Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/080519

Prime (m,n) Bi-Γ-Hyperideals in Γ-Semihypergroups

Naveed Yaqoob[1,*] *and Muhammad Aslam*[1,2]

[1] Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan
[2] Department of Mathematics, College of Science, King Khalid University, Abha, Saudi Arabia

Received: 22 Aug. 2013, Revised: 19 Nov. 2013, Accepted: 20 Nov. 2013
Published online: 1 Sep. 2014

Abstract: Relations between rough sets and algebraic structures have been already considered by many mathematicians. Motivated by studying the properties of rough (m,n) bi-Γ-hyperideals in Γ-semihypergroups, we now introduced the notion of prime (m,n) bi-Γ-hyperideals in Γ-semihypergroups and investigated several properties of these prime (m,n) bi-Γ-hyperideals. Also we applied the rough set theory to these prime (m,n) bi-Γ-hyperideals and proved that the lower and upper approximation of a prime (m,n) bi-Γ-hyperideal is a prime (m,n) bi-Γ-hyperideal in a Γ-semihypergroup. In the end we established some results on rough prime (m,n) bi-Γ-hyperideals in the quotient Γ-semihypergroups.

Keywords: Γ-semihypergroups, Prime (m,n) bi-Γ-hyperideals, Rough prime (m,n) bi-Γ-hyperideals.

1 Introduction

Prime bi-ideals of groupoids was studied by Lee [1]. Further, many other authors studied the prime bi-ideals in different structures. Shabir and Kanwal [2], studied prime bi-ideals of semigroups and proved interesting results on strongly prime, prime, semiprime, strongly irreducible and irreducible bi-ideals of semigroups. The notion of (m,n)-ideals of semigroups was introduced by Lajos [3, 4]. Further Ansari et al. [5,6] added some results on (m,n)-ideals in semigroups and Γ-semigroups.

Hyperstructure theory was introduced in 1934, when Marty [7] defined hypergroups, began to analyze their properties and applied them to groups. Nowadays, hyperstructures have a lot of applications to several domains of mathematics and computer science and they are studied in many countries of the world. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. A lot of papers and several books have been written on hyperstructure theory, see [8], [9], [10]. A recent book on hyperstructures [11] points out on their applications in rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs.

Recently, Davvaz, Hila and et. al. [12], [13], [14], [15], [16], [17], introduced the notion of Γ-semihypergroup as a generalization of a semigroup, a generalization of a semihypergroup and a generalization of a Γ-semigroup. They presented many interesting examples and obtained a several characterizations of Γ-semihypergroups.

The notion of a rough set was proposed by Pawlak [18] as a formal tool for modeling and processing incomplete information in information systems. Some authors have studied the algebraic properties of rough sets, for instance Aslam et al. [19,20], Chinram [21], Kuroki [22], Yaqoob et al. [23,24,25,26,27,28,29], Ansari and Khan [30,31], Anvariyeh et al. [32] and Davvaz [33,34,35].

In this paper we introduced the concept of prime (m,n) bi-Γ-hyperideals in Γ-semihypergroup and apply the rough set theory to prime (m,n) bi-Γ-hyperideals.

2 Some notions in Γ-semihypergroups

Here we recall the basic terms and definitions from the theory of Γ-semihypergroups. Throughout the paper S denote a Γ-semihypergroup.

Definition 1. *[15] A map $\circ : S \times S \rightarrow \mathscr{P}^*(S)$ is called a hyperoperation or join operation on the set S, where S is a non-empty set and $\mathscr{P}^*(S)$ denotes the set of all non-empty subsets of S. A hypergroupoid is a set S with together a (binary) hyperoperation. A hypergroupoid $(S,\circ)$, which is*

* Corresponding author e-mail: nayaqoob@ymail.com

associative, that is $x\circ(y\circ z)=(x\circ y)\circ z$, $\forall x,y,z\in S$, *is called a semihypergroup.*

Let A and B be two non-empty subsets of S. Then, we define

$$A\Gamma B=\bigcup_{\gamma\in\Gamma}A\gamma B=\bigcup\{a\gamma b\mid a\in A,\ b\in B \text{ and } \gamma\in\Gamma\}.$$

Let $(S,\circ)$ be a semihypergroup and let $\Gamma=\{\circ\}$. Then, S is a Γ-semihypergroup. So, every semihypergroup is Γ-semihypergroup.

Let S be a Γ-semihypergroup and $\gamma\in\Gamma$. A non-empty subset A of S is called a sub Γ-semihypergroup of S if $x\gamma y\subseteq A$ for every $x,y\in A$. A Γ-semihypergroup S is called *commutative* if for all $x,y\in S$ and $\gamma\in\Gamma$, we have $x\gamma y=y\gamma x$.

Example 1. [15] Let $S=[0,1]$ and $\Gamma=\mathbb{N}$. For every $x,y\in S$ and $\gamma\in\Gamma$, we define $\gamma: S\times S\longrightarrow \wp^*(S)$ by $x\gamma y=\left[0,\frac{xy}{\gamma}\right]$. Then, γ is a hyperoperation. For every $x,y,z\in S$ and $\alpha,\beta\in\Gamma$, we have $(x\alpha y)\beta z=\left[0,\frac{xyz}{\alpha\beta}\right]=x\alpha(y\beta z)$. This means that S is a Γ-semihypergroup.

Example 2. [15] Let $(S,\circ)$ be a semihypergroup and Γ be a non-empty subset of S. We define $x\gamma y=x\circ y$ for every $x,y\in S$ and $\gamma\in\Gamma$. Then, S is a Γ-semihypergroup.

