Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/080320

λ-Sequence Spaces of Interval Numbers

Ayhan Esi[*]

Department of Mathematics, Science and Art Faculty, Adiyaman University, 02040, Adiyaman, Turkey

Received: 22 May. 2013, Revised: 26 Sep. 2013, Accepted: 27 Sep. 2013
Published online: 1 May. 2014

Abstract: In this paper we introduce and study the concepts of strongly λ-convergence and statistically λ-convergence for interval numbers. We give some relations related to these concepts. The results are more general than the Mursaleen's results in [5].

Keywords: λ-sequence, statistical convergence, interval numbers

1 Introduction

Interval arithmetic was first suggested by Dwyer [7] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [8] in 1959 and Moore and Yang [9] 1962. Furthermore, Moore and others [8], [9], [10] and [11] have developed applications to differential equations.

Chiao in [4] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. Şengönül and Eryilmaz in [6] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete metric space. Recently, Esi [1] introduced and studied lacunary sequence spaces of interval numbers.

The idea of statistical convergence for single sequences was introduced by Fast [2] in 1951. Schoenberg [3] studied statistical convergence as a summability method and listed some of elemantary properties of statistical convergence. Both of these authors noted that if bounded sequence is statistically convergent, then it is Cesaro summable. In 2000, Mursaleen [5] defined and studied by λ-statistically convergent sequences. The notion was further investigated and different properties in the field of summability theory has been investigated by Rath and Tripathy [12], Tripathy [13], Tripathy and Sen [14] and many others.

2 Preliminaries

A set consisting of a closed interval of real numbers x such that $a \leq x \leq b$ is called an interval number. A real interval can also be considered as a set. Thus we can investigate some properties of interval numbers, for instance arithmetic properties or analysis properties.We denote the set of all real valued closed intervals by $\mathrm{I}\mathbb{R}$. Any elements of $\mathrm{I}\mathbb{R}$ is called closed interval and denoted by $\bar{x}$. That is $\bar{x} = \{x \in \mathbb{R} : \ a \leq x \leq b\}$. An interval number $\bar{x}$ is a closed subset of real numbers $[4]$. Let x_l and x_r be first and last points of $\bar{x}$ interval number, respectively. For $\bar{x}_1, \bar{x}_2 \in \mathrm{I}\mathbb{R}$, we have $\bar{x}_1 = \bar{x}_2 \Leftrightarrow x_{1_l}{=}x_{2_l}$, $x_{1_r}{=}x_{2_r}$. $\bar{x}_1 + \bar{x}_2 = \{x \in \mathbb{R} : x_{1_l} + x_{2_l} \leq x \leq x_{1_r} + x_{2_r}\}$ and if $\alpha \geq 0$, then $\alpha\bar{x} = \left\{x \in \mathbb{R} : \ \alpha x_{1_l} \leq x \leq \alpha x_{1_r}\right\}$ and if $\alpha < 0$, then $\alpha\bar{x} = \left\{x \in \mathbb{R} : \ \alpha x_{1_r} \leq x \leq \alpha x_{1_l}\right\}$,

$$\bar{x}_1.\bar{x}_2 = \left\{ \begin{array}{l} x \in \mathbb{R} : \min\left\{x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r}\right\} \leq x \\ \quad \leq \max\left\{x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r}\right\} \end{array} \right\}.$$

The set of all interval numbers $\mathrm{I}\mathbb{R}$ is a complete metric space defined by

$$d\left(\bar{x}_1, \bar{x}_2\right) = \max\left\{\left|x_{1_l} - x_{2_l}\right|, \left|x_{1_r} - x_{2_r}\right|\right\} \ [6].$$

In the special case $\bar{x}_1 = [a,a]$ and $\bar{x}_2 = [b,b]$, we obtain usual metric of $\mathbb{R}$.

Let us define transformation $f : \mathbb{N} \to \mathbb{R}$ by $k \to f(k) = \bar{x}$, $\bar{x} = (\bar{x}_k)$. Then $\bar{x} = (\bar{x}_k)$ is called sequence of interval numbers. The $\bar{x}_k$ is called k^{th} term of sequence $\bar{x} = (\bar{x}_k)$. w^i denotes the set of all interval numbers with real terms and the algebraic properties of w^i can be found in [6].

Now we give the definition of convergence of interval numbers:

[*] Corresponding author e-mail: aesi23@adiyaman.edu.tr

Definition 2.1. [4] A sequence $\bar{x} = (\bar{x}_k)$ of interval numbers is said to be convergent to the interval number $\bar{x}_o$ if for each $\varepsilon > 0$ there exists a positive integer k_o such that $d(\bar{x}_k, \bar{x}_o) < \varepsilon$ for all $k \geq k_o$ and we denote it by $\lim_k \bar{x}_k = \bar{x}_o$.

