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Fixed Points of Fuzzy Soft Mappings

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Abstract: In this paper, the concept of a fuzzy soft mapping on a fuzzy soft set is introduced and the study of fixed points of such mappings is initiated.

Keywords: Fuzzy soft set, fuzzy soft mapping, fixed point

1 Introduction

The concept of fuzzy soft set, introduced by Molodstov in [12], is a recent development to deal with uncertainties. The contribution made by probability theory, fuzzy set theory, vague sets, rough sets and interval mathematics to deal with uncertainty is of vital importance but the problem of inadequacy of parameters has been successfully resolved by Soft set theory. Maji et al. ([9], [10]) and Maji and Roy ([11]) elaborated on the theory of soft sets, fuzzy soft sets and intuitionistic fuzzy soft sets and highlighted some of their applications. Some basic operations of fuzzy soft union and intersection and other algebraic properties were studied by Ahmad and Kharal ([1]). Babitha and Sunil ([3]) and Sut ([16]) defined soft set relations and fuzzy soft relations and applied the theory to decision making problems. Biwas and Samanta ([6]) introduced relations on intuitionistic fuzzy soft sets.

The notion of soft topology on a soft set was introduced by Cagman et. al ([4]) and several properties of soft topological spaces have been discussed, among others, by Shabir and Naz ([15]), Hussain and Ahmad ([7]), and Chen ([5]). Fuzzy soft topological spaces were studied by Tridiv ([13]) and Mahanta ([8]).

Recently, Wardowski ([17]) introduced a notion of soft mapping and obtained its fixed point. Motivated by his work, we initiate the study of fixed point in fuzzy soft set theory. For this purpose we discuss some properties of a fuzzy soft element in Section 3 of this paper. In Section 4 we introduce fuzzy soft mappings with the help of cartesian product and relations on fuzzy soft sets. Concepts of fuzzy soft elements and fuzzy soft mappings to study fixed point theorems in the framework of fuzzy soft topological spaces are introduced in Section 5. Section 6 concludes the paper and gives insight to some possible future work.

2 Preliminaries

Throughout this section, by U, E and $P(U)$, we denote an initial universe, a set of parameters, and the collection of all subsets of U, respectively.

Definition 1.([18]) *A fuzzy set A in U is characterized by a function with domain as U and values in $[0,1]$. The collection of all fuzzy sets in U is denoted by I^U.*

Definition 2.([18]) *An empty fuzzy set denoted by $\widetilde{0}$ is a function which maps each $x \in U$ to 0. That is, $\widetilde{0}(x) = 0$ for all $x \in U$. A universal fuzzy set denoted by $\widetilde{1}$ is a function which maps each $x \in U$ to 1. That is, $\widetilde{1}(x) = 1$ for all $x \in U$.*

If $A, B \in I^U$ we write $A \preceq B$ whenever $A(x) \leq B(x)$ for each $x \in U$, and $A = B$ whenever $A \preceq B$ and $B \preceq A$ for all $x \in U$.

Definition 3. *([18]) Let A and B be two fuzzy sets. Then (a) their union $A \cup B$ is defined as $(A \cup B)(x) = \max\{A(x), B(x)\}$; (b) their intersection $A \cap B$ is defined as $(A \cap B)(x) = \min\{A(x), B(x)\}$, and (c) difference of B from A is denoted by $A\ /B$ and is defined by $(A\ /B)(x) = A(x) - B(x)$ for all $x \in U$.*

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Note that an implicit assumption $B \preceq A$ has been imposed to make the operation $A\,/B$ well defined.

Definition 4. *([18]) The complement of a fuzzy set A is denoted by A^c and is defined by $A^c(x) = 1 - A(x)$.*

Definition 5.*([12]) If F is a mapping on E taking values in $P(U)$, then a pair $(F,E)_s$ is called a soft set over (U,E).*

Definition 6.*([9]) Let A be a subset of E. A pair (F,A) is called a fuzzy soft set over (U,E) if $F : A \to I^U$ is a mapping from A into I^U. The collection of all fuzzy soft sets over (U,E) is denoted by $\mathscr{F}(U,E)$.*

A fuzzy soft set (F,A) over (U,E) is said to be:

(a) null fuzzy soft set if for each $e \in A$, $F(e)$ is a null fuzzy set $\widetilde{0}$ over U. We denote it by $\widetilde{\Phi}$.

(b) absolute fuzzy soft set if for each $e \in A, F(e)$ is a fuzzy universal set $\widetilde{1}$ over U. We denote it by $\widetilde{E}$.

Definition 7.*([9]) For two fuzzy soft sets (F,A) and (G,B) in $\mathscr{F}(U,E)$, we say that $(F,A)\widetilde{\subseteq}(G,B)$ if $A \subseteq B$ and $F(e) \preceq G(e)$ for each $e \in A$.*

Definition 8.*([9]) Two fuzzy soft sets (F,A) and (G,B) in $\mathscr{F}(U,E)$ are equal if $F\widetilde{\subseteq}G$ and $G\widetilde{\subseteq}F$.*

Definition 9.*([9]) The difference between two fuzzy soft sets $(F,E),(G,E)$ in $\mathscr{F}(U,E)$ is a fuzzy soft set $(F\widetilde{/}G,E)$ (say) defined by $(F\widetilde{/}G)(e) = F(e)/G(e)$ for each $e \in E$.*

Definition 10.*([9]) The complement of a fuzzy soft set (F,E) is a fuzzy soft set $(F^{\tilde{c}},E)$ defined by $F^{\tilde{c}}(e) = \widetilde{1}\,/\,F(e)$ for each $e \in E$.*

Clearly $F^{\tilde{c}} = \widetilde{E}\widetilde{/}F$, $\widetilde{\Phi}^{\tilde{c}} = \widetilde{E}$, and $((F)^{\tilde{c}})^{\tilde{c}} = F$.