Definition 2. *[15] A non-empty subset A of a Γ-semihypergroup S is a right (left) Γ-hyperideal of S if $A\Gamma S\subseteq A$ ($S\Gamma A\subseteq A$), and is a Γ-hyperideal of S if it is both a right and a left Γ-hyperideal.*

Definition 3. *[15] A sub Γ-semihypergroup B of a Γ-semihypergroup S is called a bi-Γ-hyperideal of S if $B\Gamma S\Gamma B\subseteq B$.*

Definition 4. *[23] A subset A of a Γ-semihypergroup S is called an $(m,0)$ Γ-hyperideal ($(0,n)$ Γ-hyperideal) of S if $A^m\Gamma S\subseteq A$ ($S\Gamma A^n\subseteq A$).*

Definition 5. *[23] A sub Γ-semihypergroup A of a Γ-semihypergroup S is called an (m,n) bi-Γ-hyperideal of S, if $A^m\Gamma S\Gamma A^n\subseteq A$, where m, n are non-negative integers (A^m is suppressed if $m=0$).*

Here if $m=n=1$ then A is called bi-Γ-hyperideal of S. By a proper (m,n) bi-Γ-hyperideal we mean an (m,n) bi-Γ-hyperideal, which is a proper subset of S.

Example 3. [23] Let $S=[0,1]$ and $\Gamma=\mathbb{N}$. Then, S together with the hyperoperation $x\gamma y=\left[0,\frac{xy}{\gamma}\right]$ is a Γ-semihypergroup. Let $t\in[0,1]$ and set $T=[0,t]$. Then, clearly it can be seen that T is a sub Γ-semihypergroup of S. Since $T^m\Gamma S=[0,t^m]\subseteq[0,t]=T$ ($S\Gamma T^n=[0,t^n]\subseteq[0,t]=T$), so T is an $(m,0)$ Γ-hyperideal ($(0,n)$ Γ-hyperideal) of S. Since $T^m\Gamma S\Gamma T^n=[0,t^{m+n}]\subseteq[0,t]=T$, then T is an (m,n) bi-Γ-hyperideal of Γ-semihypergroup S.

Example 4. [23] Let $S=[-1,0]$ and $\Gamma=\{-1,-2,-3,\cdots\}$. Define the hyperoperation $x\gamma y=\left[\frac{xy}{\gamma},0\right]$ for all $x,y\in S$ and $\gamma\in\Gamma$. Then, clearly S is a Γ-semihypergroup. Let $\lambda\in[-1,0]$ and the set $B=[\lambda,0]$. Then, clearly B is a sub Γ-semihypergroup of S. Since $B^m\Gamma S=[\lambda^{2m+1},0]\subseteq[\lambda,0]=B$ ($S\Gamma B^n=[\lambda^{2n+1},0]\subseteq[\lambda,0]=B$), so B is an $(m,0)$ Γ-hyperideal ($(0,n)$ Γ-hyperideal) of S. Since $B^m\Gamma S\Gamma B^n=[\lambda^{2(m+n)+1},0]\subseteq[\lambda,0]=B$, then B is an (m,n) bi-Γ-hyperideal of Γ-semihypergroup S.

3 Prime (m,n) bi-Γ-hyperideals

In this section we will define prime (m,n) bi-Γ-hyperideals of a Γ-semihypergroup and discuss some related properties.

Definition 6. *An (m,n) bi-Γ-hyperideal B of a Γ-semihypergroup S is called prime if for $x,y\in S$, $x^m\alpha S\beta y^n\subseteq B$ (or $x^m\alpha z\beta y^n\subseteq B$, for all $z\in S$) implies $x\in B$ or $y\in B$, for all $\alpha,\beta\in\Gamma$.*

Definition 7. *An (m,n) bi-Γ-hyperideal B of a Γ-semihypergroup S is called semiprime if for $x\in S$, $x^m\alpha S\beta x^n\subseteq B$ (or $x^m\alpha z\beta x^n\subseteq B$, for all $z\in S$) implies $x\in B$, for all $\alpha,\beta\in\Gamma$.*

Example 5. Let $S=M_2(\mathbb{Z})$ be the set of all 2×2 matrices, then S is a semigroup under usual multiplication. Let

$$T_1=\left\{\begin{pmatrix}a & -b\\ -c & d\end{pmatrix}: a,b,c,d\in\mathbb{Z}\right\},$$

$$T_2=\left\{\begin{pmatrix}a & b\\ -c & d\end{pmatrix}: a,b,c,d\in\mathbb{Z}\right\},$$

$$T_3=\left\{\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}: a,b,d\in\mathbb{Z}\right\},$$

$$T_4=\left\{\begin{pmatrix}a & 0\\ c & d\end{pmatrix}: a,c,d\in\mathbb{Z}\right\},$$

be non-empty subsets of S. Let $\Gamma=\{\beta_1,\beta_2,\beta_3,\beta_4\}$. We define $A_1\beta_iA_2=A_1T_iA_2$ for every $A_1,A_2\in S$, and $\beta_i\in\Gamma$, $1\leq i\leq 4$. Then S is a Γ-semihypergroup. Let $B=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}: a,b,c,d\in 2\mathbb{Z}\right\}$. Since $B^m\Gamma S\Gamma B^n\subseteq B$. Then B is a prime (m,n) bi-Γ-hyperideal of S.

Example 6. Let $S=\{a,b,c,d,e,f\}$ and $\Gamma=\{\gamma,\beta\}$ be the sets of binary hyperoperations defined below:

γ	a	b	c	d	e	f
a	a	b	a	a	a	a
b	b	b	b	b	b	b
c	a	b	$\{a,c\}$	a	a	$\{a,f\}$
d	a	b	$\{a,e\}$	a	a	$\{a,d\}$
e	a	b	$\{a,e\}$	a	a	$\{a,d\}$
f	a	b	$\{a,c\}$	a	a	$\{a,f\}$

β	a	b	c	d	e	f
a	a	b	a	a	a	a
b	b	b	b	b	b	b
c	a	b	a	a	a	a
d	a	b	a	$\{a,d\}$	$\{a,e\}$	a
e	a	b	a	a	a	a
f	a	b	a	$\{a,f\}$	$\{a,c\}$	a

Then S is a Γ-semihypergroup. The $(m,0)$ Γ-hyperideals are $\{a,b\}$, $\{b\}$, $\{a,b,c,f\}$, $\{a,b,d,e\}$ and S. The $(0,n)$ Γ-hyperideals are $\{a,b\}$, $\{b\}$, $\{a,b,c,e\}$, $\{a,b,d,f\}$ and S. The (m,n) bi-Γ-hyperideals are $\{a,b\}$, $\{b\}$, $\{a,b,c\}$, $\{a,b,f\}$, $\{a,b,d\}$, $\{a,b,c,e\}$, $\{a,b,d,f\}$, $\{a,b,c,f\}$, $\{a,b,d,e\}$ and S.

