second cause called Mode D; on the other hand, is the degradation of the organic material which typically occurs at a later stage. However, the total number of observed failures due to causes Mode E and D are 18 and 27, respectively. Also, there are 13 unfilled electrodes were still running, we have denoted the missing cause by (+). The results of the insulation voltage endurance test are shown in Table 11. In this study, from the given dataset, we considered only those observations which were completely observed and left those observations which were still running.

Table 11: Voltage endurance life-test results (in hours) of 58 electrodes.

Mode	Hours
E	2, 3, 5, 8, 21, 28, 31, 64, 69, 76, 104, 119, 144, 160, 221, 236, 282, 303
D	168, 191, 203, 211, 226, 261, 264, 278, 284, 286, 298, 314, 317, 318, 320, 327, 328, 328, 348, 350, 360, 369, 377, 387, 392, 412, 446
+	13, 31, 52, 53, 67, 78, 113, 135, 157, 179, 241, 257, 348

Firstly, to investigate whether the competing risks generalized inverted exponential model can provide a reasonable fit for the given data set, Kolmogorov-Smirnov (K-S) goodness-of-fit test statistics is used. The values of MLEs of the

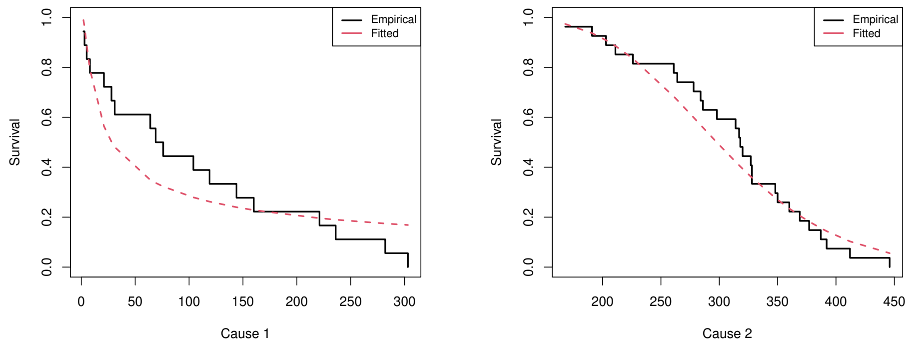


Fig. 1: Plots of the empirical and fitted survival functions under voltage endurance data.

unknown parameters α, λ_1 and λ_2 with their standard errors, the K-S distance between empirical and fitted distribution functions along with the corresponding p-value for each cause of failure are calculated and reported in Table 12.

Table 12: The MLEs and fitting K-S of GIED under voltage endurance data.

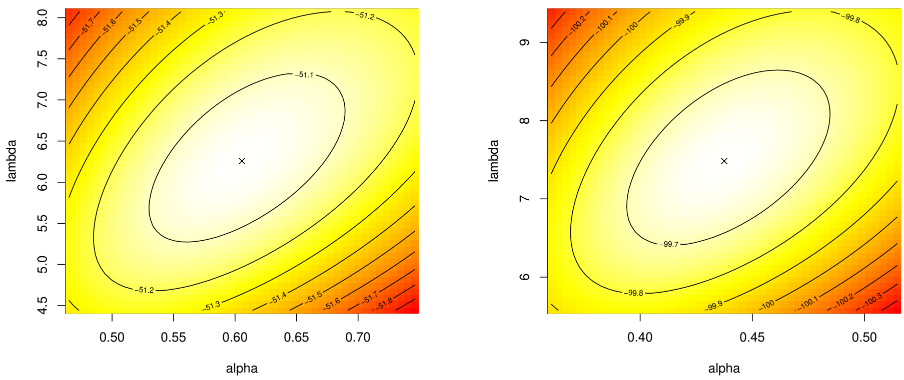
Data	Parameter	MLE(standard error)	K-S	
			Statistic	p-value
Mode E (Cause 1)	α	0.4827(0.1314)	0.2620	0.1405
	λ	7.6773(2.6871)		
Mode D (Cause 2)	α	47.095(4.1940)	0.1459	0.6132
	λ	1258.6(5.9322)		

