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On the Ulam Stability of Cauchy Functional Equation in IFN-Spaces

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Abstract: The aim of this paper is to establish some stability results concerning the Cauchy functional equation $f(x+y) = f(x) + f(y)$ in the framework of intuitionistic fuzzy normed spaces.

Keywords: t-norm, t-conorm, intuitionistic fuzzy normed space, Cauchy functional equation, Hyers-Ulam stability

1 Introduction and Preliminaries

The notion of fuzzy sets was first introduced by Zadeh [40] in 1965 which is a powerful hand set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. Among various developments of this new theory, a progressive development has been made to find the fuzzy analogues of the classical set theory. In fact the fuzzy theory has become an area of active researches for the last four decades. It has a wide range of applications in the field of science and engineering, e.g. population dynamics [4], chaos control [6], computer programming [7], nonlinear dynamical systems [9], fuzzy physics [15], etc. The fuzzy topology [11] proves to be a very useful tool to deal with such situations where the use of classical theories breaks down.

Stability problem of a functional equation was first posed by Ulam [37] which was answered by Hyers [10] and then generalized by Aoki [3] and Rassias [35] for additive mappings and linear mappings, respectively. Since then several stability problems for various functional equations have been investigated in [12], [13], [14], [16], [20], [22], [23], [33], [34] and [35]. Recently, the stability problem for mixed type quadratic-additive functional equation, Jensen functional equation, additive functional equation, Pexiderized quadratic functional equation, cubic functional equation and mixed type additive cubic functional equations have been considered in [1], [17], [19], [25], [27] [30] and [38] in the intuitionistic fuzzy normed spaces. Note that the idea of intuitionistic fuzzy normed space was introduced in [36] and further studied in [24], [28] [29], [31], [32] and [39]. Quite recently, Chang [5] has established the stability of higher ring derivation in intuitionistic fuzzy Banach algebras associated to the Jensen type functional equation. In the recent past, Alotaibi and Mohiuddine [2] established the Ulam stability of a cubic functional equation in random 2-normed spaces, while the notion of random 2-normed spaces was introduced by Goleţ [8] and further studied in [18,26,21].

Now, we recall some notations and basic definitions which will be used throughout the paper.

Definition 1.1. A binary operation $*: [0,1]\times[0,1] \to [0,1]$ is said to be a *continuous t-norm* if it satisfies the following conditions:
(a) $*$ is associative and commutative, (b) $*$ is continuous, (c) $a*1 = a$ for all $a \in [0,1]$, (d) $a*b \le c*d$ whenever $a \le c$ and $b \le d$ for each $a,b,c,d \in [0,1]$.

Definition 1.2. A binary operation $\diamondsuit : [0,1]\times[0,1] \to [0,1]$ is said to be a *continuous t-conorm* if it satisfies the following conditions:
(a') $\diamondsuit$ is associative and commutative, (b') $\diamondsuit$ is continuous, (c') $a\diamondsuit 0 = a$ for all $a \in [0,1]$, (d') $a\diamondsuit b \le c\diamondsuit d$ whenever $a \le c$ and $b \le d$ for each

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$a,b,c,d \in [0,1]$.

Using the notions of continuous t-norm and t-conorm, Saadati and Park [36] introduced the concept of intuitionistic fuzzy normed space as follows:

Definition 1.3. The five-tuple $(X,\mu,\nu,*,\diamondsuit)$ is said to be an *intuitionistic fuzzy normed spaces* (for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\diamondsuit$ is a continuous t-conorm, and μ,ν are fuzzy sets on $X\times(0,\infty)$ satisfying the following conditions for each $x,y\in X$ and $s,t>0$
(i) $\mu(x,t)+\nu(x,t)\le 1$, (ii) $\mu(x,t)>0$, (iii) $\mu(x,t)=1$ if and only if $x=0$, (iv) $\mu(\alpha x,t)=\mu(x,\frac{t}{|\alpha|})$ for each $\alpha\neq 0$, (v) $\mu(x,t)*\mu(y,s)\le\mu(x+y,t+s)$, (vi) $\mu(x,\cdot):(0,\infty)\to[0,1]$ is continuous, (vii) $\lim_{t\to\infty}\mu(x,t)=1$ and $\lim_{t\to 0}\mu(x,t)=0$, $(viii)$ $\nu(x,t)<1$, (ix) $\nu(x,t)=0$ if and only if $x=0$, (x) $\nu(\alpha x,t)=\nu(x,\frac{t}{|\alpha|})$ for each $\alpha\neq 0$, (xi) $\nu(x,t)\diamondsuit\nu(y,s)\ge\nu(x+y,t+s)$, (xii) $\nu(x,\cdot):(0,\infty)\to[0,1]$ is continuous, $(xiii)$ $\lim_{t\to\infty}\nu(x,t)=0$ and $\lim_{t\to 0}\nu(x,t)=1$.

In this case (μ,ν) is called an *intuitionistic fuzzy norm*.