Thus, $\lim_k \bar{x}_k = \bar{x}_o \Leftrightarrow \lim_k x_{k_l} = x_{o_l}$ and $\lim_k x_{k_r} = x_{o_r}$.

3 Main Results

In this paper, we introduce and study the concepts of strongly λ-convergence and statistically λ-convergence for interval numbers and these concepts will generalize λ-convergence for single sequences defined by Mursaleen [5] earlier.

Definition 3.1. Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1, \lambda_n \to \infty$ as $n \to \infty$ and $I_n = [n - \lambda_n + 1, n]$. The sequence $\bar{x} = (\bar{x}_k)$ of interval numbers is said to be strongly λ-summable if there is an interval number $\bar{x}_o$ such that

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} d(\bar{x}_k, \bar{x}_o) = 0.$$

In which case we say that the sequence $\bar{x} = (\bar{x}_k)$ of interval numbers is said to be strongly λ-summable to interval number $\bar{x}_o$. If $\lambda_n = n$, then strongly λ-summable reduces to strongly Cesaro summable defined as follows:

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} d(\bar{x}_k, \bar{x}_o) = 0.$$

Example 3.1. Let $\lambda_n = n$ for all $n \in \mathbb{N}$. Consider the sequence of interval numbers $\bar{x} = (\bar{x}_k)$ defined by

$$\bar{x}_k = \left[-\frac{k^2+1}{k^2} \right].$$

Let $\bar{x}_o$ be defined by

$$\bar{x}_o = [-1, 0].$$

Then we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} d(\bar{x}_k, \bar{x}_o) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} k^{-2} = 0$$

since $\sum_{k=1}^{n} k^{-2}$ is convergent.

Definition 3.2. A sequence $\bar{x} = (\bar{x}_k)$ of interval numbers is said to be statistically λ-convergent to interval number $\bar{x}_o$ if for every $\varepsilon > 0$

$$\lim_n \frac{1}{\lambda_n} \left| \{ k \in I_n : \ d(\bar{x}_k, \bar{x}_o) \geq \varepsilon \} \right| = 0.$$

In this case we write $s_\lambda - \lim \bar{x}_k = \bar{x}_o$. If $\lambda_n = n$, then statistically λ-convergence reduces to statistically convergence as follows:

$$\lim_n \frac{1}{n} \left| \{ k \leq n : \ d(\bar{x}_k, \bar{x}_o) \geq \varepsilon \} \right| = 0.$$

In this case we write $s - \lim \bar{x}_k = \bar{x}_o$.

Theorem 3.1. Let $\bar{x} = (\bar{x}_k)$ and $\bar{y} = (\bar{y}_k)$ be sequences of interval numbers.

(i) If $\bar{s}_\lambda - \lim \bar{x}_k = \bar{x}_o$ and $\alpha \in \mathbb{R}$, then $\bar{s}_\lambda - \lim \alpha \bar{x}_k = \alpha \bar{x}_o$.

(ii) If $\bar{s}_\lambda - \lim \bar{x}_k = \bar{x}_o$ and $\bar{s}_\lambda - \lim \bar{y}_k = \bar{y}_o$, then $\bar{s}_\lambda - \lim (\bar{x}_k + \bar{y}_k) = \bar{x}_o + \bar{y}_o$.

Proof. **(i)** Let $\alpha \in \mathbb{R}$. We have $d(\alpha \bar{x}_k, \alpha \bar{x}_o) = |\alpha| d(\bar{x}_k, \bar{x}_o)$. For a given $\varepsilon > 0$

$$\frac{1}{\lambda_n} \left| \{ k \in I_n : \ d(\alpha \bar{x}_k, \alpha \bar{x}_o) \geq \varepsilon \} \right|$$

$$\leq \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \ d(\bar{x}_k, \bar{x}_o) \geq \frac{\varepsilon}{|\alpha|} \right\} \right|.$$

Hence$\bar{s}_\lambda - \lim \alpha \bar{x}_k = \alpha \bar{x}_o$.

(ii) Suppose that $\bar{s}_\lambda - \lim \bar{x}_k = \bar{x}_o$ and $\bar{s}_\lambda - \lim \bar{y}_k = \bar{y}_o$. We have

$$d(\bar{x}_k + \bar{y}_k, \bar{x}_o + \bar{y}_o)$$
$$\leq d(\bar{x}_k, \bar{x}_o) + d(\bar{y}_k, \bar{y}_o).$$

Therefore given $\varepsilon > 0$, we have

$$\frac{1}{\lambda_n} \left| \{ k \in I_n : d(\bar{x}_k + \bar{y}_k, \bar{x}_o + \bar{y}_o) \geq \varepsilon \} \right|$$

$$\leq \frac{1}{\lambda_n} \left| \{ k \in I_n : d(\bar{x}_k, \bar{x}_o) + d(\bar{y}_k, \bar{y}_o) \geq \varepsilon \} \right|$$

$$\leq \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \ d(\bar{x}_k, \bar{x}_o) \geq \frac{\varepsilon}{2} \right\} \right|$$
$$+ \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \ d(\bar{y}_k, \bar{y}_o) \geq \frac{\varepsilon}{2} \right\} \right|.$$

Thus, $\bar{s}_\lambda - \lim (\bar{x}_k + \bar{y}_k) = \bar{x}_o + \bar{y}_o$.