From Table 12, since the p-values in both causes are quite higher than the significance level 0.05, we cannot reject the null hypothesis that the data is coming from GIED. For further clarification, both the empirical and the estimated survival functions for each cause of failure are plotted in Figure 1. It shows that the fitted survival functions are quite close to the corresponding empirical survival functions for both time of failure due to causes 1 and 2. Hence, the given data set can be analyzed using this distribution. From data set given in Table 11, using two different choices of T with fixed $r = 13$, $R_i = 2,\ i = 1,2,\ldots,12$ and $R_{13} = 8$, some adaptive Type-I progressive hybrid censored samples are generated and provided in Table 13.

Table 13: Different AT-I PHCS samples generated from voltage endurance data.

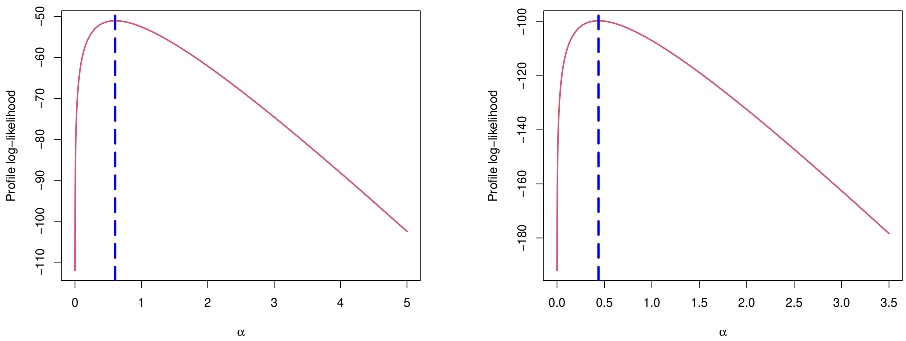
Sample	$T(D)$	Failure time (Cause)	D_1	D_2	R_D^*
A	100(10)	2(1), 3(1), 8(1), 21(1), 28(1), 31(1), 64(1), 76(1), 104(1), 168(2)	9	1	15
B	200(16)	2(1), 3(1), 8(1), 21(1), 28(1), 31(1), 64(1), 76(1), 104(1), 168(2), 191(2), 211(2), 221(1), 284(2), 320(2), 387(2)	10	6	5

Before calculating the proposed estimates, one of the major problems in the case of MLEs is that it is not often possible to prove the existence and uniqueness of the MLEs. To overcome this problem, we propose to provide the contour plot of the log-likelihood function with respect to the two-parameter GIED using the generated samples from voltage endurance



(Sample A) (Sample B)

Fig. 2: Contour plot of log-likelihood function of α and λ from voltage endurance data.



(Sample A) (Sample B)

Fig. 3: The profile log-likelihood function of α from voltage endurance data.

dataset as displayed in Figure 2. The maximum of the log-likelihood function is denoted by point x in the innermost contour.

For more illustration, to solve the starting point problem of running iterations, in Figure 3, we plot the profile log-likelihood function of α. It is clear that the profile log-likelihood is unimodal and the corresponding MLE of α due to samples A and B are close to 0.606 and 0.438, respectively. It also indicates that the MLE $\hat{\alpha}$ of α is unique and supports our findings using contour plot. The coordinates of x-point provide the MLEs of α and λ which become $(\hat{\alpha},\hat{\lambda}) \simeq (0.606, 6.258)$ and $(\hat{\alpha},\hat{\lambda}) \simeq (0.438, 7.478)$ for samples A and B, respectively. Further, it shows that the MLEs existed and are also unique. So, we suggest taking these estimates as initial values to start the computational iteration.