Definition 11.*([1]) Let (F,A) and (G,B) be two fuzzy soft sets in $\mathscr{F}(U,E)$ with $A \cap B \neq \widetilde{\Phi}$, then (d) their intersection $(F\widetilde{\cap}G,C)$ is a fuzzy soft set, where $C = A \cap B$ and, $(F\widetilde{\cap}G)e = F(e) \cap G(e)$ for each $e \in$ C, and (e) their union $(F\widetilde{\cup}G,C)$ is a fuzzy soft set, where $\mathbf{C} = A\cup B$ and $(F\widetilde{\cup}G)e = F(e) \cup G(e)$ for each $e \in \mathbf{C}$.*

Definition 12. *([14]) A fuzzy soft topology τ on $F \in \mathscr{F}(U,E)$ is a collection of fuzzy soft subsets of F satisfying:*

1. *$\widetilde{\Phi}, F \in \tau$ (this means that $\widetilde{E}$ is fuzzy soft subset of F, that is, $\widetilde{1}\,(e) \preceq F(e)$, that is $1 \leq F(e)(x)$*
2. *If $F_1,F_2 \in \tau$ then $F_1\widetilde{\cap}F_2 \in \tau$.*
3. *If $F_\alpha \in \tau$ for all $\alpha \in \Lambda$, with Λ an index set, then $\widetilde{\cup}_{\alpha\in\Lambda}F_\alpha \in \tau$.*

If τ is a fuzzy soft topology on F then the pair (F,τ) is called a fuzzy soft topological space.

3 Fuzzy soft elements

Fuzzy soft element is defined as follows.

Definition 13. *([13], [8]) Let e be any element in a set $A \subseteq E$. A fuzzy soft set F over A is called a fuzzy soft element if $F(e')$ is a null fuzzy set for each $e' \in A - \{e\}$. We denote it by (F^e,A) or simply by F^e*

A fuzzy soft element F^e is said to be in fuzzy soft set (G,B) if $(F^e,A)\widetilde{\subseteq}(G,B)$. That is, $A \subseteq B$ and $F^e(e') \preceq G(e')$ for each $e' \in A$, that is, $F^e(e) \preceq G(e')$ for each $e' \in A$. We write it as $F^e\widetilde{\in}G$. It is straightforward to check that union of all fuzzy soft elements corresponding to each parameter $e \in A$ is equal to the approximate fuzzy soft set $F(e)$ and therefore the collection of all such unions, corresponding to each parameter, results in the original fuzzy soft set (F,A).

Note that if F is a fuzzy soft set in $\mathscr{F}(U,E)$ and $F^e\widetilde{\in}F$ then $F = \{\widetilde{\cup}_{F^e\widetilde{\in}F}F^e : e \in E\}$.

*Example 1.*Let F be the fuzzy soft set in $\mathscr{F}(U,E)$ defined as

$$F = \{(e_1,\{\frac{u_1}{0.5},\frac{u_2}{0.3}\}),(e_2,\{\frac{u_1}{0.7},\frac{u_2}{0.4}\})\}$$

Then some of the fuzzy soft elements of F are

$F^{e_1} = \{(e_1,\{\frac{u_1}{0.3},\frac{u_2}{0.1}\})\}$, $F'^{e_1} = \{(e_1,\{\frac{u_1}{0.5},\frac{u_2}{0.3}\})\}$ and

$F^{e_2} = \{(e_2,\{\frac{u_1}{0.7},\frac{u_2}{0.4}\})\}$.

Note that $F^{e_1}\widetilde{\cup}F'^{e_1} = \{(e_1,\{\frac{u_1}{0.5},\frac{u_2}{0.3}\})\} = F(e_1)$. Similarly,

$$\widetilde{\cup}F^{e_2} = \{(e_2,\{\frac{u_1}{0.7},\frac{u_2}{0.4}\}) = F(e_2).$$

Therefore, $\{\widetilde{\cup}_{F^{e_1}\in F}F^{e_1},\widetilde{\cup}_{F^{e_2}}F^{e_2}\} = F$.

Basic properties with held by fuzzy soft elements are stated in the following proposition.

Proposition 1. *Let F_1,F_2 be two fuzzy soft sets over (U,E) and $e \in E$ The following holds.*

1. *$\widetilde{\Phi}$ is an empty fuzzy soft element of every fuzzy soft set.*
2. *If F is a fuzzy soft set such that $F \neq \widetilde{\Phi}$, then F contains at least one non empty fuzzy soft element.*
3. *If $F^e\widetilde{\in}F_1\widetilde{\cup}F_2$ then F^e is a fuzzy soft element of F_1 or F_2.*
4. *$F^e\widetilde{\in}F_1\widetilde{\cap}F_2$ if and only if F^e is a fuzzy soft element of F_1 and F_2.*
5. *If $F^e\widetilde{\in}F_1\widetilde{\setminus}F_2$ then F^e is a fuzzy soft element of F_1 but not necessarily a fuzzy soft element of F_2.*

Proof. 1. Let e be an element of E and F a fuzzy soft set over E. Obviously, $\widetilde{\Phi}(e) \preceq F(e)$ as $\widetilde{\Phi}(e)(x) = 0$ for each $x \in U$. Therefore $\widetilde{\Phi}$ is an empty fuzzy soft element of every fuzzy soft set.

2. If $F \neq \widetilde{\Phi}$, then there exists at least one $e^* \in E$ such that $F(e^*) \neq \widetilde{0}$, that is, there exists an $x \in U$ for which $F(e^*)(x) \neq 0$. Let $F(e^*)(x) = \varepsilon$ for some $\varepsilon \in (0,1]$. Then we define F_1 such that

$$F_1(e^*)(x) = \frac{\varepsilon}{2} \text{ and } F_1(e)(x) = 0 \text{ whenever } e \neq e^*.$$

This implies that $F_1(e^*) \preceq F(e^*)$. If $e \neq e^*$, then $\widetilde{0} = F_1(e) \preceq F(e)$. Hence fuzzy soft set F_1 is a non empty fuzzy soft element of F.

