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Derivative-Free Iterative Methods for Solving Nonlinear Equations

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Abstract: In this paper, we suggest and analyze some new derivative free iterative methods for solving nonlinear equation $f(x) = 0$ using a suitable transformation. We also give several examples to illustrate the efficiency of these methods. Comparison with other similar method is also given. These new methods can be considered as alternative to the developed methods. This technique can be used to suggest a wide class of new iterative methods for solving nonlinear equations.

Keywords: Nonlinear equation, Convergence, Steffensen's method, derivative-free method, Examples.

1 Introduction

One of the most frequently occurring problems in scientific work is to locate the approximate solution of a nonlinear equation

$$f(x) = 0 \tag{1}$$

Analytical methods for solving such equations are almost nonexistent and therefore, it is only possible to obtain approximate solutions by relying on numerical techniques based on iteration procedures [1,2,3,4,5,6,8,9]. If the function is not known explicitly or the derivative is difficult to compute, a method that uses only computed values of the function is more appropriate.

Some of the more classical numerical methods for solving nonlinear equations without using derivative [9] include the bisection method, secent method and regula falsi method. These are the basic methods but have slow convergence toward the solution.

Newton's method, which is simple and converges quadratically [9], is probably the best known and most widely used algorithm which includes the derivative of the function. However, Steffensen's method [3,9]

$$x_{n+1} = x_n - \frac{[f(x_n)]^2}{f(x_n + f(x_n)) - f(x_n)} \qquad n = 0,1,2,3,\ldots$$

is variation of Newton's method which does not employ the derivative of the function. In this method the derivative is approximated by the forward difference scheme. Steffensen's method has same order of convergence as Newton's method. Based on the approximation of the first derivative, we construct some derivative-free iterative methods for solving nonlinear equations.

2 Iterative methods

In this section, we construct some iterative methods for solving nonlinear equations. We use approximation of first derivative of the function to obtain derivative-free methods.

Let us approximate the first derivative of the function $f(x_n) = 0$, at the current iteration x_n by

$$f'(x_n) \approx g(x_n) = \frac{f(x_n + bf(x_n)) - f(x_n)}{bf(x_n)}, \tag{2}$$

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where $b \in R$, and $b \neq 0$. Using (2), in well known Newton method, we obtain the following derivative-free iterative method for solving nonlinear equation as:

Algorithm2.1. For a given x_0, find the approximation solution x_{n+1} by the following iterative scheme:

$$x_{n+1} = x_n - \frac{b[f(x_n)]^2}{f(x_n + bf(x_n)) - f(x_n)}, \quad n = 0, 1, 2, 3, \ldots$$

If $b = 1$, then the Algorithm 2.1 reduces to the well known Steffensen's method.

He [2], Noor [5] and Noor and Noor [7], suggested the iterative method for solving nonlinear equation which involves the first derivative of the function is described as:

Algorithm2.2 [2,5]. For a given x_0, find the approximation solution x_{n+1} by the following iterative scheme:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - \alpha f(x_n)}, \quad n = 0, 1, 2, 3, \ldots$$

where α is a parameter.

Algorithm2.3 [7]. For a given x_0, find the approximation solution x_{n+1} by the following iterative scheme:

$$y_n = x_n - \frac{f(x_n)}{f(x_n) - \alpha f(x_n)},$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n) - \alpha f(x_n)}.$$

Algorithm2.4 [7]. For a given x_0, find the approximation solution x_{n+1} by the following iterative scheme:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n) - \alpha f(x_n)},$$

$$x_{n+1} = y_n - \frac{2f(y_n)}{f'(x_n) - \alpha f(x_n)} + \frac{f(x_n)f(y_n)}{(f'(x_n) - \alpha f(x_n))^2}.$$

Since the evaluation of derivatives is difficult task, we replace the derivative of $f(x)$ by (2). That is,

$$f'(x_n) = \frac{f(x_n + bf(x_n)) - f(x_n)}{bf(x_n)},$$

where $0 \neq b \in \mathbb{R}$.

Consequently, we obtain the following new derivative free iterative methods for solving the nonlinear equations (1).

Algorithm2.5. For a given x_0, find the approximation solution x_{n+1} by the following iterative scheme:

$$y_n = x_n - \frac{b[f(x_n)]^2}{f(x_n + bf(x_n)) - f(x_n)},$$

$$x_{n+1} = y_n - \frac{bf(y_n)f(x_n)}{f(x_n + bf(x_n)) - f(x_n) - \alpha b[f(x_n)]^2},$$

$$n = 0, 1, 2, 3, \ldots$$

which is a third-order derivative-free method for solving nonlinear equations.

Remark2.1. We would like to point out that the Algorithm 2.3 and Algorithm 2.5 have at least second and third-order convergence respectively for all values of α. If we take $\alpha = 0, \alpha = \frac{1}{2}, \alpha = 1, \ldots$. in above derived methods, we can obtain various classes of iterative methods for solving nonlinear equations.

Remark2.2. It is important to point out that never choose such a value of α which makes the denominator zero. It is necessary that sign of α should be chosen so as to keep the denominator largest in magnitude in above derived Algorithms.