Based on both samples A and B, the MLEs (with their standard errors) and 95% two-sided ACIs (with their ACLs) of the parameters α, λ_1 and λ_2 have been calculated. Also, Bayes estimates of α, λ_1 and λ_2 relative to square error and LINEX loss functions (with $c = (-5, -0.05, +5)$) are developed. Because we have no prior information about the unknown parameters α, λ_1 and λ_2, we assume the non-informative priors (all hyper-parameters equal 0). However, due to calculation reasons, we take 0.0001 for all hyper-parameters.

Using the MCMC algorithm proposed in Section 4, we generate 20,000 MCMC samples and then first 5000 iterations have been discarded as a burn-in. The initial values of the unknown parameters α, λ_1 and λ_2 for running the MCMC sampler algorithm were taken to be their MLEs. However, the MLEs and Bayes estimators (with their standard errors) based on both samples A and B are calculated and reported in Table 14.

In addition, using the invariance property of the MLEs of α, λ_1 and λ_2, the relative risk rate due to causes 1 and 2 from sample A becomes 0.730 and 0.270, respectively. Similarly, the relative risk rate due to causes 1 and 2 from sample B becomes 0.681 and 0.319, respectively. Moreover, two-sided 95% ACIs/BCIs (with their ACLs) of the parameters α, λ_1 and λ_2 are calculated and listed in Table 15.

Tables 14-15 showed that the point estimates obtained by maximum likelihood and Bayesian methods of the unknown parameters α, λ_1 and λ_2 are quite close to each other. However, the point estimates will not be sufficient to judge which method provides better estimates because we do not know the actual values of unknown parameters. Regarding interval estimates, the results of Table 15 indicate that the Bayesian credible intervals are slightly shorter than the other confidence intervals in respect of their interval lengths. Furthermore, the classical and Bayes estimates obtained based on the observed sample B perform better than those obtained based on the observed sample A in terms of their standard errors. This result due to the pre-specified time T plays an important role in estimation problems of any function of the unknown parameters.

Table 14: The MLEs and Bayes estimators with their standard errors (in parenthesis) Based on AT-I PHCS under voltage endurance data.

Sample	Parameter	MLE	SE	LINEX		
$c \rightarrow$				-5	-0.05	+5
A	α	0.9916	0.0117	0.0117	0.0117	0.0116
		(0.2592)	(3.04×10^{-5})	(8.00×10^{-3})	(8.00×10^{-3})	(8.00×10^{-3})
	λ_1	0.0366	0.0363	0.0363	0.0363	0.0363
		(0.0114)	(4.10×10^{-6})	(2.58×10^{-6})	(2.58×10^{-6})	(2.59×10^{-6})
	λ_2	0.0995	0.0998	0.0998	0.0998	0.0997
		(0.0352)	(6.43×10^{-6})	(2.39×10^{-6})	(2.38×10^{-6})	(2.37×10^{-6})
B	α	2.3948	0.0181	0.0182	0.0181	0.0180
		(0.0102)	(3.72×10^{-5})	(1.94×10^{-2})	(1.94×10^{-2})	(1.94×10^{-2})
	λ_1	0.0017	0.0016	0.0016	0.0016	0.0016
		(0.0002)	(6.66×10^{-7})	(6.35×10^{-7})	(6.35×10^{-7})	(6.35×10^{-7})
	λ_2	0.0027	0.0028	0.0028	0.0028	0.0028
		(0.0004)	(8.17×10^{-7})	(7.74×10^{-7})	(7.74×10^{-7})	(7.73×10^{-7})

Table 15: The 95% ACIs/BCIs with their ACLs Based on AT-I PHCS under voltage endurance data.