3. Let F^e be a fuzzy sof element of $F_1 \widetilde{\cup} F_2$, that is, $F^e \widetilde{\in} (F_1 \widetilde{\cup} F_2)$ which implies that $F^e(e) \preceq F_1(e') \cup F_2(e')$ for each $e' \in E$. So, for each $x \in U$, $F^e(e)(x) \leq \max\{F_1(e')(x), F_2(e')(x)\}$. Now if $F_1(e')(x) \leq F_2(e')(x)$ then for each $e' \in E$, $F^e(e) \preceq F_2(e')$. Hence $F^e \widetilde{\in} F_2$. If $F_2(e')(x) \leq F_1(e')(x)$ then $F^e(e) \preceq F_1(e')$ for each $e' \in E$ which implies that $F^e \widetilde{\in} F_1$. So, $F^e \widetilde{\in} F_1$ or $F^e \widetilde{\in} F_2$. Conversely, suppose that $F^e \widetilde{\in} F_1$ or $F^e \widetilde{\in} F_2$. Then $F^e(e) \preceq F_1(e')$ or $F^e(e) \preceq F_2(e')$ for each $e' \in E$, that is, for all $x \in U$, $F^e(e)(x) \leq F_1(e')(x)$ or $F^e(e)(x) \leq F_2(e')(x)$. Thus $F^e(e)(x) \leq \max\{F_1(e')(x), F_2(e')(x)\}$. Therefore $F^e \widetilde{\in} F_1 \widetilde{\cup} F_2$.

4. Let $F^e \widetilde{\in} (F_1 \widetilde{\cap} F_2)$ which implies that $F^e(e) \preceq F_1(e') \cap F_2(e')$ for each $e' \in E$. So for each $x \in U$,

$$F^e(e)(x) \leq \min\{F_1(e')(x), F_2(e')(x)\}.$$

If $F_1(e')(x) \leq F_2(e')(x)$ then $F^e(e)(x) \leq F_1(e')(x) \leq F_2(e')(x)$ implies that F^e is a fuzzy sof element of F_1 and F_2. Similarly if $F_2(e')(x) \leq F_1(e')(x)$ then $F^e(e)(x) \leq F_2(e')(x) \leq F_1(e')(x)$ means that F^e is a fuzzy sof element of F_2 and F_1. Conversely, suppose that $F^e \widetilde{\in} F_1$ and $F^e \widetilde{\in} F_2$. Then, for each $e' \in E, F^e(e) \preceq F_1(e')$ and $F^e(e) \preceq F_2(e')$ which implies that

$$F^e(e)(x) \leq \min\{F_1(e')(x), F_2(e')(x)\}$$

for each x in U. Therefore, $F^e \widetilde{\in} F_1 \widetilde{\cap} F_2$.

5. Let $F^e \widetilde{\in} F_1 \widetilde{\setminus} F_2$. Then, $F^e(e) \preceq F_1(e') \setminus F_2(e')$ for each $e' \in E$, that is, $F^e(e)(x) \leq F_1(e')(x) - F_2(e')(x)$ for each $x \in U$. Then $F^e(e)(x) \leq F_1(e')(x)$ but the real number $F^e(e)(x)$ is not necessarily less than $F_2(e')(x)$ for each x. Therefore, F^e is a fuzzy soft element of F_1 but F^e is not necessarily a fuzzy soft element of F_2.

Example 2. Suppose that $U = \{u_1, u_2, u_3\}$ and $E = \{e_1, e_2\}$. Let F and $G \in \mathscr{F}(U,E)$ be of the form

$$F = \{(e_1, \{\frac{u_1}{0.6}, \frac{u_2}{0.8}, \frac{u_3}{0.3}\}, (e_2, \{\frac{u_1}{0.4}, \frac{u_2}{0.6}, \frac{u_3}{0.7}\})\} \text{ and}$$

$$G = \{(e_1, \{\frac{u_1}{0.5}, \frac{u_2}{0.8}, \frac{u_3}{0.3}\}, (e_2, \{\frac{u_1}{0.2}, \frac{u_2}{0.4}, \frac{u_3}{0.3}\})\}.$$

Note that

$$F \widetilde{\cup} G = \{(e_1, \{\frac{u_1}{0.6}, \frac{u_2}{0.8}, \frac{u_3}{0.3}\}, (e_2, \{\frac{u_1}{0.4}, \frac{u_2}{0.6}, \frac{u_3}{0.7}\})\},$$

$$F \widetilde{\cap} G = \{(e_1, \{\frac{u_1}{0.5}, \frac{u_2}{0.8}, \frac{u_3}{0.3}\}, (e_2, \{\frac{u_1}{0.2}, \frac{u_2}{0.4}, \frac{u_3}{0.3}\})\}, \text{ and}$$

$$F \widetilde{\setminus} G = \{(e_1, \{\frac{u_1}{0.1}\}, (e_2, \{\frac{u_1}{0.2}, \frac{u_2}{0.2}, \frac{u_3}{0.4}\})\}.$$

$F^{e_1} = \{(e_1, \{\frac{u_1}{0.4}, \frac{u_2}{0.1}, \frac{u_3}{0.3}\})\}$ is a soft fuzzy element of F.

Note that $F^{e_1} \widetilde{\in} F \widetilde{\cup} G$. Similarly, $F^{e_1} \widetilde{\in} F \widetilde{\cap} G$. Also, $F^{e_2} = \{(e_2, \{\frac{u_1}{0.1}, \frac{u_2}{0.1}, \frac{u_3}{0.4}\})\}$ is a soft fuzzy point of $F \widetilde{\setminus} G$ then $F^{e_2} \widetilde{\in} F$ but F^{e_2} is not a fuzzy soft element of G.