Example 1.4. Let $(X,\|.\|)$ be a normed space, $a*b=ab$ and $a\diamondsuit b=\min\{a+b,1\}$ for all $a,b\in[0,1]$. For all $x\in X$ and every $t>0$, consider

$$\mu(x,t)=\begin{cases}\frac{t}{t+\|x\|} & \text{if } t>0\\ 0 & \text{if } t\le 0;\end{cases} \quad\text{and}\quad \nu(x,t)=\begin{cases}\frac{\|x\|}{t+\|x\|} & \text{if } t>0\\ 0 & \text{if } t\le 0;\end{cases}$$

Then $(X,\mu,\nu,*,\diamondsuit)$ is an IFNS.

The concepts of convergence and Cauchy sequences in intuitionistic fuzzy normed space are studied in [36].

Let $(X,\mu,\nu,*,\diamondsuit)$ be an IFNS. A sequence $x=(x_k)$ is said to be *intuitionistic fuzzy convergent* to $L\in X$ if, for every $\varepsilon>0$, there exists $k_0\in\mathbb{N}$ such that $\mu(x_k-L,t)>1-\varepsilon$ and $\nu(x_k-L,t)<\varepsilon$ for all $k\ge k_0$. In this case we write $(\mu,\nu)-lim x_k=L$ or $x_k\xrightarrow{(\mu,\nu)}L$ as $k\to\infty$

Let $(X,\mu,\nu,*,\diamondsuit)$ be an IFNS. A $x=(x_k)$ is said to be *intuitionistic fuzzy Cauchy sequence* if, for every $\varepsilon>0$ and $t>0$, there exists $k_0\in\mathbb{N}$ such that $\mu(x_k-x_l,t)>1-\varepsilon$ and $\nu(x_k-x_l,t)<\varepsilon$ for all $k,l\ge k_0$

An IFNS $(X,\mu,\nu,*,\diamondsuit)$ is said to be *complete* if every intuitionistic fuzzy Cauchy sequence is intuitionistic fuzzy convergent in $(X,\mu,\nu,*,\diamondsuit)$. In this case (X,μ,ν) is called intuitionistic fuzzy Banach space.

2 Main Results

We start our work with an intuitionistic fuzzy version of the Hyers-Ulam-Rassias stability in which we uniformly approximate a 'uniform' approximate additive mapping.

Theorem 2.1. Let X be a linear space and (Y,μ,ν) be an intuitionistic fuzzy Banach space. Let $\varphi: X\times X\to[0,\infty)$ be a control function such that

$$\tilde{\varphi}(x,y)=\sum_{n=0}^{\infty}2^{-n}\varphi(2^n x,2^n y)<\infty, \tag{2.1.1}$$

for all $x,y\in X$. Let $f:X\to Y$ be a uniformly approximately additive function with respect to φ in the sense that

$$\lim_{t\to\infty}\mu(f(x+y)-f(x)-f(y),t\varphi(x,y))=1$$

and

$$\lim_{t\to\infty}\nu(f(x+y)-f(x)-f(y),t\varphi(x,y))=0 \tag{2.1.2}$$

uniformly in $X\times X$. Then $T(x)=(\mu,\nu)\text{-}\lim_{n\to\infty}\frac{f(2^n x)}{2^n}$ for each $x\in X$ exists and defines an additive mapping $T:X\to Y$ such that if for some $\delta>0,\alpha>0$ and all $x,y\in X$,

$$\mu(f(x+y)-f(x)-f(y),\delta\varphi(x,y))>\alpha$$

and

$$\nu(f(x+y)-f(x)-f(y),\delta\varphi(x,y))<1-\alpha, \tag{2.1.3}$$

then

$$\mu\left(T(x)-f(x),\frac{\delta}{2}\tilde{\varphi}(x,x)\right)>\alpha$$

and

$$\nu\left(T(x)-f(x),\frac{\delta}{2}\tilde{\varphi}(x,x)\right)<1-\alpha.$$

Proof. Given $\varepsilon>0$. By (2.1.2), we can find some $t_\circ>0$ such that

$$\mu(f(x+y)-f(x)-f(y),t\varphi(x,y))\ge 1-\varepsilon$$

and

$$\nu(f(x+y)-f(x)-f(y),t\varphi(x,y))\le\varepsilon \tag{2.1.4}$$

for all $x,y\in X$ and all $t\ge t_\circ$. By induction on n, we shall show that

$$\left.\begin{array}{l}\mu\left(f(2^n x)-2^n f(x),t\sum_{k=0}^{n-1}2^{n-k-1}\varphi(2^k x,2^k x)\right)\ge 1-\varepsilon\\ \text{and}\\ \nu\left(f(2^n x)-2^n f(x),t\sum_{k=0}^{n-1}2^{n-k-1}\varphi(2^k x,2^k x)\right)\le\varepsilon,\end{array}\right\} \tag{2.1.5}$$

for all $x\in X$, $t\ge t_\circ$ and all positive integers n. Putting $y=x$ in (2.1.4), we get (2.1.5) for $n=1$. Let (2.1.5) hold for some positive integer n. Then

$$\mu\left(f(2^{n+1}x)-2^{n+1}f(x),t\sum_{k=0}^{n}2^{n-k}\varphi(2^k x,2^k x)\right)$$