Sample	Parameter	ACI			BCI		
		Lower	Upper	Length	Lower	Upper	Length
A	α	0.4836	1.4996	1.0160	0.0050	0.0191	0.0141
	λ_1	0.0142	0.0589	0.0447	0.0353	0.0372	0.0019
	λ_2	0.0304	0.1685	0.1380	0.0982	0.1012	0.0030
B	α	2.3748	2.4148	0.0400	0.0095	0.0271	0.0176
	λ_1	0.0012	0.0021	0.0009	0.0014	0.0017	0.0003
	λ_2	0.0019	0.0034	0.0015	0.0026	0.0030	0.0004

Table 16: Vital MCMC characteristics for some posterior characteristics of α, λ_1 and λ_2.

Sample	Parameter	Mean	Median	Mode	SD	Sk.
A	α	0.01169	0.01129	0.00207	3.73×10^{-3}	0.65901
	λ_1	0.03628	0.03628	0.03505	4.89×10^{-4}	0.02377
	λ_2	0.09979	0.09981	0.10010	7.87×10^{-4}	-0.01116
B	α	0.01812	0.01769	0.00514	4.56×10^{-3}	0.51969
	λ_1	0.00162	0.00162	0.00152	8.16×10^{-5}	0.00351
	λ_2	0.00280	0.00279	0.00295	1.00×10^{-4}	0.10673

Moreover, for each generated sample, some important characteristics such as: mean, median, mode, standard deviation (SD) and skewness (Sk.) for MCMC posterior distributions of α, λ_1 and λ_2 after bun-in; are computed and provided in Table 16. To show the MCMC convergence, using both generated samples, trace plots of the posterior distributions of α, λ_1 and λ_2 are plotted in Figure 4. It displays 15,000 outputs for the unknown parameters α, λ_1 and λ_2 with their sample mean (represented by soled lines (—)) and 95% two bounds BCIs (represented by dashed lines (- - -)). Further, it indicates that the MCMC sampling procedure has converged well. It also shows that the burn-in sample has an appropriate size to erase the effect of the initial values.

Further, using the Gaussian kernel, the marginal posterior density estimates of α, λ_1 and λ_2 with their histograms and sample means (vertical dashed lines (:)), based on MCMC samples of size 15,000 are represented in Figure 5. It is evident from the estimates that the generated posteriors of λ_1 and λ_2 for both generated samples A and B are fairly symmetric while the generated posteriors of α are positive quite skewed. Thus, the results of the proposed methods under voltage endurance dataset give a good explanation to our model.