The only prime (m,n) bi-Γ-hyperideals of S are $\{b\}$ and S, and hence these are semiprime.

Furthermore $\{a,b\}$, $\{a,b,c\}$, $\{a,b,f\}$, $\{a,b,d\}$, $\{a,b,c,e\}$, $\{a,b,d,f\}$, $\{a,b,c,f\}$, $\{a,b,d,e\}$ are not prime (m,n) bi-Γ-hyperideals. Indeed

$e^m\Gamma S\Gamma f^n \subseteq \{a,b\}$, but $e,f \notin \{a,b\}$,
$e^m\Gamma S\Gamma f^n \subseteq \{a,b,c\}$, but $e,f \notin \{a,b,c\}$,
$c^m\Gamma S\Gamma d^n \subseteq \{a,b,f\}$, but $c,d \notin \{a,b,f\}$,
$e^m\Gamma S\Gamma f^n \subseteq \{a,b,d\}$, but $e,f \notin \{a,b,d\}$,
$d^m\Gamma S\Gamma f^n \subseteq \{a,b,c,e\}$, but $d,f \notin \{a,b,c,e\}$,
$c^m\Gamma S\Gamma e^n \subseteq \{a,b,d,f\}$, but $c,e \notin \{a,b,d,f\}$,
$d^m\Gamma S\Gamma e^n \subseteq \{a,b,c,f\}$, but $d,e \notin \{a,b,c,f\}$,
$c^m\Gamma S\Gamma f^n \subseteq \{a,b,d,e\}$, but $c,f \notin \{a,b,d,e\}$.

Theorem 1. *If an (m,n) bi-Γ-hyperideal B of a Γ-semihypergroup S is prime, then for an $(m,0)$ Γ-hyperideal R and a $(0,n)$ Γ-hyperideal L of S, $R\Gamma L \subseteq B$ implies $R \subseteq B$ or $L \subseteq B$.*

Proof. Suppose that $R\Gamma L \subseteq B$ for an $(m,0)$ Γ-hyperideal R and a $(0,n)$ Γ-hyperideal L of S and $R \nsubseteq B$. Then there exists $x \in R\backslash B$. Let $y \in L$. Then

$$x^m\Gamma S\Gamma y^n \subseteq R^m\Gamma S\Gamma L^n \subseteq R\Gamma L \subseteq B.$$

Since B is a prime (m,n) bi-Γ-hyperideal and $x \notin B$, we have $y \in B$. Thus $L \subseteq B$. □

Proposition 1. *If an (m,n) bi-Γ-hyperideal B of a Γ-semihypergroup S is prime, then B is a $(0,n)$ Γ-hyperideal or an $(m,0)$ Γ-hyperideal of S.*

Proof. Since $B^m\Gamma S$ is an $(m,0)$ Γ-hyperideal of S and $S\Gamma B^n$ a $(0,n)$ Γ-hyperideal of S such that

$$(B^m\Gamma S)\Gamma(S\Gamma B^n) \subseteq B^m\Gamma S\Gamma B^n \subseteq B,$$

we get $B^m\Gamma S \subseteq B$ or $S\Gamma B^n \subseteq B$ by Theorem 1. Hence B is a $(0,n)$ Γ-hyperideal or an $(m,0)$ Γ-hyperideal of S. □

Definition 8. *An (m,n) bi-Γ-hyperideal B of a Γ-semihypergroup S is called a strongly prime (m,n) bi-Γ-hyperideal if $B_1\Gamma B_2 \cap B_2\Gamma B_1 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ for any (m,n) bi-Γ-hyperideals B_1 and B_2 of S.*

Every strongly prime (m,n) bi-Γ-hyperideal of a Γ-semihypergroup S is a prime (m,n) bi-Γ-hyperideal and every prime (m,n) bi-Γ-hyperideal is a semiprime (m,n) bi-Γ-hyperideal. A prime (m,n) bi-Γ-hyperideal is not necessarily strongly prime.

Example 7. Let $S = \{e,a,b,c,d\}$ and $\Gamma = \{\gamma,\beta\}$ be the sets of binary hyperoperations defined below:

γ	e	a	b	c	d
e	e	e	e	e	e
a	e	$\{a,b\}$	b	b	b
b	e	b	b	b	b
c	e	c	c	c	c
d	e	d	d	d	d

β	e	a	b	c	d
e	e	e	e	e	e
a	e	a	a	a	a
b	e	a	$\{a,b\}$	a	a
c	e	c	c	c	c
d	e	d	d	d	d

Then S is a Γ-semihypergroup. The (m,n) bi-Γ-hyperideals of S are $\{e\}$, $\{e,c\}$, $\{e,d\}$, $\{e,a,b\}$, $\{e,c,d\}$ and S. Here all (m,n) bi-Γ-hyperideals of S are prime and hence semiprime. However, the prime (m,n) bi-Γ-hyperideal $\{e\}$ is not strongly prime (m,n) bi-Γ-hyperideal of S because

$$\{e,c\}\Gamma\{e,d\} \cap \{e,d\}\Gamma\{e,c\} = \{e\} \subseteq \{e\},$$

but neither $\{e,c\}$ nor $\{e,d\}$ is contained in $\{e\}$.

Definition 9. *An (m,n) bi-Γ-hyperideal B of a Γ-semihypergroup S is called an irreducible (resp. strongly irreducible) (m,n) bi-Γ-hyperideal if $B_1 \cap B_2 = B$ (resp. $B_1 \cap B_2 \subseteq B$) implies $B_1 = B$ or $B_2 = B$ (resp. $B_1 \subseteq B$ or $B_2 \subseteq B$).*

In Example 7, the irreducible (m,n) bi-Γ-hyperideals of S are $\{e,c\}$, $\{e,d\}$, $\{e,a,b\}$, $\{e,c,d\}$ and S. But the (m,n) bi-Γ-hyperideal $\{e\}$ is not irreducible, because $\{e,c\} \cap \{e,d\} = \{e\}$ but neither $\{e,c\} = \{e\}$ nor $\{e,d\} = \{e\}$.

