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On Finding Exact and Approximate Solutions to Some PDEs Using the Reduced Differential Transform Method

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Abstract: In this research article, we give analytic approximate solution to the Sharma Tasso Olver (STO) equation and exact solutions to both the Schrodinger equation and the Telegraph equation. Also, the approximate analytical and exact solutions we present in this paper are calculated in the form of power series with easily computable components. The obtained results are in a good agreement with the exact solutions. We present an algorithm called the Reduced Differential Transform Method (RDTM) to find approximate solution and we compare the results with the exact solutions. This method reduces significantly the numerical computations compare with the existing methods such as the perturbation technique, differential transform method (DTM) and the Adomian decomposition method (ADM).

Keywords: Reduced Differential Transform Method (RDTM), Sharma Tasso Olver (STO) equation, Schrodinger equation, Telegraph equation, Approximate solutions, Analytical solutions.

1 Introduction

There are many wave equations which are quite useful in physics and engineering. These equations are represented usually by linear and nonlinear PDEs and solving such equations is very important and sometime it is difficult to handle the nonlinear part of these PDEs. Many authors applied numerical methods to find solutions of these equations and to name few of these methods: The Differential Transform Method (DTM) [7,12], the Adomian Decomposition Method (ADM) [5,6], the Variational Iteration Method (VIM) [8,11] and the sine–cosine method [9]. The RDTM was first introduced by Y. Keskin in his Ph.D. [4]. This method based on the use of the traditional DTM techniques. Usually, a few numbers of iteration needed of the series solution for numerical purposes with high accuracy.

The RDTM has been used by many authors to obtain analytical approximate and in some cases exact solutions to nonlinear wave equations. Keskin and Oturanc [1,2,3] used the RDTM to solve linear and nonlinear wave equations and they showed the effectiveness, and the accuracy of the proposed method. Moreover, they showed that it takes only few iterations to get an approximate solutions with high accuracy. In addition, M. Rawashdeh [14] used the RDTM to find exact and approximate solution for Gardner equation, Variant Nonlinear Water Wave equation (VNWW), and the Fifth-Order Korteweg-de Vries (FKdV) equation. Finally, Abazari. R, Soltanalizadeh [16] used the RDTM to find approximate solutions for the Kawahara Equations.

In this paper, we apply the RDTM to the Sharma Tasso Olver (STO) equation which is a good example to show fission and fusion of the soliton solutions. The SOT was studied by many authors using different methods such as Hirota's direct method [17] and extended tanh method [10]. Also, A. M. Wazwaz [10] found solitons and kinks solutions to the Sharma Tasso Olver equation.
The standard form of the Sharma Tasso Olver equation [10,17] is given by

$$u_t + \alpha\left(u^3\right)_x + \frac{3}{2}\alpha(u^2)_{xx} + \alpha u_{xxx} = 0, \qquad (1)$$

where α is a constant.

In this paper, we present analytic approximate solution to Eq.(1) and exact solutions for both Eq.(4) and Eq.(6).

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First, the Sharma Tasso Olver equation of the form:

$$u_t + \alpha \left(u^3\right)_x + \frac{3}{2}\alpha (u^2)_{xx} + \alpha u_{xxx} = 0, \tag{2}$$

subject to the initial condition

$$u(x,0) = \sqrt{\frac{1}{\alpha}} \tanh\left(\sqrt{\frac{1}{\alpha}}\, x\right). \tag{3}$$

Second, the Schrodinger equation of the form:

$$u_t = i u_{xx}, \tag{4}$$

subject to the initial condition

$$u(x,0) = \sinh(x), \tag{5}$$

Third, the homogeneous Telegraph equation of the form:

$$u_{xx} = u_{tt} - 2u_t - u, \tag{6}$$

subject to the initial condition

$$u(x,0) = \cosh(x) - 1; \qquad u_t(x,0) = 1. \tag{7}$$

The rest of this paper is organized as follows: In Section 2, the reduced differential transform method is introduced. Section 3 is devoted to apply the RDTM to three test problems to show the effectiveness of the RDTM. In section 4, we present a table to show the comparison between the approximate and exact solutions using the RDTM. Section 5 discussion and conclusion of this paper.