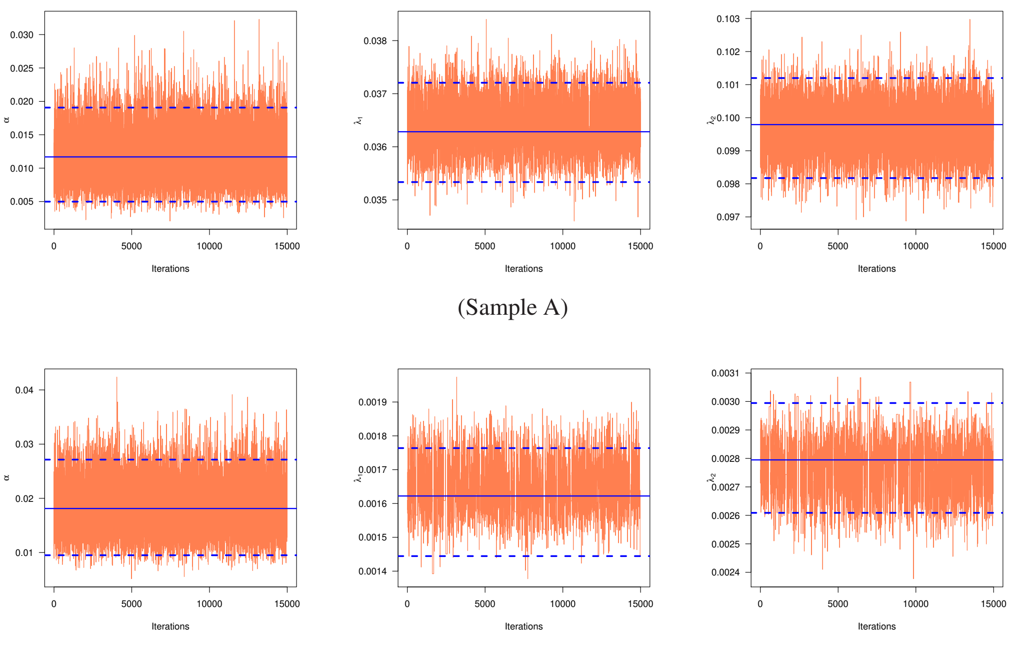
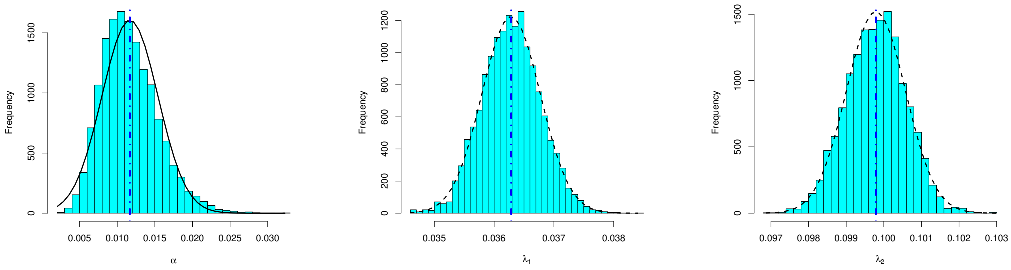
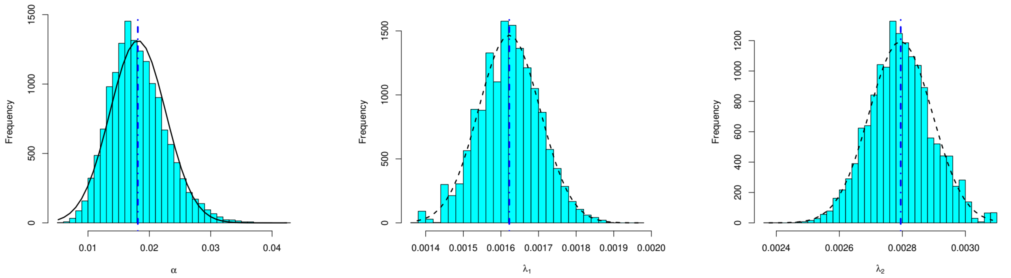


Fig. 4: MCMC trace plots of α, λ_1 and λ_2 from voltage endurance data.



(Sample A)



(Sample B)

Fig. 5: Histogram and kernel density estimates of α, λ_1 and λ_2 from voltage endurance data.

7 Conclusions

This paper has proposed several statistical inference methods to estimate the parameters and reliability of GIED when the adaptive Type-I progressive censoring scheme with independent competing risks, including maximum likelihood and Bayesian estimation and also the fixed number of causes of failure is unknown. We have derived MLEs and Bayesian estimation based on SE and LINEX loss functions for the unknown parameters of GIED. Then, using a simulation study, the efficiency of the MLEs and Bayes estimators of the parameters are compared to each other and a real data has also been presented to illustrate all the inferential results established here. In general, from the simulation it can be seen that the classical and Bayes estimates of the unknown parameters are good in terms of minimum RMSEs and RABs. Also, Bayes MCMC estimates using gamma informative prior are better as they include prior information than MLEs in terms of RMSEs and RABs. Finally, a numerical example is provided to illustrate the inference methods described in the paper. An assumption is made in this paper that the competing risks are statistically independent. The case, however, where the competing risks are dependent, is very common in practice and related statistical inference with dependent competing risks model is possible future work.

Acknowledgements

The authors thanks the editor, Associate Editor, and the reviewers for their constructive comments and suggestions on the original version of the manuscript, which led to this improved version.

Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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