Lemma 1. *The intersection of any family of prime (m,n) bi-Γ-hyperideals of a Γ-semihypergroup is a semiprime (m,n) bi-Γ-hyperideal.*

Proof. The proof is straightforward. □

Theorem 2. *Every strongly irreducible, semiprime (m,n) bi-Γ-hyperideal of a Γ-semihypergroup S is a strongly prime (m,n) bi-Γ-hyperideal.*

Proof. Let B be a strongly irreducible semiprime (m,n) bi-Γ-hyperideal of S. Let B_1, B_2 be any (m,n) bi-Γ-hyperideals of S such that $B_1\Gamma B_2 \cap B_2\Gamma B_1 \subseteq B$. Since

$$(B_1 \cap B_2)^2 \subseteq B_1\Gamma B_2 \text{ and } (B_1 \cap B_2)^2 \subseteq B_2\Gamma B_1,$$

$$(B_1 \cap B_2)^2 \subseteq B_1\Gamma B_2 \cap B_2\Gamma B_1 \subseteq B.$$

Since B is a semiprime (m,n) bi-Γ-hyperideal, $B_1 \cap B_2 \subseteq B$. Because B is a strongly irreducible (m,n) bi-Γ-hyperideal of S, so either $B_1 \subseteq B$ or $B_2 \subseteq B$. Thus B is a strongly prime (m,n) bi-Γ-hyperideal of S. □

Theorem 3. *Let B be an (m,n) bi-Γ-hyperideal of a Γ-semihypergroup S and $a \in S$ such that $a \notin B$ for a positive integer m. Then there exists an irreducible (m,n) bi-Γ-hyperideal I of S such that $B \subseteq I$ and $a \notin I$.*

Proof. Let $\mathscr{A}$ be the collection of all (m,n) bi-Γ-hyperideals of S which contain B and do not contain a. Then $\mathscr{A}$ is nonempty, because $B \in \mathscr{A}$. The collection $\mathscr{A}$ is a partially ordered set under inclusion. If $\mathscr{C}$ is any totally ordered subset of $\mathscr{A}$ then $\cup\mathscr{C}$ is an (m,n) bi-Γ-hyperideal of S containing B. Hence by Zorn's Lemma, there exists a maximal element I in $\mathscr{A}$. We show that I is an irreducible (m,n) bi-Γ-hyperideal. Let C and D be two (m,n) bi-Γ-hyperideals of S such that $I = C \cap D$. If both C and D properly contain I then $a \in C$ and $a \in D$. Hence $a \in C \cap D = I$. This contradicts the fact that $a \notin I$. Thus $I = C$ or $I = D$. □

Definition 10. *An element $x \in S$ is called regular if there exist a in S and $\alpha, \beta \in \Gamma$ such that $x \in x\alpha a\beta x$. If every element of a Γ-semihypergroup S is regular then S called regular Γ-semihypergroup.*

Definition 11. *An element a of a Γ-semihypergroup S is called intra-regular if there exist $x, y \in S$ such that $a \in x\alpha a^2\beta y$, for all $\alpha, \beta \in \Gamma$ and S is called intra-regular, if every element of S is intra-regular.*

Theorem 4. *Let S be a regular and intra-regular Γ-semihypergroup. Then the following assertions, for an (m,n) bi-Γ-hyperideal B of S, are equivalent:*
(i) B is strongly irreducible.
(ii) B is strongly prime.

Proof. The proof is straightforward. □

Theorem 5. *For a Γ-semihypergroup S the following assertions are equivalent:*
(i) The set of (m,n) bi-Γ-hyperideals of S is totally ordered under inclusion,
(ii) Each (m,n) bi-Γ-hyperideal of S is strongly irreducible,
(iii) Each (m,n) bi-Γ-hyperideal of S is irreducible.

Proof. $(i) \Longrightarrow (ii)$ Let B be an arbitrary (m,n) bi-Γ-hyperideal of S and B_1, B_2 be two (m,n) bi-Γ-hyperideals of S such that $B_1 \cap B_2 \subseteq B$. Since the set of (m,n) bi-Γ-hyperideals is totally ordered, either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Thus either $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$. Hence $B_1 \cap B_2 \subseteq B$ implies either $B_1 \subseteq B$ or $B_2 \subseteq B$. This shows that B is a strongly irreducible (m,n) bi-Γ-hyperideal.

$(ii) \Longrightarrow (iii)$ Let B be an arbitrary (m,n) bi-Γ-hyperideal of S and B_1, B_2 be two (m,n) bi-Γ-hyperideals of S such that $B_1 \cap B_2 = B$. Then $B \subseteq B_1$ and $B \subseteq B_2$. By hypothesis, either $B_1 \subseteq B$ or $B_2 \subseteq B$. Hence either $B_1 = B$ or $B_2 = B$. That is, B is an irreducible (m,n) bi-Γ-hyperideal.

$(iii) \Longrightarrow (i)$ Let B_1 and B_2 be any two (m,n) bi-Γ-hyperideals of S. Then $B_1 \cap B_2$ is an (m,n) bi-Γ-hyperideal of S. So by hypothesis, either $B_1 = B_1 \cap B_2$ or $B_2 = B_1 \cap B_2$, that is, either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. □

Proposition 2. *If every (m,n) bi-Γ-hyperideal of S is semiprime then S is regular.*

Proof. Suppose that every (m,n) bi-Γ-hyperideal of S is semiprime. Let $B = x^m\Gamma S\Gamma x^n$ for $x \in S$. Then using [23, Lemma 2.1], we have

$$\begin{aligned} B^m\Gamma S\Gamma B^n &= (x^m\Gamma S\Gamma x^n)^m\Gamma S\Gamma(x^m\Gamma S\Gamma x^n)^n \\ &\subseteq x^m\Gamma S\Gamma x^n = B, \end{aligned}$$

thus B is an (m,n) bi-Γ-hyperideal of S. As B is semiprime for all $x \in S$. Since $B = x^m\Gamma S\Gamma x^n$, we get $x \in x^m\Gamma S\Gamma x^n = B$. Now for any $x \in S$,

$$x \in x^m\Gamma S\Gamma x^n \subseteq x\Gamma S\Gamma x.$$

Hence for all $x \in S$, there exists $a \in S$ such that $x \in x\alpha a\beta x$, for all $\alpha, \beta \in \Gamma$. Therefore S is regular. □

Lemma 2. *A Γ-semihypergroup S is completely regular if and only if*

$$A \subseteq (A\Gamma A)\Gamma S\Gamma(A\Gamma A)$$

for every $A \subseteq S$. Equivalently, if $x \in x\Gamma x\Gamma S\Gamma x\Gamma x$ for all $x \in S$.