Proposition 2. *Let F_1, F_2 be two fuzzy soft sets over E. Then $F_1 \widetilde{\subseteq} F_2$ if and only if $F^e \widetilde{\in} F_1$ implies that $F^e \widetilde{\in} F_2$.*

Proof. Let $F_1 \widetilde{\subseteq} F_2$ then $F_1(e) \preceq F_2(e)$ for each $e \in E$, that is $F_1(e)(x) \leq F_2(e)(x)$ for each $x \in U$. Suppose that $F^e \widetilde{\in} F_1$. That is, for each $e' \in E$, $F^e(e) \preceq F_1(e')$ and hence $F^e(e) \preceq F_2(e')$ for each $e' \in E$. Therefore, $F^e \widetilde{\in} F_2$. Conversely, suppose that every fuzzy soft element F^e in F_1 is also a fuzzy soft element of F_2. Let $\overline{F}_1^e$ to be the largest fuzzy soft element of F_1 for each $e \in E$ then $\overline{F}_1^e \widetilde{\in} F_2$. Let $\varepsilon \in (0,1]$ and $\overline{F}_1^e(e)(x) + \varepsilon$ be such that $\overline{F}_1^e(e)(x) + \varepsilon \leq F_2(e')(x)$ for each $x \in U$. That is, $\overline{F}_1^e(e)(x) \leq F_2(e')(x)$ for each $e' \in E$. Therefore, $F_1 \widetilde{\subseteq} F_2$.

Definition 14.([8]) *A fuzzy soft topological space (F, τ) is said to be a fuzzy soft Hausdorff space if for distinct fuzzy soft elements $F^e, F^{e'}$ of F, there exists disjoint fuzzy soft open sets (F_1, A) and (F_2, A) such that $F^e \widetilde{\in} F_1$ and $F^{e'} \widetilde{\in} F_2$.*

Proposition 3.*Let (F, τ) be a fuzzy soft topological space. A fuzzy soft set $V \widetilde{\subseteq} F$ is fuzzy soft open if and only if for each $F^e \widetilde{\in} V$ there exists a fuzzy soft set $W \widetilde{\in} \tau$ such that $F^e \widetilde{\in} W \widetilde{\subseteq} V$.*

Proof. Let $V \in \tau$. Then clearly for each $F^e \widetilde{\in} V$ we have $F^e \widetilde{\in} V \widetilde{\subseteq} V$. Let $V \widetilde{\subseteq} F$ be such that for each $F^e \widetilde{\in} V$ there exists a fuzzy soft open set W_{F^e} such that $F^e \widetilde{\in} W_{F^e} \widetilde{\subseteq} V$ which means that $F^e(e) \preceq W_{F^e}(e') \preceq V(e')$ for each $e' \in E$. Since for each $e \in E$, $V(e) = \widetilde{\cup}\{F^e : F^e \widetilde{\in} V\} \widetilde{\subseteq} \widetilde{\cup} W_{F^e}(e) \widetilde{\subseteq} V(e)$, we deduce that $V = \{\widetilde{\cup} W_{F^e} : e \in E\} \in \tau$.

4 Fuzzy soft mapping

In this section, a concept of fuzzy soft mapping is introduced. Relevant definitions are formulated and some properties of fuzzy soft mappings are studied.

Definition 15.([2]) *The cartesian product of two fuzzy soft sets (F,A) and (G,B) is defined as a fuzzy soft set $(H,C) = (F,A) \widehat{\times} (G,B)$, where $C = A \times B$ and $H : C \to \mathscr{F}(U,E)$ is defined by*

$$H(e,e') = F(e) \widetilde{\times} G(e')$$

for all $(e,e') \in C$, where

$$F(e) \widetilde{\times} G(e') = \{\frac{x}{\min\{F(e')(x), G(e')(x)\}} : x \in U\}.$$

Example 3. Let $U=\{u_1,u_2\}$ and $A=\{e_1,e_2,e_3\}$. Define fuzzy soft sets F_1 and F_2 as follows:

$$(F_1,A)=\{(e_1,\{\frac{u_1}{0.6},\frac{u_2}{0.5}\}),(e_2,\{\frac{u_1}{0.3},\frac{u_2}{0.5}\}),$$
$$(e_3,\{\frac{u_1}{0.2},\frac{u_2}{0.7}\})\},$$

and

$$(F_2,A)=\{(e_1,\{\frac{u_1}{0.3},\frac{u_2}{0.4}\}),(e_2,\{\frac{u_1}{0.6},\frac{u_2}{0.7}\}),$$
$$(e_3,\{\frac{u_1}{0.5},\frac{u_2}{0.4}\})\}.$$

Then $(F_1,A)\widehat{\times}(F_2,A)=(H,C)$ where $C=A\times A$ and H is given by

$$H(e_1,e_1)=F_1(e_1)\widetilde{\times}F_2(e_1)=\{\frac{u_1}{0.3},\frac{u_2}{0.4}\},$$
$$H(e_1,e_2)=F_1(e_1)\widetilde{\times}F_2(e_2)=\{\frac{u_1}{0.6},\frac{u_2}{0.5}\},$$
$$H(e_1,e_3)=F_1(e_1)\widetilde{\times}F_2(e_3)=\{\frac{u_1}{0.5},\frac{u_2}{0.4}\},$$
$$H(e_2,e_1)=F_1(e_2)\widetilde{\times}F_2(e_1)=\{\frac{u_1}{0.3},\frac{u_2}{0.4}\},$$
$$H(e_2,e_2)=F_1(e_2)\widetilde{\times}F_2(e_2)=\{\frac{u_1}{0.3},\frac{u_2}{0.5}\},$$
$$H(e_2,e_3)=F_1(e_2)\widetilde{\times}F_2(e_3)=\{\frac{u_1}{0.3},\frac{0.4}{u_2}\},$$
$$H(e_3,e_1)=F_1(e_1)\widetilde{\times}F_2(e_1)=\{\frac{u_1}{0.2},\frac{u_2}{0.4}\},$$
$$H(e_3,e_2)=F_1(e_1)\widetilde{\times}F_2(e_1)=\{\frac{u_1}{0.2},\frac{u_2}{0.7}\},$$
$$H(e_3,e_3)=F_1(e_1)\widetilde{\times}F_2(e_1)=\{\frac{u_1}{0.2},\frac{u_2}{0.4}\}.$$