$$\geq \mu\left(f(2^{n+1}x)-2f(2^nx),t_\circ\varphi(2^nx,2^nx)\right)*$$

$$\mu\left(2f(2^nx)-2^{n+1}f(x),t_\circ\sum_{k=0}^{n-1}2^{n-k}\varphi(2^kx,2^kx)\right)$$

$$\geq (1-\varepsilon)*(1-\varepsilon)=1-\varepsilon$$

and

$$\nu\left(f(2^{n+1}x)-2^{n+1}f(x),t\sum_{k=0}^{n}2^{n-k}\varphi(2^kx,2^kx)\right)$$

$$\leq \nu\left(f(2^{n+1}x)-2f(2^nx),t_\circ\varphi(2^nx,2^nx)\right)\diamondsuit$$

$$\nu\left(2f(2^nx)-2^{n+1}f(x),t_\circ\sum_{k=0}^{n-1}2^{n-k}\varphi(2^kx,2^kx)\right)$$

$$\leq \varepsilon\diamondsuit\varepsilon=\varepsilon.$$

This completes the induction argument. Let $t=t_\circ$ and put $n=p$ then by replacing x with 2^nx in (2.1.5), we obtain

$$\mu\left(\frac{f(2^{n+p}x)}{2^{n+p}}-\frac{f(2^nx)}{2^n},\frac{t_\circ}{2^{n+p}}\sum_{k=0}^{p-1}2^{p-k-1}\varphi(2^{n+k}x,2^{n+k}x)\right)$$

$$\geq 1-\varepsilon$$

and

$$\nu\left(\frac{f(2^{n+p}x)}{2^{n+p}}-\frac{f(2^nx)}{2^n},\frac{t_\circ}{2^{n+p}}\sum_{k=0}^{p-1}2^{p-k-1}\varphi(2^{n+k}x,2^{n+k}x)\right)$$

$$\leq \varepsilon, \tag{2.1.6}$$

for all integers $n\geq 0$ and $p>0$. The convergence of (2.1.1) and

$$\sum_{k=0}^{p-1}2^{-n-k-1}\varphi(2^{n+k}x,2^{n+k}x)=\frac{1}{2}\sum_{k=n}^{n+p-1}2^{-k}\varphi(2^kx,2^kx)$$

imply that for given $\delta>0$ there is $n_\circ\in\mathbb{N}$ such that

$$\frac{t_\circ}{2}\sum_{k=n}^{n+p-1}2^{-k}\varphi(2^kx,2^kx)<\delta,$$

for all $n\geq n_\circ$ and all $p>0$. Now we deduce that from (2.1.6) that

$$\mu\left(\frac{f(2^{n+p}x)}{2^{n+p}}-\frac{f(2^nx)}{2^n},\delta\right)\geq$$

$$\mu\left(\frac{f(2^{n+p}x)}{2^{n+p}})-\frac{f(2^nx)}{2^n},\frac{t_\circ}{2^{n+p}}\textstyle\sum_{k=0}^{p-1}2^{p-k-1}\varphi(2^{n+k}x,2^{n+k}x)\right)$$

$$\geq 1-\varepsilon$$

and

$$\nu\left(\frac{f(2^{n+p}x)}{2^{n+p}}-\frac{f(2^nx)}{2^n},\delta\right)$$

$$\leq \nu\left(\frac{f(2^{n+p}x)}{2^{n+p}})-\frac{f(2^nx)}{2^n},\frac{t_\circ}{2^{n+p}}\textstyle\sum_{k=0}^{p-1}2^{p-k-1}\varphi(2^{n+k}x,2^{n+k}x)\right)$$

$$\leq \varepsilon,$$

for all $n\geq n_\circ$ and all $p>0$. Hence $(\frac{f(2^nx)}{2^n})$ is a Cauchy sequence in Y. Since Y is an intuitionistic fuzzy Banach space, $(\frac{f(2^nx)}{2^n})$ converges to some $T(x)\in Y$. Hence, we can define a mapping $T:X\to Y$ such that $T(x)=(\mu,\nu)\text{-}\lim_{n\to\infty}\frac{f(2^nx)}{2^n}$, namely, for each $t>0$, and $x\in X$,

$$\mu\left(T(x)-\frac{f(2^nx)}{2^n},t\right)=1 \text{ and } \nu\left(T(x)-\frac{f(2^nx)}{2^n},t\right)=0.$$

Now, let $x,y\in X$. Choose any fix value of $t>0$, and $\varepsilon\in(0,1)$. Since $\lim_{n\to\infty}2^{-n}\varphi(2^nx,2^ny)=0$, there exists $n_1>n_\circ$ such that $t_\circ\varphi(2^nx,2^ny)<\frac{2^nt}{4}$ for all $n\geq n_1$. Hence for each $n\geq n_1$, we have