2 Analysis of the Method

In this section, we will give the methodology of the RDTM. So let's start with a function of two variables $u(x,t)$ which is analytic and $k-$times continuously differentiable with respect to time t and space x in the domain of our interest. Assume we can represent this function as a product of two single-variable functions $u(x,t) = f(x).g(t)$. From the definitions of the DTM, the function can be represented as follows:

$$u(x,t) = \left(\sum_{i=0}^{\infty} F(i)x^i\right)\left(\sum_{j=0}^{\infty} G(j)t^j\right) = \sum_{k=0}^{\infty} U_k(x)t^k \tag{8}$$

where $U_k(x)$ is the transformed function of $u(x,t)$ which can be defined as:

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,t)\right]_{t=0}. \tag{9}$$

From equations (8) and (9) one can deduce

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,t)\right]_{t=0} t^k = \sum_{k=0}^{\infty} U_k(x)t^k. \tag{10}$$

In this work, the lowercase $u(x,t)$ represent the original function while the uppercase $U_k(x)$ stand for the transformed function. Note that from the above discussion, one can realize that the RDTM is derived from the power series expansion.
Some basic operations of the reduced differential transformation obtained from equations (8) and (9) are given in the table below:

Table 1. Basic operations of the RDTM [1, 2, 3, 4]

Functional Form	Transformed form
$u(x,t)$	$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,t)\right]_{t=0}$
$w(x,t) = \alpha u(x,t) \pm \beta v(x,t)$	$W_k(x) = \alpha U_k(x) \pm \beta V_k(x)$, α and β are constants.
$w(x,t) = u(x,t)v(x,t)$	$W_k(x) = \Sigma_{i=0}^{k} U_i(x)V_{k-i}(x)$
$f(x,t) = u(x,t)v(x,t)w(x,t)$	$F_k(x) = \Sigma_{i=0}^{k} \Sigma_{j=0}^{i} U_j(x)V_{i-j}(x)W_{k-i}(x)$
$w(x,t) = \frac{\partial^n}{\partial t^n} u(x,t)$	$W_k(x) = \frac{(k+n)!}{K!} U_{k+n}(x)$
$w(x,t) = \frac{\partial^n}{\partial x^n} u(x,t)$	$W_k(x) = \frac{\partial^n}{\partial x^n} U_k(x)$
$w(x,t) = x^m t^n u(x,t)$	$W_k(x) = x^m U_{k-n}(x)$
$w(x,t) = x^m t^n$	$W_k(x) = x^m \delta(k-n)$, where $\delta(k-n) = \begin{cases} 1, & n=k \\ 0, & n \neq k \end{cases}$
$w(x,t) = \frac{\partial^{n+m}}{\partial x^n \partial t^m} u(x,t)$	$W_k(x) = \frac{\partial^n}{\partial x^n}\left[\frac{(k+m)!}{k!} U_{k+m}(x)\right]$

Remark. It is worth mentioning here that table 1 was derived by Y. Keskin in his Ph.D [4]. The proofs of theses theorems are also available in [4].

Now, we illustrate the RDTM by using Eq.(1) in standard form

$$L(u(x,t)) + R(u(x,t)) + N(u(x,t)) = 0 \tag{11}$$

with initial conditions

$$u(x,0) = f(x); \quad u_t(x,0) = g(x), \tag{12}$$

where $L = \frac{\partial}{\partial t}$ is a linear operator, $N(u(x,t)) = \alpha\left(u^3\right)_x + \frac{3}{2}\alpha(u^2)_{xx}$ is the nonlinear term and $R(u(x,t))$ is the remaining linear term.