Definition 16. *Let $(F_1,A),(F_2,A)$ be fuzzy soft sets in $\mathscr{F}(U,E)$. A fuzzy soft set R is called a fuzzy soft relation from F_1 to F_2 if $R=(G,D)$ where $D\subseteq C$ and $G=H$ on D.*

Example 4. Let F_1,F_2 be as given in Example 3. Then

$$R=\{F_1(e_1)\widetilde{\times}F_2(e_2),F_1(e_2)\widetilde{\times}F_2(e_3),F_1(e_3)\widetilde{\times}F_2(e_3)\}$$

is a fuzzy soft relation from F_1 to F_2 which itself is a fuzzy soft set with $\{(e_1,e_1),(e_2,e_3),(e_3,e_3)\}$ as a set of parameters. By F_1RF_2, we mean that $F_1(e_1)\widetilde{\times}F_2(e_2)\in R$.

We now introduce a fuzzy soft mapping.

Definition 17. *Let F,G be fuzzy soft sets in $\mathscr{F}(U,E)$. A fuzzy soft relation T from F to G is called a fuzzy soft mapping from F to G denoted by $T:F\to G$ if the following conditions are satisfied.*

C1 *for each fuzzy soft element $F^e\widetilde{\in}F$, there exists only one fuzzy soft element $G^e\widetilde{\in}G$ such that F^eTG^e which will be denoted as $T(F^e)=G^e$.*

C2 *for each fuzzy soft empty element $F^e\widetilde{\in}F, T(F^e)$ is a empty fuzzy soft element of G.*

Definition 18. *Let F,G be fuzzy soft sets in $\mathscr{F}(U,E)$ and $T:F\to G$ a fuzzy soft mapping. The image of $X\widetilde{\subseteq}F$ under fuzzy soft mapping T is the fuzzy soft set $T(X)$ defined by*

$$T(X)=\{\widetilde{\cup}_{F^e\widetilde{\in}X}\,T(F^e):e\in E\}.$$

It is clear that $T(\widetilde{\Phi})=\widetilde{\Phi}$ for each fuzzy soft mapping T.

Definition 19. *Let $F,G\in\mathscr{F}(U,E)$ and $T:F\to G$ a fuzzy soft mapping. The inverse image of $Y\widetilde{\subseteq}G$ under fuzzy soft mapping T is the fuzzy soft set denoted by $T^{-1}(Y)$ and defined as:*

$$T^{-1}(Y)=\{\{\widetilde{\cup}_{F^e\widetilde{\in}F}F^e:e\in E\}:T(F^e)\widetilde{\in}Y \text{ for each } e\in E\}.$$

Example 5. Let F and G be defined as:

$F=\{(e_1,\{\frac{u_1}{0.6},\frac{u_2}{0.4}\}),(e_2,\{\frac{u_1}{0.3},\frac{u_2}{0.7}\})\}$ and

$G=\{(e_1,\{\frac{u_1}{0.2},\frac{u_2}{0.6}\}),(e_2,\{\frac{u_1}{0.7},\frac{u_2}{0.8}\})\}.$

Define T as $T(F^e)=\widehat{G^e}$ for each $e\in E$, where $\widehat{G^e}$ is the largest fuzzy soft element corresponding to each parameter $e\in E$, that is, if G^e is any fuzzy soft element in G then $G^e\widetilde{\subseteq}\widehat{G^e}$. So, $T(F^{e_1})=\widehat{G^{e_1}}=\{\frac{u_1}{0.2},\frac{u_2}{0.6}\}$ for all $F^{e_1}\widetilde{\in}F$ and $T(F^{e_2})=\widehat{G^{e_2}}=\{\frac{u_1}{0.7},\frac{u_2}{0.8}\}$ for all $F^{e_2}\widetilde{\in}F$. Moreover,

$$T(F)=\{\{\cup_{F^{e_1}\widetilde{\in}X}T(F^{e_1})\},\{\cup_{F^{e_2}\widetilde{\in}X}T(F^{e_2})\}\}$$
$$=\{\widehat{G^{e_1}},\widehat{G^{e_2}}\}=G.$$

Proposition 4. *Let $F,G\in\mathscr{F}(U,E)$, $(X,E),(X_1,E),(X_2,E)\widetilde{\subseteq}(F,E)$, and $(Y,E),(Y_1,E),(Y_2,E)\widetilde{\subseteq}(G,E)$. Let $T:F\to G$ be a fuzzy soft mapping. Then following hold.*

i. $X_1\widetilde{\subseteq}X_2\Rightarrow T(X_1)\widetilde{\subseteq}T(X_2)$,
ii. $Y_1\widetilde{\subseteq}Y_2\Rightarrow T^{-1}(Y_1)\widetilde{\subseteq}T^{-1}(Y_2)$,
iii. $X\widetilde{\subseteq}T^{-1}(T(X))$,
iv. $T(T^{-1}(Y))\widetilde{\subseteq}Y$,
v. $T(X_1\widetilde{\cup}X_2)=T(X_1)\widetilde{\cup}T(X_2)$,
vi. $T(X_1\widetilde{\cap}X_2)=T(X_1)\widetilde{\cap}T(X_2)$,
vii. $T^{-1}(Y_1\widetilde{\cup}Y_2)=T^{-1}(Y_1)\widetilde{\cup}T^{-1}(Y_2)$, and
viii. $T^{-1}(Y_1\widetilde{\cap}Y_2)=T^{-1}(Y_1)\widetilde{\cap}T^{-1}(Y_2)$.