$$\mu(T(x+y)-T(x)-T(y),t)$$

$$\geq \mu\left(T(x+y)-\frac{f(2^n(x+y))}{2^n},\frac{t}{4}\right)*$$

$$\mu\left(T(x)-\frac{f(2^n(x))}{2^n},\frac{t}{4}\right)*$$

$$\mu\left(T(y)-\frac{f(2^n(y))}{2^n},\frac{t}{4}\right)*$$

$$\mu\left(f(2^n(x+y))-f(2^nx)-f(2^ny),\frac{2^nt}{4}\right) \tag{2.1.7}$$

and also

$$\mu(f(2^n(x+y))-f(2^nx)-f(2^ny),2^nt/4)$$

$$\geq \mu(f(2^n(x+y))-f(2^nx)-f(2^ny),t_\circ\varphi(2^nx,2^ny)). \tag{2.1.8}$$

Letting $n\to\infty$ in (2.1.7) and using (2.1.4), (2.1.8), we get

$$\mu(T(x+y)-T(x)-T(y),t)\geq 1-\varepsilon$$

for all $t>0$ and $\varepsilon\in(0,1)$. Similarly, we obtain $\nu(T(x+y)-T(x)-T(y),t)\leq\varepsilon$ for all $t>0$ and $\varepsilon\in(0,1)$. It follows that

$$\mu(T(x+y)-T(x)-T(y),t)=1$$

and

$$\nu(T(x+y)-T(x)-T(y),t)=0,$$

for all $t>0$. Therefore $T(x+y)=T(x)+T(y)$.

Lastly, suppose that for some positive δ and α, (2.1.3) holds and

$$\varphi_n(x,y)=\sum_{k=0}^{n-1}2^{-k-1}\varphi(2^kx,2^ky),$$

for all $x,y\in X$. By a similar argument as in the beginning of the proof one can deduce from (2.1.3)

$$\left.\begin{array}{l}\mu\left(f(2^n x)-2^n f(x),\delta\sum_{k=0}^{n-1}2^{n-k-1}\varphi(2^k x,2^k x)\right)\geq\alpha\\ \text{and}\\ \nu\left(f(2^n x)-2^n f(x),\delta\sum_{k=0}^{n-1}2^{n-k-1}\varphi(2^k x,2^k x)\right)\leq 1-\alpha,\end{array}\right\} \tag{2.1.9}$$

for all positive integers n. For $s>0$, we have

$$\mu(f(x)-T(x),\delta\varphi_n(x,x)+s)$$
$$\geq\mu\left(f(x)-\frac{f(2^n x)}{2^n},\delta\varphi_n(x,x)\right)*$$
$$\mu\left(\frac{f(2^n x)}{2^n}-T(x),s\right)$$

and

$$\nu(f(x)-T(x),\delta\varphi_n(x,x)+s)$$
$$\leq\nu\left(f(x)-\frac{f(2^n x)}{2^n},\delta\varphi_n(x,x)\right)\diamondsuit$$
$$\nu\left(\frac{f(2^n x)}{2^n}-T(x),s\right) \tag{2.1.10}$$

Combining (2.1.9), (2.1.10) and using the fact that

$$\lim_{n\to\infty}\mu\left(\frac{f(2^n x)}{2^n}-T(x),s\right)=1$$

and

$$\lim_{n\to\infty}\nu\left(\frac{f(2^n x)}{2^n}-T(x),s\right)=0,$$

we obtain

$$\mu(f(x)-T(x),\delta\varphi_n(x,x)+s)\geq\alpha$$

and

$$\nu(f(x)-T(x),\delta\varphi_n(x,x)+s)\leq 1-\alpha,$$

for sufficiently large n. From the (upper semi) continuity of real functions $\mu(f(x)-T(x),.)$ and $\nu(f(x)-T(x),.)$, we see that

$$\mu\left(f(x)-T(x),\frac{\delta}{2}\tilde{\varphi}(x,x)+s\right)\geq\alpha$$

and

$$\nu\left(f(x)-T(x),\frac{\delta}{2}\tilde{\varphi}(x,x)+s\right)\leq 1-\alpha.$$

Taking the limit $s\to\infty$, we get

$$\mu\left(f(x)-T(x),\frac{\delta}{2}\tilde{\varphi}(x,x)\right)\geq\alpha$$

and

$$\nu\left(f(x)-T(x),\frac{\delta}{2}\tilde{\varphi}(x,x)\right)\leq 1-\alpha. \square$$

Theorem 2.2. Let X be a linear space and (Y,μ,ν) be an intuitionistic fuzzy Banach space. Let $\varphi:X\times X\to[0,\infty)$ be a control function satisfying (2.1.1). Let $f:X\to Y$ be an uniformly approximately additive function with respect to φ. Then there is a unique additive mapping $T:X\to Y$ such that

$$\lim_{n\to\infty}\mu(f(x)-T(x),t\tilde{\varphi}(x,x))=1$$

and

$$\lim_{n\to\infty}\nu(f(x)-T(x),t\tilde{\varphi}(x,x))=0 \tag{2.2.1}$$

uniformly in X.