Theorem 6. *If every (m,n) bi-Γ-hyperideal of S is semiprime then S is completely regular.*

Proof. Let $x \in S$. Let

$$B = (x\Gamma x)^m\Gamma S\Gamma(x\Gamma x)^n = x^m\Gamma x^m\Gamma S\Gamma x^n\Gamma x^n.$$

Then using [23, Lemma 2.1], we have

$$\begin{aligned} &B^m\Gamma S\Gamma B^n \\ &= (x^m\Gamma x^m\Gamma S\Gamma x^n\Gamma x^n)^m\Gamma S\Gamma(x^m\Gamma x^m\Gamma S\Gamma x^n\Gamma x^n)^n \\ &\subseteq x^m\Gamma x^m\Gamma S\Gamma x^n\Gamma x^n = (x\Gamma x)^m\Gamma S\Gamma(x\Gamma x)^n = B, \end{aligned}$$

thus B is an (m,n) bi-Γ-hyperideal of S. As B is semiprime for all $x \in S$. Since $B = (x\Gamma x)^m\Gamma S\Gamma(x\Gamma x)^n$, we get $x \in (x\Gamma x)^m\Gamma S\Gamma(x\Gamma x)^n = B$. Now for any $x \in S$, $x \in (x\Gamma x)^m\Gamma S\Gamma(x\Gamma x)^n \subseteq x\Gamma x\Gamma S\Gamma x\Gamma x$. Hence for all $x \in S$, there exists $a \in S$ such that $x \in x\alpha x\gamma a\delta x\beta x$, for all $\alpha, \beta, \gamma, \delta \in \Gamma$. Therefore S is completely regular. □

4 Rough prime (m,n) bi-Γ-hyperideals

In this section we will study rough prime (m,n) bi-Γ-hyperideals.

Definition 12. *Let S be a Γ-semihypergroup. An equivalence relation ρ on S is called regular on S if*

$$(a,b) \in \rho \text{ implies } (a\gamma x, b\gamma x) \in \rho \text{ and } (x\gamma a, x\gamma b) \in \rho,$$

for all $x \in S$ and $\gamma \in \Gamma$.

If ρ is a regular relation on S, then, for every $x \in S$, $[x]_\rho$ stands for the class of x with the represent ρ. A regular relation ρ on S is called complete if $[a]_\rho \gamma [b]_\rho = [a\gamma b]_\rho$ for all $a, b \in S$ and $\gamma \in \Gamma$. In addition, ρ on S is called congruence if, for every $(a,b) \in S$ and $\gamma \in \Gamma$, we have $c \in [a]_\rho \gamma [b]_\rho \Longrightarrow [c]_\rho \subseteq [a]_\rho \gamma [b]_\rho$.

Let A be a non-empty subset of a Γ-semihypergroup S and ρ be a regular relation on S. Then, the sets

$$\underline{Apr}_\rho(A) = \left\{x \in S : [x]_\rho \subseteq A\right\}$$

and $$\overline{Apr}_\rho(A) = \left\{x \in S : [x]_\rho \cap A \neq \emptyset\right\}$$

are called ρ-lower and ρ-upper approximations of A, respectively. For a non-empty subset A of S, $Apr_\rho(A) = (\underline{Apr}_\rho(A), \overline{Apr}_\rho(A))$ is called a rough set with respect to ρ if $\underline{Apr}_\rho(A) \neq \overline{Apr}_\rho(A)$.

Theorem 7. *[32] Let ρ be a regular relation on a Γ-semihypergroup S and let A and B be non-empty subsets of S. Then,*
(1) $\overline{Apr}_\rho(A)\Gamma\overline{Apr}_\rho(B) \subseteq \overline{Apr}_\rho(A\Gamma B)$;
(2) If ρ is complete, then $\underline{Apr}_\rho(A)\Gamma\underline{Apr}_\rho(B) \subseteq \underline{Apr}_\rho(A\Gamma B)$.

A subset A of a Γ-semihypergroup S is called a ρ-upper (resp. ρ-lower) rough (m,n) bi-Γ-hyperideal of S if $\overline{Apr}_\rho(A)$ (resp. $\underline{Apr}_\rho(A)$) is an (m,n) bi-Γ-hyperideal of S.

Theorem 8. *[23] Let ρ be a regular relation on a Γ-semihypergroup S. If A is an (m,n) bi-Γ-hyperideal of S, then it is a ρ-upper rough (m,n) bi-Γ-hyperideal of S.*

Theorem 9. *[23] Let ρ be a complete regular relation on a Γ-semihypergroup S. If A is an (m,n) bi-Γ-hyperideal of S, then $\underline{Apr}_\rho(A)$ is, if it is nonempty, an (m,n) bi-Γ-hyperideal of S.*

Let ρ be a regular relation on a Γ-semihypergroup S. Then a subset A of S is called a ρ-lower rough prime (m,n) bi-Γ-hyperideal of S if $\underline{Apr}_\rho(A)$ is a prime (m,n) bi-Γ-hyperideal of S. A ρ-upper rough prime (m,n) bi-Γ-hyperideal of S is defined analogously. A is called a rough prime (m,n) bi-Γ-hyperideal of S if A is a ρ-lower and a ρ-upper rough prime (m,n) bi-Γ-hyperideal of S.