Proof. i. Let F^e be an arbitrary fuzzy soft element in $T(X_1)$ then there exists a fuzzy soft element F^e in X_1 such that $T(F^e)=F^e$. As $X_1\widetilde{\subseteq}X_2$ so F^e is a fuzzy soft element of X_2. So for every fuzzy soft element F^e in $T(X_1)$, F^e is a fuzzy soft element in $T(X_2)$. Hence the result.

v. Let $F^e\ \widetilde{\in}T(X_1\widetilde{\cup}X_2)$. Then $F^e=T(F^e)$ for some $F^e\widetilde{\in}X_1\widetilde{\cup}X_2$. If $F^e\widetilde{\in}X_1$ then $F^e\widetilde{\in}T(X_1)\widetilde{\subseteq}T(X_1)\widetilde{\cup}\ T(X_2)$ and if $F^e\widetilde{\in}X_2$ then $F^e\widetilde{\in}T(X_2)\widetilde{\subseteq}T(X_1)\widetilde{\cup}T(X_2)$. Therefore, $T(X_1\widetilde{\cup}X_2)\widetilde{\subseteq}\ T(X_1)\widetilde{\cup}T(X_2)$. Now let $F^e\widetilde{\in}T(X_1)\widetilde{\cup}T(X_2)$, that is, F^e is fuzzy soft element of $T(X_1)$ or $T(X_2)$. If $F^e\widetilde{\in}T(X_1)$, then $T(X_1)\widetilde{\subseteq}T(X_1\widetilde{\cup}X_2)$ gives $F^e\widetilde{\in}T(X_1\widetilde{\cup}X_2)$.

Similarly, If $F^e\tilde{\in}T(X_2)$, then $T(X_2)\tilde{\subseteq}T(X_1\tilde{\cup}X_2)$ gives $F^e\tilde{\in}T(X_1\tilde{\cup}X_2)$. Therefore $T(X_1)\tilde{\cup}T(X_2)\tilde{\subseteq}T(X_1\tilde{\cup}X_2)$. So we conclude that

$$T(X_1\tilde{\cup}X_2)=T(X_1)\tilde{\cup}T(X_2).$$

viii. If $F^e\tilde{\in}T^{-1}(Y_1\tilde{\cap}Y_2)$ then $T(F^e)\tilde{\in}Y_1\tilde{\cap}Y_2$. Since for each $e\in E$, $T(F^e)\tilde{\subseteq}Y_1(e)\cap Y_2(e)$,then, for all x, $T(F^e)(x)$ is less than the minimum of $Y_1(e)(x)$ and $Y_2(e)(x)$. Hence, $F^e\tilde{\in}\ T^{-1}(Y_1)\tilde{\cap}T^{-1}(Y_2)$ and therefore,

$$T^{-1}(Y_1\tilde{\cap}Y_2)\tilde{\subseteq}T^{-1}(Y_1)\tilde{\cap}T^{-1}(Y_2).$$

Now, let $F^e\tilde{\in}T^{-1}(Y_1)\tilde{\cap}T^{-1}(Y_2)$. Then following similar arguments to those given above it follows that $T(F^e)\tilde{\in}Y_1$and $T(F^e)\tilde{\in}Y_2$. It follows from here that $F^e\tilde{\in}T^{-1}(Y_1\tilde{\cap}Y_2)$. So, $T^{-1}(Y_1)\tilde{\cap}T^{-1}(Y_2)\tilde{\subseteq}T^{-1}(Y_1\tilde{\cap}Y_2)$.

Proofs of the rest of the properties follow on similar lines.

Definition 20.*Let (F,τ) be a fuzzy soft topological space and $K\tilde{\subseteq}F$. A fuzzy soft open cover for K is a collection of fuzzy soft open sets $\{V_i\}_{i\in I}\subseteq\tau$ whose union contains K.*

Definition 21.*A fuzzy soft topological space (F,τ) is compact if for each fuzzy soft open cover $\{V_i\}_{i\in I}$ of K there exists $i_1,i_2,..,i_k\in I,k\in\mathbb{N}$ such that $K\tilde{\subseteq}\cup_{n=1}^{k}V_{i_n}$.*

Definition 22.*Let $(F,\tau),(G,v)$ be fuzzy soft topological spaces and $T:F\to G$ a fuzzy soft mapping. Then T is a fuzzy soft continuous mapping (with respect to the fuzzy soft topologies τ and v) if for each $V\in v,T^{-1}(V)\in\tau$, that is, the inverse image of a fuzzy soft open set is a fuzzy soft open set.*

We say that the fuzzy soft set $K\tilde{\subseteq}F$ is fuzzy soft compact in (F,τ) if the fuzzy soft topological space $(K,\tau_{|K})$ is fuzzy soft compact.

*Example 6.*Let $U=\{u_1,u_2,u_3\},E=\{e_1,e_2,e_3\}$. Suppose $F\in\mathscr{F}(U,E)$ is of the form

$$F=\{(e_1,\{\frac{u_1}{1},\frac{u_2}{1},\frac{u_3}{0.7}\}),(e_2,\{\frac{u_1}{0.6},\frac{u_2}{0.9},\frac{u_3}{0.7}\})\}.$$

Consider the family τ of all fuzzy soft subsets of F and let $V=\widehat{F^{e_1}}\in\tau$ where $\widehat{F^{e_1}}$ is the largest fuzzy soft element of F. Define $T:F\to F$ as $T(F^e)=F^e$ for each $e\in E$. Then, $T^{-1}(\widehat{F^{e_1}})=\widehat{F^{e_1}}\in\tau$.