Proof. The existence of uniform limit (2.2.1) immediately follows from Theorem 2.1. It remains to prove the uniqueness assertion. Let S be another additive mapping satisfying (2.2.1). Choose any fix value of $c>0$. Given $\varepsilon>0$, there is some $t_\circ>0$ such that (2.2.1) for T and S

$$\mu(f(x)-T(x),\frac{t}{2}\tilde{\varphi}(x,x))\geq 1-\varepsilon,\ \mu(f(x)-S(x),\frac{t}{2}\tilde{\varphi}(x,x))$$
$$\geq 1-\varepsilon \text{ and}$$
$$\nu(f(x)-T(x),\frac{t}{2}\tilde{\varphi}(x,x))\leq\varepsilon,\ \nu(f(x)-S(x),\frac{t}{2}\tilde{\varphi}(x,x))\leq\varepsilon$$

for all $x\in X$ and all $t\geq t_\circ$. For some $x\in X$, we can find some integer $n_\circ$ such that

$$t_\circ\sum_{k=n}^{\infty}2^{-k}\varphi(2^k x,2^k x)<\frac{c}{2},$$

for all $n\geq n_\circ$. Since

$$\sum_{k=n}^{\infty}2^{-k}\varphi(2^k x,2^k x)$$
$$=$$
$$\frac{1}{2^n}\sum_{k=n}^{\infty}2^{-(k-n)}\varphi(2^{(k-n)}(2^n x),2^{(k-n)}(2^n x))$$
$$=\frac{1}{2^n}\sum_{m=0}^{\infty}2^{-m}\varphi(2^m(2^n x),2^m(2^n x))=\frac{1}{2^n}\tilde{\varphi}(2^n x,2^n x),$$

we have

$$\mu(S(x)-T(x),c)\geq\mu\left(\frac{f(2^n x)}{2^n}-T(x),c/2\right)*$$
$$\mu\left(S(x)-\frac{f(2^n x)}{2^n},c/2\right)$$
$$=\mu(f(2^n x)-T(2^n x),2^{n-1}c)*\mu(S(2^n x)-f(2^n x),2^{n-1}c)$$
$$\geq\mu\left(f(2^n x)-T(2^n x),2^n t_\circ\sum_{k=n}^{\infty}2^{-k}\varphi(2^k x,2^k x)\right)*$$
$$\mu\left(S(2^n x)-f(2^n x),2^n t_\circ\sum_{k=n}^{\infty}2^{-k}\varphi(2^k x,2^k x)\right)$$

$$= \mu(f(2^n x) - T(2^n x), t_\circ \tilde{\varphi}(2^n x, 2^n x)) *$$
$$\mu(S(2^n x) - f(2^n x), t_\circ \tilde{\varphi}(2^n x, 2^n x)) \geq 1 - \varepsilon$$

and similarly

$$\nu(S(x) - T(x), c) \leq \nu\left(\frac{f(2^n x)}{2^n} - T(x), c/2\right) \diamondsuit$$
$$\nu\left(S(x) - \frac{f(2^n x)}{2^n}, c/2\right)$$
$$\leq \nu\left(f(2^n x) - T(2^n x), 2^n t_\circ \sum_{k=n}^{\infty} 2^{-k} \varphi(2^k x, 2^k x)\right) \diamondsuit$$
$$\nu\left(S(2^n x) - f(2^n x), 2^n t_\circ \sum_{k=n}^{\infty} 2^{-k} \varphi(2^k x, 2^k x)\right)$$
$$= \nu(f(2^n x) - T(2^n x), t_\circ \tilde{\varphi}(2^n x, 2^n x)) \diamondsuit$$
$$\nu(S(2^n x) - f(2^n x), t_\circ \tilde{\varphi}(2^n x, 2^n x)) \leq \varepsilon.$$

It follows that

$$\mu(S(x) - T(x), c) = 1 \text{ and } \nu(S(x) - T(x), c) = 0$$

for all $c > 0$. Hence $T(x) = S(x)$ for all $x \in X$. □

In the next result, we consider the control function $\varphi(x,y) = \theta(\|x\|^q + \|y\|^q)$ for some $\theta > 0$.

Corollary 2.3. Let X be a normed linear space and (Y, μ, ν) be an intuitionistic fuzzy Banach space. Let $f : X \to Y$ be a function such that for all $\theta \geq 0$, $0 \leq q \leq 1$

$$\lim_{t\to\infty} \mu(f(x+y) - f(x) - f(y), t\theta(\|x\|^q + \|y\|^q)) = 1$$

and

$$\lim_{t\to\infty} \nu(f(x+y) - f(x) - f(y), t\theta(\|x\|^q + \|y\|^q)) = 0,$$

uniformly in $X \times X$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\lim_{t\to\infty} \mu\left(T(x) - f(x), \frac{2\theta t\|x\|^q}{1 - 2^{q-1}}\right) = 1$$

and

$$\lim_{t\to\infty} \nu\left(T(x) - f(x), \frac{2\theta t\|x\|^q}{1 - 2^{q-1}}\right) = 0,$$

uniformly in X.