Theorem 10. *Let ρ be a complete regular relation on a Γ-semihypergroup S. If A is a prime (m,n) bi-Γ-hyperideal of S, then A is a ρ-upper rough prime (m,n) bi-Γ-hyperideal of S.*

Proof. Since A is an (m,n) bi-Γ-hyperideal of S, then by Theorem 8, $\overline{Apr}_\rho(A)$ is an (m,n) bi-Γ-hyperideal of S. Let w be any element of S. Let $x, y \in S$ and $\beta, \gamma \in \Gamma$ such that $x^m \beta w \gamma y^n \subseteq \overline{Apr}_\rho(A)$. Thus

$$[x^m \beta w \gamma y^n]_\rho \cap A = [x^m]_\rho \beta [w]_\rho \gamma [y^n]_\rho \cap A \neq \phi.$$

Thus there exist $a^m \subseteq [x^m]_\rho = [x]_\rho^m$, $w' \in [w]_\rho$ and $b^n \subseteq [y^n]_\rho = [y]_\rho^n$ such that $a^m \beta w' \gamma b^n \subseteq A$. Since A is a prime (m,n) bi-Γ-hyperideal, we have $a \in A$ or $b \in A$. Now

$$a^m \subseteq [x]_\rho^m \Longrightarrow a \in [x]_\rho \text{ also } b^n \subseteq [y]_\rho^n \Longrightarrow b \in [y]_\rho.$$

Thus $a \in [x]_\rho \cap A$ or $b \in [y]_\rho \cap A$. So $[x]_\rho \cap A \neq \phi$ or $[y]_\rho \cap A \neq \phi$, and so $x \in \overline{Apr}_\rho(A)$ or $y \in \overline{Apr}_\rho(A)$. Therefore $\overline{Apr}_\rho(A)$ is a prime (m,n) bi-Γ-hyperideal of S. □

Theorem 11. *Let ρ be a complete regular relation on a Γ-semihypergroup S and A is a prime (m,n) bi-Γ-hyperideal of S. Then $\underline{Apr}_\rho(A)$ is, if it is nonempty, a prime (m,n) bi-Γ-hyperideal of S.*

Proof. Since A is an (m,n) bi-Γ-hyperideal of S, by Theorem 9, we know that $\underline{Apr}_\rho(A)$ is an (m,n) bi-Γ-hyperideal of S. We suppose that $\underline{Apr}_\rho(A)$ is not a prime (m,n) bi-Γ-hyperideal, then for $\beta, \gamma \in \Gamma$ there exists $x, y \in S$ and any element $w \in S$, such that $x^m \beta w \gamma y^n \subseteq \underline{Apr}_\rho(A)$, but $x \notin \underline{Apr}_\rho(A)$ and $y \notin \underline{Apr}_\rho(A)$. Thus $[x]_\rho \nsubseteq A$ and $[y]_\rho \nsubseteq A$. Then there exist

$$a \in [x]_\rho \text{ but } a \notin A \quad \text{and} \quad b \in [y]_\rho \text{ but } b \notin A.$$

We have for all $w \in S$ and $\beta, \gamma \in \Gamma$,

$$\begin{aligned} a^m \beta w \gamma b^n &\subseteq [x]_\rho^m \beta [w]_\rho \gamma [y]_\rho^n = [x^m]_\rho \beta [w]_\rho \gamma [y^n]_\rho \\ &= [x^m \beta w \gamma y^n]_\rho \subseteq A. \end{aligned}$$

This implies that $a^m \beta w \gamma b^n \subseteq A$. Since A is a prime (m,n) bi-Γ-hyperideal, we have $a \in A$ or $b \in A$. It contradicts the supposition. This means that $\underline{Apr}_\rho(A)$ is, if it is nonempty, a prime (m,n) bi-Γ-hyperideal of S. □

The following example shows that the converse of Theorem 10 and Theorem 11 does not hold.

Example 8. Let $S = \{a,b,c,d,e\}$ and $\Gamma = \{\gamma, \beta\}$ be the sets of binary hyperoperations defined below:

γ	a	b	c	d	e
a	$\{a,b\}$	$\{b,c\}$	c	$\{d,e\}$	e
b	$\{b,c\}$	c	c	$\{d,e\}$	e
c	c	c	c	$\{d,e\}$	e
d	$\{d,e\}$	$\{d,e\}$	$\{d,e\}$	d	e
e	e	e	e	e	e

β	a	b	c	d	e
a	$\{b,c\}$	c	c	$\{d,e\}$	e
b	c	c	c	$\{d,e\}$	e
c	c	c	c	$\{d,e\}$	e
d	$\{d,e\}$	$\{d,e\}$	$\{d,e\}$	d	e
e	e	e	e	e	e

Then S is a Γ-semihypergroup. Let ρ be a complete regular relation on S such that ρ-regular classes are the subsets $\{a,b,c\}$, $\{d,e\}$. Then for $A = \{c,d,e\} \subseteq S$,

$\overline{Apr}_\rho(A) = \{a,b,c,d,e\}$, and $\underline{Apr}_\rho(A) = \{d,e\}$. It is clear that $\overline{Apr}_\rho(A)$, $\underline{Apr}_\rho(A)$ are prime (m,n) bi-Γ-hyperideals of S. The (m,n) bi-Γ-hyperideal A is not a prime (m,n) bi-Γ-hyperideal for $b^m\Gamma c\Gamma a^n = c \in A$ but $b \notin A$ and $a \notin A$.

5 Rough prime (m,n) bi-Γ-hyperideals in the quotient Γ-semihypergroups

Let ρ be a regular relation on a Γ-semihypergroup S. We put $\widehat{\Gamma} = \{\widehat{\gamma} : \gamma \in \Gamma\}$. For every $[a]_\rho, [b]_\rho \in S/\rho$, we define $[a]_\rho \widehat{\gamma}[b]_\rho = \{[z]_\rho : z \in a\gamma b\}$.

Theorem 12. *[32, Theorem 4.1] If S is a Γ-semihypergroup, then S/ρ is a $\widehat{\Gamma}$-semihypergroup.*

Definition 13. *Let ρ be a regular relation on a Γ-semihypergroup S. The ρ-lower approximation and ρ-upper approximation of a non-empty subset A of S can be presented in an equivalent form as shown below:*

$$\underline{\underline{Apr}}_\rho(A) = \left\{[x]_\rho \in S/\rho : [x]_\rho \subseteq A\right\}$$

and $$\overline{\overline{Apr}}_\rho(A) = \left\{[x]_\rho \in S/\rho : [x]_\rho \cap A \neq \emptyset\right\},$$

respectively.