Using the RDTM formulas in Table 1, we can derive the following recursive relation:

$$(k+1)U_k(x) = R(U_k(x)) - N(U_k(x)) + U_k(x) \tag{13}$$

where, $R(U_k(x))$, $U_k(x)$ and $N(U_k(x))$ are the transformations of $R(u(x,t))$, $u(x,t)$ and $N(u(x,t))$ respectively.
Now from equation (12), we can write the initial condition as:

$$U_0(x) = f(x); \quad U_1(x) = g(x) \tag{14}$$

To find all other iterations, we first substitute equation (14) into equation (13) and then we find the values of $U_k(x)$. Finally, we apply the inverse transformation to all values $\{U_k(x)\}_{k=0}^{n}$ to obtain the approximate solution:

$$\widehat{u}(x,t) = \sum_{k=0}^{n} U_k(x)t^k. \tag{15}$$

where n is the number of iterations we need to find the intended approximate solution.
Hence, the exact solution of our problem is given by $u(x,t) = \lim_{n\to\infty} \widehat{u}\,(x,t)$.

3 Numerical Examples

In this section, we apply the RDTM to three numerical examples and then compare our approximate solutions to the exact solutions to show the efficiency of the RDTM.

3.1 Sharma Tasso Olver (STO) equation

First, consider the Sharma Tasso Olver (STO) equation:

$$u_t + \alpha \left(u^3\right)_x + \frac{3}{2}\alpha(u^2)_{xx} + \alpha u_{xxx} = 0, \qquad (16)$$

where α is a constant.

In the case when $\alpha = 4$, the STO becomes

$$u_t + 4\left(u^3\right)_x + 6(u^2)_{xx} + 4u_{xxx} = 0, \qquad (17)$$

subject to the initial conditions

$$u(x,0) = \frac{1}{2}tanh\left(\frac{x}{2}\right); \;\; u_t(x,0) = \frac{-1}{4}sech^2\left(\frac{x}{2}\right), \qquad (18)$$

where the exact solution is

$$u(x,t) = \frac{1}{2}tanh\left(\frac{x-t}{2}\right). \qquad (19)$$

Applying the RDTM to (18) and (17), we obtain the recursive relation

$$\begin{aligned} U_{k+1}(x) &= \frac{-1}{k+1}\left(4\frac{\partial^3 U_k(x)}{\partial x^3} + 4\frac{\partial}{\partial x}\left(\sum_{i=0}^{k}\sum_{j=0}^{i} U_{i-j}(x)U_j(x)U_{k-i}(x)\right)\right) \\ &+ \frac{-6}{k+1}\left(\frac{\partial^2}{\partial x^2}\left(\sum_{i=0}^{k} U_i(x)U_{k-i}(x)\right)\right). \end{aligned} \qquad (20)$$

where the $U_k(x)$, is the transform function of the $t-$dimensional spectrum. Note that

$$U_0(x) = \frac{1}{2}tanh\left(\frac{x}{2}\right); \;\; U_1(x) = \frac{-1}{4}sech^2\left(\frac{x}{2}\right). \qquad (21)$$

Now, substitute Eq. (21) into Eq. (20) to obtain the following:

$$U_2(x) = \frac{sinh(x)}{4(1+cosh(x))^2}$$

$$U_3(x) = -\frac{1}{48}(cosh(x)-2)sech^4\left(\frac{x}{2}\right)$$

$$U_4(x) = -\frac{(cosh(x)-5)tanh\left(\frac{x}{2}\right)}{48(1+cosh(x))^2}.$$

We continue in this manner and after a few iterations, the differential inverse transform of $\{U_k(x)\}_{k=0}^{\infty}$ will provide us with the following approximate solution:

$$\begin{aligned} \widehat{u}\,(x,t) &= \sum_{k=0}^{\infty} U_k(x)t^k \\ &= U_0(x) + U_1(x)t + U_2(x)t^2 + U_3(x)t^3 + ... \\ &= \frac{1}{2}tanh\left(\frac{x}{2}\right) - \frac{1}{4}sech^2\left(\frac{x}{2}\right)t - \frac{sinh(x)}{4(1+cosh(x))^2}t^2 \\ &- \frac{1}{48}(cosh(x)-2)sech^4\left(\frac{x}{2}\right)t^3 + \ldots. \end{aligned}$$

Hence, the approximate solution converges rapidly to the exact solution and the exact solution of the problem is given by $u(x,t) = \lim_{n\to 0} \widehat{u}_n\,(x,t)$.