Example 2.4. Let X be a Banach space and α and β be real numbers. Write

$$f(x) = \alpha x + \beta\|x\|^q x_\circ$$

and

$$\varphi(x,y) = \|x\|^q + \|y\|^q,$$

for all $x_\circ, x, y \in X$ and $0 \leq q \leq 1$. Then

$$\tilde{\varphi}(x,y) \sum_{n=0}^{\infty} 2^{-n} \varphi(2^n x, 2^n y) = \frac{2^{1-q}(\|x\|^q + \|y\|^q)}{2^{1-q} - 1},$$

for all $x, y \in X$. For each intuitionistic fuzzy norm (μ, ν), we have

$$\mu(f(x+y) - f(x) - f(y), t\varphi(x,y))$$
$$= \mu(\beta x_\circ(\|x+y\|^q - \|x\|^q - \|y\|^q), (\|x\|^q + \|y\|^q)t)$$

and

$$\nu(f(x+y) - f(x) - f(y), t\varphi(x,y))$$
$$= \nu(\beta x_\circ(\|x+y\|^q - \|x\|^q - \|y\|^q), (\|x\|^q + \|y\|^q)t),$$

for all $x, y \in X$ and $t \in \mathbb{R}$. Thus

$$\mu(f(x+y) - f(x) - f(y), t\varphi(x,y)) \geq \mu(\beta x_\circ, t/2)$$

and

$$\nu(f(x+y) - f(x) - f(y), t\varphi(x,y)) \leq \nu(\beta x_\circ, t/2),$$

for all $x, y \in X$ and $t \in \mathbb{R}$. Hence

$$\lim_{t\to\infty} \mu(f(x+y) - f(x) - f(y), t\varphi(x,y)) = 1$$

and

$$\lim_{t\to\infty} \nu(f(x+y) - f(x) - f(y), t\varphi(x,y)) = 0$$

uniformly in $X \times X$. Therefore the condition of Corollary 2.3 are fulfilled. □

Now, we are giving our second intuitionistic fuzzy Hyers-Ulam-Rassias type theorem (non-uniform version).

Theorem 2.5. Let X be a linear space and (Z, μ', ν') be an intuitionistic fuzzy normed space. Let $\psi : X \times X \to Z$ be a function such that for some $0 < \alpha < 2$,

$$\mu'(\psi(2x, 2y), t) \geq \mu'(\alpha\psi(x,y), t)$$

and

$$\nu'(\psi(2x, 2y), t) \leq \nu'(\alpha\psi(x,y), t), \tag{2.5.1}$$

for all $x, y \in X$ and $t > 0$. Let (Y, μ, ν) be an intuitionistic fuzzy Banach space and let $f : X \to Y$ be a ψ-approximately additive mapping in the sense that

$$\mu(f(x+y) - f(x) - f(y), t) \geq \mu'(\psi(x,y), t)$$

and

$$\nu(f(x+y) - f(x) - f(y), t) \leq \nu'(\psi(x,y), t), \tag{2.5.2}$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\mu(f(x) - T(x), t) \geq \mu'\left(\frac{2\psi(x,x)}{2 - \alpha}, t\right)$$

and

$$\nu(f(x)-T(x),t) \le \nu'\left(\frac{2\psi(x,x)}{2-\alpha},t\right).$$

for all $x \in X$ and $t > 0$.

Proof. Put $y = x$ in (2.5.2), we get

$$\mu(f(2x)-2f(x),t) \ge \mu'(\psi(x,x),t)$$

and

$$\nu(f(2x)-2f(x),t) \le \nu'(\psi(x,x),t), \tag{2.5.3}$$

for all $x \in X$ and $t > 0$. Using (2.5.1) and induction on n, we obtain

$$\mu'(\psi(2^n x,2^n x),t) \ge \mu'(\alpha^n \psi(x,x),t)$$

and

$$\nu'(\psi(2^n x,2^n x),t) \le \nu'(\alpha^n \psi(x,x),t), \tag{2.5.4}$$

for all $x \in X$ and $t > 0$. Replacing x by $2^{n-1}x$ in (2.5.3) and using (2.5.4), we get

$$\left.\begin{array}{l} \mu(f(2^n x)-2f(2^{n-1}x),t) \ge \mu'(\alpha^{n-1}\psi(x,x),t) \ \text{ and} \\ \nu(f(2^n x)-2f(2^{n-1}x),t) \le \nu'(\alpha^{n-1}\psi(x,x),t). \end{array}\right\} \tag{2.5.5}$$

It follows that

$$\mu\left(\frac{f(2^n x)}{2^n}-\frac{f(2^{n-1}x)}{2^{n-1}},\frac{t}{2^n}\right) \ge \mu'\left(\frac{1}{\alpha}\psi(x,x),\frac{t}{\alpha^n}\right)$$

and

$$\nu\left(\frac{f(2^n x)}{2^n}-\frac{f(2^{n-1}x)}{2^{n-1}},\frac{t}{2^n}\right) \le \nu'\left(\frac{1}{\alpha}\psi(x,x),\frac{t}{\alpha^n}\right).$$

Thus

$$\mu\left(\frac{f(2^n x)}{2^n}-\frac{f(2^{n-1}x)}{2^{n-1}},\left(\frac{\alpha}{2}\right)^n t\right) \ge \mu'\left(\frac{1}{\alpha}\psi(x,x),t\right)$$

and

$$\nu\left(\frac{f(2^n x)}{2^n}-\frac{f(2^{n-1}x)}{2^{n-1}},\left(\frac{\alpha}{2}\right)^n t\right) \le \nu'\left(\frac{1}{\alpha}\psi(x,x),t\right),$$