Theorem 13. *[23] Let ρ be a regular relation on a Γ-semihypergroup S. If A is an (m,n) bi-Γ-hyperideal of S. Then,*

(1) $\overline{\overline{Apr}}_\rho(A)$ is an (m,n) bi-$\widehat{\Gamma}$-hyperideal of S/ρ.

(2) $\underline{\underline{Apr}}_\rho(A)$ is, if it is non-empty, an (m,n) bi-$\widehat{\Gamma}$-hyperideal of S/ρ.

Theorem 14. *Let ρ be a complete regular relation on a Γ-semihypergroup S. If A is a ρ-upper rough prime (m,n) bi-Γ-hyperideal of S, then $\overline{\overline{Apr}}_\rho(A)$ is a prime (m,n) bi-$\widehat{\Gamma}$-hyperideal of S/ρ.*

Proof. Let A be a ρ-upper rough prime (m,n) bi-Γ-hyperideal of S, by Theorem 13(1), we know that $\overline{\overline{Apr}}_\rho(A)$ is an (m,n) bi-Γ-hyperideal of S/ρ. Suppose for any $w \in S$, $\beta,\gamma \in \Gamma$ and $[x]_\rho, [y]_\rho \in S/\rho$, such that

$$\begin{aligned}[x]_\rho^m \widehat{\beta} w \widehat{\gamma} [y]_\rho^n &= [x^m]_\rho \widehat{\beta} w \widehat{\gamma} [y^n]_\rho \\ &= [x^m \beta w \gamma y^n]_\rho \subseteq \overline{\overline{Apr}}_\rho(A),\end{aligned}$$

for $\widehat{\beta}, \widehat{\gamma} \in \widehat{\Gamma}$. Thus $[x^m\beta w\gamma y^n]_\rho \cap A \neq \phi$. Now there exist t, such that $t \in x^m\beta w\gamma y^n \subseteq \overline{Apr}_\rho(A)$. Since A is a ρ-upper rough prime (m,n) bi-Γ-hyperideal of S, that is $\overline{Apr}_\rho(A)$ is a prime (m,n) bi-Γ-hyperideal, we have $x \in \overline{Apr}_\rho(A)$ or $y \in \overline{Apr}_\rho(A)$. Now as $t \in x^m\beta w\gamma y^n$, we obtain $[t]_\rho \in [x]_\rho^m \widehat{\beta} w \widehat{\gamma} [y]_\rho^n$. On the other hand, since $t \in \overline{Apr}_\rho(A)$, we have $[t]_\rho \cap A \neq \phi$. So $[x]_\rho \cap A \neq \phi$ or $[y]_\rho \cap A \neq \phi$. Hence $[x]_\rho \in \overline{\overline{Apr}}_\rho(A)$ or $[y]_\rho \in \overline{\overline{Apr}}_\rho(A)$. Therefore $\overline{\overline{Apr}}_\rho(A)$ is a prime (m,n) bi-$\widehat{\Gamma}$-hyperideal of S/ρ. $\square$

Theorem 15. *Let ρ be a complete regular relation on a Γ-semihypergroup S. If A is a ρ-lower rough prime (m,n) bi-Γ-hyperideal of S, then $\underline{\underline{Apr}}_\rho(A)$ is a prime (m,n) bi-$\widehat{\Gamma}$-hyperideal of S/ρ.*

Proof. Let A be a ρ-lower rough prime (m,n) bi-Γ-hyperideal of S, by Theorem 13(2), we know that $\underline{\underline{Apr}}_\rho(A)$ is an (m,n) bi-$\widehat{\Gamma}$-hyperideal of S/ρ. Suppose for any $w \in S$, $\beta,\gamma \in \Gamma$ and $[x]_\rho, [y]_\rho \in S/\rho$, such that

$$\begin{aligned}[x]_\rho^m \widehat{\beta} w \widehat{\gamma} [y]_\rho^n &= [x^m]_\rho \widehat{\beta} w \widehat{\gamma} [y^n]_\rho \\ &= [x^m \beta w \gamma y^n]_\rho \subseteq \underline{\underline{Apr}}_\rho(A).\end{aligned}$$

for $\widehat{\beta}, \widehat{\gamma} \in \widehat{\Gamma}$. Thus $[x^m\beta w\gamma y^n]_\rho \subseteq A$. Now there exist t, such that $t \in x^m\beta w\gamma y^n \subseteq \underline{Apr}_\rho(A)$. Since A is a ρ-lower rough prime (m,n) bi-Γ-hyperideal of S, that is $\underline{Apr}_\rho(A)$ is a prime (m,n) bi-Γ-hyperideal, we have $x \in \underline{Apr}_\rho(A)$ or $y \in \underline{Apr}_\rho(A)$. Now as $t \in x^m\beta w\gamma y^n$, we obtain $[t]_\rho \in [x]_\rho^m \widehat{\beta} w \widehat{\gamma} [y]_\rho^n$. On the other hand, since $t \in \underline{Apr}_\rho(A)$, we have $[t]_\rho \subseteq A$. So $[x]_\rho \subseteq A$ or $[y]_\rho \subseteq A$. Hence $[x]_\rho \in \underline{\underline{Apr}}_\rho(A)$ or $[y]_\rho \in \underline{\underline{Apr}}_\rho(A)$. Therefore $\underline{\underline{Apr}}_\rho(A)$ is a prime (m,n) bi-$\widehat{\Gamma}$-hyperideal of S/ρ. $\square$

6 Conclusion

In this paper, we investigated some properties of prime (m,n) bi-Γ-hyperideals in Γ-semihypergroups. Also we applied the rough set theory to these prime (m,n) bi-Γ-hyperideals.

In our future study, the following topics may be considered:

(i) Study on rough fuzzy Γ-hyperideals in Γ-semihypergroups.

(ii) Study on rough fuzzy prime Γ-hyperideals in Γ-semihypergroups.