From figure 1 below one can observe that the values of the approximate solution at different grid points obtained by RDTM are very close to the values of the exact solution with high accuracy with only five iterations and the accuracy increases as the order of approximation increases.

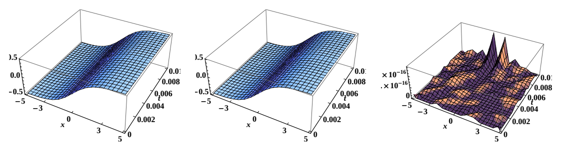

Fig. 1: The approximate, exact solutions and absolute error, respectively for example 3.1 when $-5 < x < 5$ and $0 < t < 0.01$

Also figure 2 below shows the exact solution, approximate solution of $u(x,t)$ for the values of $x = -5, -3, 3, 5$ and $t = 0.02, 0.04, 0.06, 08, 0.1$.

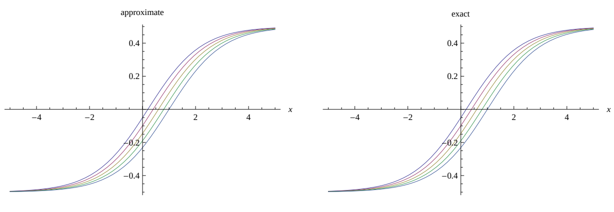

Fig. 2: The approximate and exact solutions for example 3.1 when $-5 < x < 5$ and $0 < t < 0.1$

3.2 Schrodinger equation

Consider the Schrodinger equation of the form:

$$u_t = i\,u_{xx}, \qquad (22)$$

subject to the initial condition

$$u(x,0) = \sinh(x), \tag{23}$$

where the exact solution is

$$u(x,t) = e^{it}\sinh(x). \tag{24}$$

Applying the RDTM to (22) and (23), we obtain the recursive relation

$$U_{k+1}(x) = \left(\frac{i}{k+1}\right)\left(\frac{\partial^2}{\partial x^2}\left(U_k(x)\right)\right), \tag{25}$$

where the $U_k(x)$, is the transform function of the $t-$dimensional spectrum.

$$U_0(x) = \sinh(x). \tag{26}$$

Now, substitute Eq. (26) into Eq. (25) to obtain the following:

$$U_1(x) = i\sinh(x)$$

$$U_2(x) = \tfrac{1}{2}\sinh(x)$$

$$U_3(x) = \tfrac{-i}{6}\sinh(x).$$

We continue in this manner and after a few iterations, the differential inverse transform of $\{U_k(x)\}_{k=0}^{\infty}$ will provide us with the following approximate solution:

$$\begin{aligned}
\widehat{u}\,(x,t) &= \sum_{k=0}^{\infty} U_k(x)t^k \\
&= U_0(x) + U_1(x)t + U_2(x)t^2 + \ldots. \\
&= \sinh(x) + it\sinh(x) - \frac{t^2}{2}\sinh(x) - \frac{it^3}{6}\sinh(x) + \ldots \\
&= \left(1 + it - \frac{t^2}{2} - \frac{it^3}{6} + \ldots\right)\sinh(x) \\
&= e^{it}\sinh(x).
\end{aligned}$$

This is the exact solution of Eq. (22).