In the following theorems, we exhibit some connections between strongly λ-summable and statistically λ-convergence of sequences of interval numbers.

Theorem 3.2. If an interval sequence $\bar{x} = (\bar{x}_k)$ is strongly λ-summable to interval number $\bar{x}_o$, then it is statistically λ-convergent to interval number $\bar{x}_o$.

Proof. Let $\varepsilon > 0$. Since

$$\sum_{k \in I_n} d(\bar{x}_k, \bar{x}_o) \geq \sum_{\substack{k \in I_n \\ d(\bar{x}_k, \bar{x}_o) \geq \varepsilon}} d(\bar{x}_k, \bar{x}_o)$$

$$\geq \left| \{ k \in I_n : \ d(\bar{x}_k, \bar{x}_o) \geq \varepsilon \} \right| \varepsilon$$

if $\bar{x} = (\bar{x}_k)$ is strongly λ-summable to $\bar{x}_o$, then it is statistically λ-convergent to $\bar{x}_o$.

Theorem 3.3. If $\bar{x}=(\bar{x}_k)\in\overline{m}$ and $\bar{x}=(\bar{x}_k)$ is statistically λ-convergent to interval number $\bar{x}_o$, then it is strongly λ-summable to $\bar{x}_o$ and hence $\bar{x}=(\bar{x}_k)$ is strongly Cesaro summable to $\bar{x}_o$, where $\overline{m}=\{\bar{x}=(\bar{x}_k):\ \sup_k d(\bar{x}_k,\bar{x}_o)<\infty\}$.

Proof. Suppose that $\bar{x}=(\bar{x}_k)\in\overline{m}$ and statistically λ-convergent to interval number $\bar{x}_o$. Since $\bar{x}=(\bar{x}_k)\in\overline{m}$, we write $d(\bar{x}_k,\bar{x}_o)\le A$ for all $k\in\mathbb{N}$. Given $\varepsilon>0$, we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n} d(\bar{x}_k,\bar{x}_o)$$

$$=\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ d(\bar{x}_k,\bar{x}_o)\ge\varepsilon}} d(\bar{x}_k,\bar{x}_o)+\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ d(\bar{x}_k,\bar{x}_o)<\varepsilon}} d(\bar{x}_k,\bar{x}_o)$$

$$\le\frac{A}{\lambda_n}\left|\{k\in I_n:\ d(\bar{x}_k,\bar{x}_o)\ge\varepsilon\}\right|+\varepsilon$$

which implies that $\bar{x}=(\bar{x}_k)$ is strongly λ-summable to $\bar{x}_o$. Further we have

$$\frac{1}{n}\sum_{k=1}^{n} d(\bar{x}_k,\bar{x}_o)$$

$$=\frac{1}{n}\sum_{k=1}^{n-\lambda_n} d(\bar{x}_k,\bar{x}_o)+\frac{1}{n}\sum_{k\in I_n} d(\bar{x}_k,\bar{x}_o)$$

$$\le\frac{1}{\lambda_n}\sum_{k=1}^{n-\lambda_n} d(\bar{x}_k,\bar{x}_o)+\frac{1}{\lambda_n}\sum_{k\in I_n} d(\bar{x}_k,\bar{x}_o)$$

$$\le\frac{2}{\lambda_n}\sum_{k\in I_n} d(\bar{x}_k,\bar{x}_o).$$

Hence $\bar{x}=(\bar{x}_k)$ is strongly Cesaro summable to $\bar{x}_o$.

Theorem 3.4. If a interval sequence $\bar{x}=(\bar{x}_k)$ is statistically convergent to interval number $\bar{x}_o$ and $\liminf_n \frac{\lambda_n}{n}>0$ then it is statistically λ-convergent to $\bar{x}_o$.