Algorithm2.6. For a given x_0, find the approximation solution x_{n+1} by the following iterative scheme:

$$y_n = x_n - \frac{b[f(x_n)]^2}{f(x_n + bf(x_n)) - f(x_n)},$$

$$x_{n+1} = y_n - \frac{2bf(y_n)f(x_n)}{f(x_n + bf(x_n)) - f(x_n) - \alpha b[f(x_n)]^2}$$

$$+ \frac{b^2 f(y_n)(f(x_n))^3}{(f(x_n + bf(x_n)) - f(x_n) - \alpha b[f(x_n)]^2)^2},$$

$n = 0, 1, 2, 3, \ldots$

3 Convergence analysis

One can consider the convergence criteria of the iterative methods developed in section 2, using the techniques developed in [1,4]

Theorem 1. *Assume that the function $f : \mathscr{D} \subset \mathbb{R} \to \mathbb{R}$ for an open interval in $\mathscr{D}$ with simple root $p \in \mathscr{D}$. Let $f(x)$ be a smooth sufficiently in some neighborhood of the root and then the Algorithm 2.5 has third order convergence.*

4 Numerical results

We now present some examples to illustrate the efficiency of the new developed iterative methods in Tables 4.1-4.6. We compare Steffensen's method (SM), with the newly developed methods for different values of α and b

involved in the iterative schemes. We use MAPLE for all the computations. We use the following examples for the comparison of the methods.

$$f_1(x) = sin^2 x - x^2 + 1,$$

$$f_2(x) = x^3 + 4x^2 - 15,$$
$$f_3(x) = x^2 - e^x - 3x + 2,$$

$$f_4(x) = cosx - x,$$
$$f_5(x) = (x-2) - e^{-x},$$

$$f_6(x) = x^3 + 4x^2 + 8x + 8,$$
$$f_7(x) = sinx - \frac{1}{2}x.$$

Table 4.1. Comparison of various iterative schemes.

$f(x)$	x_0	SM $b=1$	Alg 2.1 $b=-1$	Alg 2.1 $b=1/2$	Alg 2.1 $b=-1/2$
f_1	-1	10	5	7	4
f_2	2	18	10	11	14
f_3	-2	38	9	5	7
f_4	1.7	5	5	4	5
f_5	0	7	4	6	4
f_6	-1	9	8	5	5
f_7	-1	7	4	6	4

Table 4.2. Comparison for $\alpha = 1$.

$f(x)$	x_0	SM $b=1$	Alg 2.3 $b=-1$	Alg 2.3 $b=1/2$	Alg 2.3 $b=-1/2$
f_1	-1	10	4	5	6
f_2	2	18	14	9	6
f_3	-2	38	6	12	13
f_4	1.7	5	5	5	6
f_5	0	7	5	7	6
f_6	-1	9	6	8	9
f_7	-1	7	6	7	6

Table 4.3. Comparison for $\alpha = 0.5$.

$f(x)$	x_0	SM $b=1$	Alg 2.3 $b=-1$	Alg 2.3 $b=1/2$	Alg 2.3 $b=-1/2$
f_1	-1	10	5	5	6
f_2	2	18	8	10	5
f_3	-2	38	7	9	10
f_4	1.7	5	5	6	5
f_5	0	7	7	7	7
f_6	-1	9	7	7	7
f_7	-1	7	4	4	6

Table 4.4. Comparison for $\alpha = 0$.

$f(x)$	x_0	SM $b=1$	Alg 2.5 $b=-1$	Alg 2.5 $b=1/2$	Alg 2.5 $b=-1/2$
f_1	-1	10	5	3	4
f_2	2	18	6	7	4
f_3	-2	38	6	4	5
f_4	1.7	5	3	3	3
f_5	0	7	3	4	3
f_6	-1	9	5	5	3
f_7	-1	7	7	5	5

Table 4.5. Comparison for $\alpha = 1$.

$f(x)$	x_0	SM $b=1$	Alg 2.5 $b=-1$	Alg 2.5 $b=1/2$	Alg 2.5 $b=-1/2$
f_1	-1	10	4	5	4
f_2	2	18	6	7	7
f_3	-2	38	4	4	5
f_4	1.7	5	4	3	4
f_5	0	7	3	5	3
f_6	-1	9	4	4	4
f_7	-1	7	4	5	5

Table 4.6. Comparison for $\alpha = 0.5$.

$f(x)$	x_0	SM $b=1$	Alg 2.5 $b=-1$	Alg 2.5 $b=1/2$	Alg 2.5 $b=-1/2$
f_1	-1	10	4	3	4
f_2	2	18	6	7	6
f_3	-2	38	5	4	4
f_4	1.7	5	4	3	3
f_5	0	7	3	4	3
f_6	-1	9	5	5	4
f_7	-1	7	3	5	4

As convergence criteria, it was required that the distance of two consecutive approximations for the zero was less than 10^{-15}. Displayed in all Tables 4.1–4.6 is the number of iterations (IT) to approximate the zero for all the methods. The computational results presented in above Tables show that for most of the functions we tested, the presented methods are efficient and show better performance as compared with the Steffensen's method. Thus, presented methods in this contribution can be considered as an improvement of the derivative-free methods for solving nonlinear equations.

5 Conclusion

In this work, we have presented new iterative methods for solving nonlinear equations. These all are derivative-free methods. For different values of the parameter α, we can obtain different classes of derivative-free methods for solving nonlinear equations. These computed methods are compared with well known Steffensen's method and the proposed methods have been observed to have at least better performance.

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