Proposition 5.*Let (K,τ) be a fuzzy soft compact topological space and $T:K\to K$ a fuzzy soft continuous mapping. Then $T(K)$ is a fuzzy soft compact set in (K,τ).*

Proof. Suppose that $T(K)\tilde{\subseteq}\tilde{\cup}_\ell G_\ell$, where $\{G_\ell\}$ is a family of fuzzy soft open sets in K. Then taking the preimage, we have, $K\tilde{\subseteq}T^{-1}(\tilde{\cup}_\ell G_\ell)$. As $T^{-1}(G_\ell)$ is open in K so there must exist soft fuzzy open $V_\ell\tilde{\subseteq}T(K)$ such that $T^{-1}(G_\ell)=V_\ell\tilde{\cap}K$. So $K\tilde{\subseteq}\tilde{\cup}_\ell(V_\ell\tilde{\cap}K)$ implies that $K\tilde{\subseteq}\tilde{\cup}_\ell V_\ell$. Since K is compact fuzzy soft set, therefore there exist $\ell_1,\ell_2,...,\ell_N$ such that $K\tilde{\subseteq}\tilde{\cup}_{i=1}^{N}V_{\ell_i}$. Hence $K=\tilde{\cup}_\ell(V_\ell\tilde{\cap}K)$ $=\tilde{\cup}_{i=1}^{N}T^{-1}(G_{\ell_i})$ which implies that $T(K)\tilde{\subseteq}\tilde{\cup}_{i=1}^{N}G_{\ell_i}$. Hence $T(K)$ is compact.

5 Fixed points of soft fuzzy mappings

We start this section with the definition of a fixed point of a fuzzy soft mapping.

Definition 23.*Let $F\in\mathscr{F}(U,E)$ be a fuzzy soft set and $T:F\to F$ a fuzzy soft mapping. A fuzzy soft element $F^e\tilde{\in}F$ is called a fixed point of T if $T(F^e)=F^e$.*

*Example 7.*If $T:F\to F$ is defined as an identity map, then each fuzzy soft element of F is a fixed point.

Proposition 6.*Let (F,τ) be a fuzzy soft compact topological space and $\{F_n:n\in\mathbb{N}\}$ a family of fuzzy soft subsets of F satisfying:*

A1.*$F_n\neq\tilde{\Phi}$ for each $n\in\mathbb{N}$,*
A2.*F_n is fuzzy soft closed for each $n\in\mathbb{N}$,*
A3.*$F_{n+1}\tilde{\subseteq}F_n$ for each $n\in\mathbb{N}$.*

Then $\tilde{\cap}_{n\in\mathbb{N}}F_n\neq\tilde{\Phi}$.

Proof. Suppose on the contrary, that $\tilde{\cap}_{n\in\mathbb{N}}F_n=\tilde{\Phi}$. We know that $(\tilde{\cap}_{n\in\mathbb{N}}F_n)^{\tilde{c}}=\tilde{\cup}_{n\in\mathbb{N}}(F_n)^{\tilde{c}}$ (see [1]). From $(A2)$, $(F_n)^{\tilde{c}}$ is a fuzzy soft open set for each $n\in\mathbb{N}$. Hence

$$F\tilde{\subseteq}\tilde{E}=(\tilde{\Phi})^{\tilde{c}}=(\tilde{\cap}_{n\in\mathbb{N}}F_n)^{\tilde{c}}=\tilde{\cup}_{n\in\mathbb{N}}(F_n)^{\tilde{c}}.$$

As F is fuzzy soft compact, there exists $i_1,i_2,...,i_k\in\mathbb{N}$, $i_1<i_2<...<i_k$, $k\in\mathbb{N}$ such that

$$F\tilde{\subseteq}F_{i_1}^{\tilde{c}}\tilde{\cup}F_{i_2}^{\tilde{c}},...\tilde{\cup}F_{i_k}^{\tilde{c}}.$$

Hence from $(A3)$, we have, $F_{i_k}\tilde{\subseteq}F\tilde{\subseteq}(F_{i_1}\tilde{\cap}F_{i_2}\tilde{\cap}...\tilde{\cap}F_{i_k})^{\tilde{c}}=F_{i_k}^{\tilde{c}}=\tilde{E}/F_{i_k}$, which is impossible in the light of $(A1)$.

*Example 8.*Let (F,τ) be a fuzzy soft topological space where τ contains all possible subsets of

$$F=\{(e_1,\{\frac{u_1}{1},\frac{u_2}{0.7}\}),(e_2,\{\frac{u_1}{0.9},\frac{u_2}{1}\})\}.$$

Let two fuzzy soft subsets of F be defined as

$$F_1=\{(e_1,\{\frac{u_1}{0.4},\frac{u_2}{0.5}\}),(e_2,\{\frac{u_1}{0.8},\frac{u_2}{0.4}\})\}$$

and

$$F_2=\{(e_1,\{\frac{u_1}{0.6},\frac{u_2}{0.3}\}),(e_2,\{\frac{u_1}{0.8},\frac{u_2}{0.5}\})\}.$$

Note that they satisfy the conditions of Proposition 6. Moreover $F_1\tilde{\subseteq}F_2$ and $\tilde{\cap}_{j=1}^{2}F_j=F_1\neq\tilde{\Phi}$.