3.3 Telegraph equation

We consider the homogeneous Telegraph equation of the form:

$$u_{xx} = u_{tt} - 2u_t - u, \tag{27}$$

subject to the initial condition

$$u(x,0) = \cosh(x) - 1; u_t(x,0) = 1, \tag{28}$$

where the exact solution

$$u(x,t) = \cosh(x) - e^{-t}. \tag{29}$$

Now, we apply the RDTM to Eq. (27) and Eq. (28) we get

$$U_{k+2}(x) = \frac{1}{(k+2)(k+1)}\left(\frac{\partial^2 U_k(x)}{\partial x^2} - 2(k+1)U_{k+1}(x) - U_k(x)\right) \tag{30}$$

where the $U_k(x)$, is the transform function of the $t-$dimensional spectrum. Note that

$$U_0(x) = \cosh(x) - 1; \qquad U_1(x) = 1. \tag{31}$$

Now, substitute Eq. (30) into Eq. (31) to obtain the following:

$$U_2(x) = \frac{-1}{2},\ U_3(x) = \frac{1}{6}, \ldots \tag{32}$$

So after a few iterations, the differential inverse transform of $\{U_k(x)\}_{k=0}^{\infty}$ will give the following approximate solution:

$$\begin{aligned}
\widehat{u}\,(x,t) &= \sum_{k=0}^{\infty} U_k(x)t^k \\
&= U_0(x) + U_1(x)t + U_2(x)t^2 + U_3(x)t^3 + \ldots \\
&= \cosh(x) - 1 + t - \frac{t^2}{2} + \frac{t^3}{6} - \frac{t^4}{12} + \ldots \\
&= \cosh(x) - \left(1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} + \ldots\right) \\
&= \cosh(x) - e^{-t}.
\end{aligned}$$

This is an exact solution of Eq. (27).

4 Tables of Numerical Calculations

In this section, we shall illustrate the accuracy and efficiency of the RDTM. For this purpose, we can evaluate the approximate solution using the 5th-order approximation. Table 2 shows the exact solution, the approximate solution and the absolute error obtained by the RDTM. We must emphasize here only five iterations was used for different values of x and t, specifically, $x = -5, -3, 3, 5$ and $t = 0.002, 0.004, 0.006, 0.01$.

Table 2: Comparison of the absolute error of the solutions of the STO equation by RDTM

x	t	Exact Solution	RDTM Solution	Abs-error-RDTM, n=5
-5	.002	-0.4933204320	-0.4933204320	$5.55111512E^{-17}$
	.004	-0.4933336888	-0.4933336888	0
	.006	-0.4933469195	-0.4933469195	0
	.01	-0.4933733027	-0.4933733027	$5.55111512E^{-17}$
-3	.002	-0.4526643984	-0.4526643984	0
	.004	-0.4527545066	-0.4527545066	$5.55111512E^{-17}$
	.006	-0.4528444519	-0.4528444519	$5.55111512E^{-17}$
	.01	-0.4530238543	-0.4530238543	$5.55111512E^{-17}$
3	.002	0.4524836916	0.4524836916	$5.55111512E^{-17}$
	.004	0.4523930926	0.4523930926	0
	.006	0.4523023296	0.4523023296	$5.55111512E^{-17}$
	.01	0.4521203101	0.4521203101	$5.55111512E^{-17}$
5	.002	0.4932938398	0.4932938398	0
	.004	0.4932805043	0.4932805043	0
	.006	0.4932671424	0.4932671424	0
	.01	0.49324033948	0.49324033948	$5.55111512E^{-17}$

5 Conclusion

In this paper, we applied the Reduced Differential Transform Method (RDTM) to all three physical models, namely, the Sharma Tasso Olver (STO) equation, the Schrodinger equation and the Telegraph equation. We successfully found approximate solution for the STO and the results we obtained in example (3.1) were in excellent agreement with the exact solution. Also, we found exact solutions to the Schrodinger equation and the Telegraph equation. The RDTM introduces a significant improvement in the fields over existing techniques because it takes less calculations and the number of iteration is less compared by other methods. My goal in the future is to apply the RDTM to fractional nonlinear PDEs that arises in other areas of science such as Biology, Medicine and Engineering. There is no existing method in the literature that can give exact solution to fractional PDEs, so finding approximate solutions is very important. Computations of this paper have been carried out using the computer package of Mathematica 7.

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