Acknowledgement

The authors are highly grateful to referee's for their valuable comments and suggestions for improving the paper.

References

[1] S.K. Lee, Prime bi-ideals of groupoids, Kangweon Kyungki Math. J., **13**, 217-221 (2005).
[2] M. Shabir and N. Kanwal, Prime bi-ideals of semigroups, Southeast Asian Bull. Math., **31**, 757-764 (2007).
[3] S. Lajos, Generalized ideals in semigroups, Acta Sci. Math., **22**, 217–222 (1961).
[4] S. Lajos, Notes on (m,n)-ideals I, Proc. Japan Acad., **39**, 419-421 (1963).
[5] M.A. Ansari, M.R. Khan and J.P. Kaushik, Notes on generalized (m,n) bi-ideals in semigroups with involution, Int. J. Algebra, **3**, 945-952 (2009).
[6] M.A. Ansari and M.R. Khan, Notes on (m,n)-bi-Γ-ideals in Γ-semigroups, Rend. Circ. Mat. Palermo, **60**, 31-42 (2011).
[7] F. Marty, Sur une generalization de la notion de group, 8th Congres Math. Scandinaves, Stockholm, 45-49 (1934).
[8] P. Corsini, Prolegomena of hypergroup theory, Second edition, Aviani editor, (1993).
[9] B. Davvaz and V. L. Fotea, Hyperring theory and applications, International Academic Press, USA, (2007).
[10] T. Vougiouklis, Hyperstructures and their representations, Hadronic Press, Florida, (1994).
[11] P. Corsini and V. Leoreanu, Applications of hyperstructure theory, Advances in Mathematics, Kluwer Academic Publishers, Dordrecht, (2003).
[12] D. Heidari, S.O. Dehkordi and B. Davvaz, Γ-semihypergroups and their properties, U.P.B. Sci. Bull., Series A, **72**, 197-210 (2010).
[13] K. Hila, B. Davvaz and J. Dine, Study on the structure of Γ-semihypergroups, Commun. Algebra, **40**, 2932-2948 (2012).
[14] S. Mirvakili, S. M. Anvariyeh and B. Davvaz, Γ-semihypergroups and regular relations, J. Math., Article ID 915250, 9 pages (2013).
[15] S.M. Anvariyeh, S. Mirvakili, B. Davvaz, On Γ-hyperideals in Γ-semihypergroups, Carpathian J. Math., **26**, 11-23 (2010).
[16] S. Abdullah, K. Hila and M. Aslam, On bi-Γ-hyperideals of Γ-semihypergroups, U.P.B. Sci. Bull., Series A, **74**, 79-90 (2012).
[17] S. Abdullah, M. Aslam and T. Anwar, A note on M-hypersystems and N-hypersystems in Γ-semihypergroups, Quasigroups Related Systems, **19**, 101-104 (2011).
[18] Z. Pawlak, Rough sets, Int. J. Comput. Inform. Sci., **11**, 341-356 (1982).
[19] M. Aslam, M. Shabir, N. Yaqoob and A. Shabir, On rough (m,n)-bi-ideals and generalized rough (m,n)-bi-ideals in semigroups, Ann. Fuzzy Math. Inform., **2**, 141-150 (2011) .
[20] M. Aslam, S. Abdullah, B. Davvaz and N. Yaqoob, Rough M-hypersystems and fuzzy M-hypersystems in Γ-semihypergroups, Neural Comput. Appl., **21**, 281-287 (2012).
[21] R. Chinram, Rough prime ideals and rough fuzzy prime ideals in Γ-semigroups, Commun. Korean Math. Soc., **24**, 341-351 (2009).
[22] N. Kuroki, Rough ideals in semigroups, Inform. Sci., **100**, 139-163 (1997).
[23] N. Yaqoob, M. Aslam, B. Davvaz and A.B. Saeid, On rough (m,n) bi-Γ-hyperideals in Γ-semihypergroups, U.P.B. Sci. Bull., Series A, **75**, 119-128 (2013).
[24] N. Yaqoob, S. Abdullah, N. Rehman and M. Naeem, Roughness and fuzziness in ordered ternary semigroups, World Appl. Sci. J., **17**, 1683-1693 (2012).
[25] N. Yaqoob, Applications of rough sets to Γ-hyperideals in left almost Γ-semihypergroups, Neural Comput. Appl., **21**, 267-273 (2012).
[26] N. Yaqoob, M. Aslam and K. Hila, Rough fuzzy hyperideals in ternary semihypergroups, Advances in Fuzzy Systems, Article ID 595687, 9 pages (2012).
[27] N. Yaqoob and M. Aslam, On rough Quasi-Γ-hyperideals in Γ-semihypergroups, Afrika Matematika, doi: 10.1007/s13370-013-0207-x. 1-13 (2013)
[28] N. Yaqoob and R. Chinram, On prime (m,n) bi-ideals and rough prime (m,n) bi-ideals in semigroups, Far East J. Math. Sci., **62**, 145-159 (2012).
[29] N. Yaqoob, M. Aslam and R. Chinram, Rough prime bi-ideals and rough fuzzy prime bi-ideals in semigroups, Ann. Fuzzy Math. Inform., **3**, 203-211 (2012) .
[30] M.A. Ansari and M.R. Khan, On rough (m,n) quasi-Γ-ideals in Γ-semigroups, Int. J. Math. Res., **3**, 31-38 (2011).
[31] M.A. Ansari and M.R. Khan, On rough (m,n) quasi-ideals in semigroups, Indian J. Math. Math. Sci., **6**, 134-143 (2010).
[32] S.M. Anvariyeh, S. Mirvakili and B. Davvaz, Pawlak's approximations in Γ-semihypergroups, Comput. Math. Applic., **60**, 45-53(2010).
[33] B. Davvaz, Roughness in rings, Inform. Sci., **164**, 147-163 (2004).
[34] B. Davvaz, Roughness based on fuzzy ideals, Inform. Sci., **176**, 2417-2437 (2006).
[35] B. Davvaz, Approximations in a semigroup by using a neighborhood system, Int. J. Comput. Math., **88**, 709-713 (2011).