for all $x \in X$, $t > 0$ and $n \ge 1$. Therefore

$$\mu\left(\frac{f(2^n x)}{2^n}-\frac{f(2^m x)}{2^m},\sum_{k=m+1}^{n}\left(\frac{\alpha}{2}\right)^k t\right)$$
$$= \mu\left(\sum_{k=m+1}^{n}\left(\frac{f(2^k x)}{2^k}-\frac{f(2^{k-1}x)}{2^{k-1}}\right),\sum_{k=m+1}^{n}\left(\frac{\alpha}{2}\right)^k t\right)$$
$$\ge \mu'\left(\frac{1}{\alpha}\psi(x,x),t\right)$$

and

$$\nu\left(\frac{f(2^n x)}{2^n}-\frac{f(2^m x)}{2^m},\sum_{k=m+1}^{n}\left(\frac{\alpha}{2}\right)^k t\right)$$
$$= \nu\left(\sum_{k=m+1}^{n}\left(\frac{f(2^k x)}{2^k}-\frac{f(2^{k-1}x)}{2^{k-1}}\right),\sum_{k=m+1}^{n}\left(\frac{\alpha}{2}\right)^k t\right)$$
$$\le \nu'\left(\frac{1}{\alpha}\psi(x,x),t\right),$$

for all $x \in X$, $t > 0$ and $n > m \ge 0$. Hence

$$\left.\begin{array}{l} \mu\left(\frac{f(2^n x)}{2^n}-\frac{f(2^m x)}{2^m},t\right) \ge \mu'\left(\frac{1}{\alpha}\psi(x,x),\frac{t}{\sum_{k=m+1}^{n}(\frac{\alpha}{2})^k}\right) \\ \text{and} \\ \nu\left(\frac{f(2^n x)}{2^n}-\frac{f(2^m x)}{2^m},t\right) \le \nu'\left(\frac{1}{\alpha}\psi(x,x),\frac{t}{\sum_{k=m+1}^{n}(\frac{\alpha}{2})^k}\right), \end{array}\right\} \tag{2.5.6}$$

for all $x \in X$, $t > 0$ and $n > m \ge 0$. Since $\lim_{s\to\infty}\mu'(\frac{1}{\alpha}\psi(x,x),s) = 1$ and $\lim_{s\to\infty}\nu'(\frac{1}{\alpha}\psi(x,x),s) = 0$; also $\sum_{n=0}^{\infty}(\frac{\alpha}{2})^n < \infty$. This shows that $(\frac{f(2^n x)}{2^n})$ is a Cauchy sequence in the intuitionistic fuzzy Banach space (Y,μ,ν), therefore it is convergent to some $T(x) \in Y$. So we can define a mapping $T : X \to Y$ by $T(x) = (\mu,\nu)\text{-}\lim_{n\to\infty}\frac{f(2^n x)}{2^n}$. For $x,y \in X$ and $t > 0$, it follows from (2.5.2) that

$$\mu(f(2^n(x+y))-f(2^n x)-f(2^n y),t) \ge \mu'(\psi(2^n x,2^n y),t)$$
$$\ge \mu'(\alpha^n\psi(x,y),t) \ge \mu'\left(\psi(x,y),\left(\frac{1}{\alpha}\right)^n t\right)$$

and similarly

$$\nu(f(2^n(x+y))-f(2^n x)-f(2^n y),t) \le \nu'(\psi(2^n x,2^n y),t)$$
$$\le \nu'\left(\psi(x,y),\left(\frac{1}{\alpha}\right)^n t\right).$$

Thus

$$\mu\left(\frac{f(2^n(x+y))}{2^n}-\frac{f(2^n x)}{2^n}-\frac{f(2^n y)}{2^n},t\right)$$
$$\ge \mu'\left(\psi(x,y),\left(\frac{2}{\alpha}\right)^n t\right)$$

and

$$\nu\left(\frac{f(2^n(x+y))}{2^n}-\frac{f(2^n x)}{2^n}-\frac{f(2^n y)}{2^n},t\right)$$
$$\le \nu'\left(\psi(x,y),\left(\frac{2}{\alpha}\right)^n t\right) \tag{2.5.7}$$

for all n. Furthermore,

$$\mu(T(x+y)-T(x)-T(y),t)$$

$$\geq \mu\left(T(x+y)-\frac{f(2^n(x+y))}{2^n},\frac{t}{4}\right)*$$
$$\mu\left(T(x)-\frac{f(2^nx)}{2^n},\frac{t}{4}\right)*\mu\left(T(y)-\frac{f(2^ny)}{2^n},\frac{t}{4}\right)*$$
$$\mu\left(\frac{f(2^n(x+y))}{2^n}-\frac{f(2^nx)}{2^n}-\frac{f(2^ny)}{2^n},\frac{t}{4}\right)$$

and

$$\nu(T(x+y)-T(x)-T(y),t)$$
$$\leq \nu\left(T(x+y)-\frac{f(2^n(x+y))}{2^n},\frac{t}{4}\right)\diamond$$
$$\nu\left(T(x)-\frac{f(2^nx)}{2^n},\frac{t}{4}\right)\diamond\nu\left(T(y)-\frac{f(2^ny)}{2^n},\frac{t}{4}\right)$$
$$\diamond\nu\left(\frac{f(2^n(x+y))}{2^n}-\frac{f(2^nx)}{2^n}-\frac{f(2^ny)}{2^n},\frac{t}{4}\right) \qquad (2.5.8)$$