Proof. For given $\varepsilon>0$, we have

$$\{k\le n:\ d(\bar{x}_k,\bar{x}_o)\ge\varepsilon\}\supset\{k\in I_n:\ d(\bar{x}_k,\bar{x}_o)\ge\varepsilon\}.$$

Therefore

$$\frac{1}{n}\left|\{k\le n:\ d(\bar{x}_k,\bar{x}_o)\ge\varepsilon\}\right|\ge\frac{1}{n}\left|\{k\in I_n:\ d(\bar{x}_k,\bar{x}_o)\ge\varepsilon\}\right|$$

$$\ge\frac{\lambda_n}{n}.\frac{1}{\lambda_n}\left|\{k\in I_n:\ d(\bar{x}_k,\bar{x}_o)\ge\varepsilon\}\right|.$$

Taking limit as $n\to\infty$ and using $\liminf_n \frac{\lambda_n}{n}>0$, we get that $\bar{x}=(\bar{x}_k)$ is statistically λ-convergent to $\bar{x}_o$.

Finally we conclude this paper by stating a definition which generalizes Definition 3.1. of Section 3 and two theorems related to this definition.

Definition 3.3. Let $\lambda=(\lambda_n)$ be a non-decreasing sequence of positive numbers such that $\lambda_{n+1}\le\lambda_n+1,\lambda_1=1,\lambda_n\to\infty$ as $n\to\infty$ and $I_n=[n-\lambda_n+1,n]$ and $p\in(0,\infty)$. The sequence $\bar{x}=(\bar{x}_k)$ of interval numbers is said to be strongly λp-summable if there is an interval number $\bar{x}_o$ such that

$$\lim_n\frac{1}{\lambda_n}\sum_{k\in I_n}\left[d(\bar{x}_k,\bar{x}_o)\right]^p=0.$$

In which case we say that the sequence $\bar{x}=(\bar{x}_k)$ of interval numbers is said to be strongly λp-summable to interval number $\bar{x}_o$. If $\lambda_n=n$, then strongly λp-summable reduces to strongly p-Cesaro summable defined as follows:

$$\lim_n\frac{1}{n}\sum_{k=1}^{n}\left[d(\bar{x}_k,\bar{x}_o)\right]^p=0.$$

The following theorems is similar to that of Theorem 3.2. and Theorem 3.3, so the proofs omitted.

Theorem 3.5. If an interval sequence $\bar{x}=(\bar{x}_k)$ is strongly λp-summable to interval number $\bar{x}_o$, then it is statistically λ-convergent to interval number $\bar{x}_o$.

Theorem 3.6. If $\bar{x}=(\bar{x}_k)\in\overline{m}$ and $\bar{x}=(\bar{x}_k)$ is statistically λ-convergent to interval number $\bar{x}_o$, then it is strongly λp-summable to $\bar{x}_o$ and hence $\bar{x}=(\bar{x}_k)$ is strongly p-Cesaro summable to $\bar{x}_o$.

References

[1] A. Esi, Lacunary sequence spaces of interval numbers, Thai Journal of Mathematics, **10**, 445-451 (2012).
[2] H. Fast, Sur la convergence statistique, Collog.Math., **2**, 241-244 (1951).
[3] I. J. Schoenberg, The integrability of certain functions and related summability methods, Amer.Math.Monthly, **66**, 361-375 (1959).
[4] Kuo-Ping Chiao, Fundamental properties of interval vector max-norm, Tamsui Oxford Journal of Mathematics, **18**, 219-233 (2002).
[5] Mursaleen, λ-statistical convergence, Math.Slovaca, **50**, 111-115 (2000).
[6] M. Şengönül and A. Eryılmaz, On the sequence spaces of interval numbers, Thai Journal of Mathematics, **8**, 503-510 (2010).
[7] P. S. Dwyer, Linear Computation, New York, Wiley, (1951).
[8] R. E. Moore, Automatic Error Analysis in Digital Computation, LSMD-48421, Lockheed Missiles and Space Company, (1959).
[9] R. E. Moore and C. T. Yang, Interval Analysis I, LMSD-285875, Lockheed Missiles and Space Company, (1962).
[10] R. E. Moore and C. T. Yang, Theory of an interval algebra and its application to numeric analysis, RAAG Memories II, Gaukutsu Bunken Fukeyu-kai, Tokyo, (1958).
[11] S. Markov, Quasilinear spaces and their relation to vector spaces, Electronic Journal on Mathematics of Computation, **2**, (2005).
[12] D. Rath and B. C. Tripathy, Matrix maps on sequence spaces associaetd with sets of integers, Indian Jour.Pure Appl.Math., **27**, 197-206 (1996).
[13] B. C. Tripathy, On generalized difference paranormed statistically convergent sequences, Indian Jour.Pure Appl.Math., **35**, 655-663 (2004).

[14] B. C. Tripathy and M. Sen, On generalized statistically convergent sequences, Indian Jour.Pure Appl.Math., **32**, 1689-1694 (2001).

Ayhan Esi received the PhD degree in Mathematics at Firat University. His research interests are in the areas of summability theory and fuzzy mathematics. He has published research articles in reputed international journals of mathematical sciences. He is referee and editor of mathematical journals.