Proposition 7.*Let (F,τ) be a fuzzy soft topological space and $T:F\to F$ a fuzzy soft mapping such that for each nonempty fuzzy soft element $F^e\tilde{\in}F$, $T(F^e)$ is a nonempty fuzzy soft element of F. If $\tilde{\cap}_{n\in\mathbb{N}}T^n(F)$ contains only one nonempty fuzzy soft element $F^e\tilde{\in}F$, then F^e is a unique fixed point of T.*

Proof. Observe that $T^n(F)\widetilde{\subseteq}T^{n-1}(F)$ for each $n\in\mathbb{N}$. Let F^e be a fuzzy soft element of F such that $F^e\widetilde{\in}\widetilde{\cap}_{n\in\mathbb{N}}T^n(F)$. That is, $F^e\widetilde{\subseteq}\widetilde{\cap}_{n\in\mathbb{N}}T^n(F)$. Consequently

$$T(F^e)\widetilde{\subseteq}T(\widetilde{\cap}_{n\in\mathbb{N}}T^n(F))\widetilde{\subseteq}\widetilde{\cap}_{n\in\mathbb{N}}T^{n+1}(F)\widetilde{\subseteq}\widetilde{\cap}_{n\in\mathbb{N}}T^n(F)=F^e.$$

Since $T(F^e)$ is a non empty fuzzy soft element of F, therefore we obtain that $T(F^e)=F^e$.

*Example 9.*Let (F,τ) be a fuzzy soft topological space and define $T:F\to F$ as $T(F^e)=\widehat{F^e}$ for all $F^e\widetilde{\in}F$, where $F\neq\widetilde{\Phi}$ and $\widehat{F^e}$ represents the largest fuzzy soft element of F or equivalently $F^e\widetilde{\subseteq}\widehat{F^e}$ for each fuzzy soft element $F^e\widetilde{\in}F$. Then $\widetilde{\cap}_{n\in\mathbb{N}}T^n(F)$ contains only one non empty fuzzy soft element which is $\widehat{F^e}$. Note that $\widehat{F^e}$ is a unique fixed point of T.

Proposition 8.*Let (F,τ) be a fuzzy soft Hausdorff topological space. Then every fuzzy soft compact set in F is fuzzy soft closed in F.*

Proof. Let K be a fuzzy soft compact set in (F,τ). We need to show that K is fuzzy soft closed, that is, $K^{\tilde{c}}$ is fuzzy soft open. Let $F^e\widetilde{\in}K^{\tilde{c}}$. For every $F^{e'}\widetilde{\in}K$, let $U_i,V_i\in\tau$ be such that $U_i\widetilde{\cap}V_i=\widetilde{\Phi}$ and $F^e\widetilde{\in}U_i$, $F^{e'}\widetilde{\in}V_i$ where $i\in I$. Since K is fuzzy soft compact so there exists $F^{e'},F^{e'},...,F^{e'}\widetilde{\in}K$ such that $K\widetilde{\subseteq}V_{i_1}\widetilde{\cup}V_{i_2}\widetilde{\cup}...\widetilde{\cup}V_{i_k}$. Denote $U=U_{i_1}\widetilde{\cup}U_{i_2}\widetilde{\cup}...\widetilde{\cup}U_{i_k}$ and $V=V_{i_1}\widetilde{\cup}V_{i_2}\widetilde{\cup}...\widetilde{\cup}V_{i_k}$. Then $F^e\widetilde{\in}U\in\tau, U\widetilde{\cap}V=\widetilde{\Phi}$, which gives that $F^e\widetilde{\in}U\widetilde{\subseteq}K^{\tilde{c}}$. Therefore K is fuzzy soft closed.

Theorem 1.*Let (K,τ) be a fuzzy soft compact Hausdorff topological space and $T:K\to K$ a fuzzy soft continuous mapping such that*

a.for each non empty fuzzy soft element $F^e\widetilde{\in}K$, $T(F^e)$ is a non empty fuzzy soft element of K,
b.for each fuzzy soft closed set $X\widetilde{\subseteq}K$ if $T(X)=X$ then X contains only one nonempty fuzzy soft element of K.

Then there exists a unique nonempty fuzzy soft element $F^e\widetilde{\in}K$ such that $T(F^e)=F^e$.

Proof. Consider a family of fuzzy soft subsets of K of the form

$$C_1=T(K),C_2=T(C_1)=T^2(K),...,C_n=T(C_{n-1})=T^n(K)$$

for $n\in\mathbb{N}$. It is clear that $C_n\widetilde{\subseteq}C_{n-1}$ for each $n\in\mathbb{N}$. By Proposition 8, for each $n\in\mathbb{N}$, C_n is fuzzy soft closed. Using Proposition 6, we conclude that a fuzzy soft set D of the form $D=\cap_{n\in N}C_n$ is nonempty. Observe that

$$T(D)=T(\cap_{n\in N}T^n(K))\widetilde{\subseteq}\cap_{n\in N}T^{n+1}(K)\widetilde{\subseteq}\cap_{n\in N}T^n(K)=D.$$

Now we show that $D\widetilde{\subseteq}T(D)$. For this, suppose that there exists $F^e\widetilde{\in}D$ such that F^e is not a fuzzy soft element of $T(D)$. Put $E_n=T^{-1}(F^e)\widetilde{\cap}C_n$. Let us observe that $E_n\neq\widetilde{\Phi}$ and $E_n\widetilde{\subseteq}E_{n-1}$ for each $n\in\mathbb{N}$. By Proposition 6, there exists nonempty fuzzy soft element $F^{e'}\widetilde{\in}T^{-1}(F^e)\widetilde{\cap}D$ and thus $F^e=T(F^{e'})\widetilde{\in}T(D)$, a contradiction. Therefore, $T(D)=D$. Hence the result follows using Proposition 6.

6 Conclusion

In this paper we put forward the notion of fuzzy soft mappings based on the theory of fuzzy soft element of fuzzy soft set and fuzzy soft topological space. We study fixed points of fuzzy soft mappings. Employing these results, we can further study fixed point theory in the framework of fuzzy soft set theory.

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