Letting $n\to\infty$ in (2.5.7) and (2.5.8), we get

$$\mu(T(x+y)-T(x)-T(y),t)=1$$

and

$$\nu(T(x+y)-T(x)-T(y),t)=0,$$

for all $x,y\in X$ and $t>0$. Thus $T(x+y)=T(x)+T(y)$. This means that T satisfies the Cauchy equation and so it is additive. Using (2.5.6) with $m=0$, and for all $x\in X$ and $t>0$, we get

$$\mu(T(x)-f(x),t)\geq \mu\left(T(x)-\frac{f(2^nx)}{2^n},\frac{t}{2}\right)*$$
$$\mu\left(\frac{f(2^nx)}{2^n}-f(x),\frac{t}{2}\right)$$
$$\geq \mu\left(T(x)-\frac{f(2^nx)}{2^n},\frac{t}{2}\right)*\mu'\left(\frac{1}{\alpha}\psi(x,x),\frac{t}{2\sum_{k=1}^{n}(\frac{\alpha}{2})^k}\right)$$

and

$$\nu(T(x)-f(x),t)\leq \nu\left(T(x)-\frac{f(2^nx)}{2^n},\frac{t}{2}\right)\diamond$$
$$\nu\left(\frac{f(2^nx)}{2^n}-f(x),\frac{t}{2}\right)\leq \nu\left(T(x)-\frac{f(2^nx)}{2^n},\frac{t}{2}\right)$$
$$\diamond\nu'\left(\frac{1}{\alpha}\psi(x,x),\frac{t}{2\sum_{k=1}^{n}(\frac{\alpha}{2})^k}\right) \qquad (2.5.9)$$

Letting $n\to\infty$ in (2.5.9), we get

$$\mu(T(x)-f(x),t)\geq \mu'\left(\frac{2\psi(x,x)}{\alpha},\frac{t}{2\sum_{k=1}^{\infty}(\frac{\alpha}{2})^k}\right)$$
$$=\mu'\left(\frac{2\psi(x,x)}{2-\alpha},t\right)$$

and

$$\nu(T(x)-f(x),t)\leq \nu'\left(\frac{2\psi(x,x)}{\alpha},\frac{t}{2\sum_{k=1}^{\infty}(\frac{\alpha}{2})^k}\right)$$
$$=\nu'\left(\frac{2\psi(x,x)}{2-\alpha},t\right),$$

for all $x\in X$ and $t>0$. The uniqueness of T can be proved on the same lines as in Theorem 2.2.

Example 2.6. Let X be a normed space. Suppose that (μ,ν) and (μ',ν') be intuitionistic fuzzy norms on X and $\mathbb{R}$ respectively, defined in Example 1.4 and $\varphi:(0,\infty)\to(0,\infty)$ be a function such that $\varphi(2r)<\alpha\varphi(r)$ for all $r>0$ and $\alpha\in(0,2)$. Define

$$\psi(x,y)=\varphi(\|x\|)+\varphi(\|y\|)+\varphi(\|x+y\|)$$

for each $x,y\in X$. Let $x_\circ\in X$ be a unit vector. Define $f:X\to X$ by $f(x)=x+\varphi(\|x\|)x_\circ$. Then for each $x,y\in X$ and $t>0$, we have

$$\mu(f(x+y)-f(x)-f(y),t)$$
$$=\frac{t}{t+|\varphi(\|x\|)+\varphi(\|y\|)+\varphi(\|x+y\|)|.\|x_\circ\|}$$
$$\geq\frac{t}{t+|\varphi(\|x\|)+\varphi(\|y\|)+\varphi(\|x+y\|)|}=\mu'(\psi(x,y),t)$$

and similarly,

$$\nu(f(x+y)-f(x)-f(y),t)$$
$$=\frac{t}{t+|\varphi(\|x\|)+\varphi(\|y\|)+\varphi(\|x+y\|)|.\|x_\circ\|}\leq\nu'(\psi(x,y),t).$$

Furthermore,
$\mu'(\psi(2x,2y),t)=\frac{t}{t+\psi(2x,2y)}\geq\frac{t}{t+\alpha\psi(x,y)}=\mu'(\psi(x,y),t)$
and
$\nu'(\psi(2x,2y),t)=\frac{t}{t+\psi(2x,2y)}\leq\frac{t}{t+\alpha\psi(x,y)}=\nu'(\psi(x,y),t)$,
for all $x,y\in X$ and $t>0$. Therefore, by Theorem 2.5, there exists a unique additive mapping $T:X\to Y$ such that for each $x\in X$ and $t>0$

$$\mu(f(x)-T(x),t)\geq\mu'\left(\frac{2\psi(x,x)}{2-\alpha},t\right)\text{ and }\nu(f(x)-T(x),t)$$
$$\leq\nu'\left(\frac{2\psi(x,x)}{2-\alpha},t\right).$$

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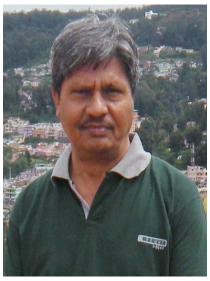

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