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# On Generalized Left Derivations in $BCI$-Algebras

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**Abstract:** In the present paper, we introduce the notion of *generalized left derivation* of a $BCI$-algebra $X$, construct several examples, and investigate related properties. Also establish some results on regular *generalized left derivation*. Furthermore, for a generalized left derivation $H$, the concept of a $H$-invariant *generalized left derivation* is introduced, and examples are discussed. Using this concept a condition for a *generalized left derivation* to be regular is provided. Finally, some results on p-semisimple $BCI$-algebra are obtained and it is shown that let $H$ be a self map in a p-semisimple $BCI$-algebra $X$. Then $H$ is a *generalized left derivation* if and only if it is a derivation on $X$.

**Keywords:** Derivations, $BCI$-Algebras

## 1 Introduction

The notion of BCK-algebras and BCI-algebras were introduced by Y. Imai and K. Iseki in 1966 [9,10]. BCK-algebras and BCI-algebras are algebraic formulation of BCK-system and BCI-system in combinatory logic. Later, the notion of BCI algebras have been extensively investigated by many researchers (see [2,3,14] and references there in). BCI-algebra is a generalization of a BCK-algebra that is every BCK-algebra is a BCI-algebra but not vice versa(see [6]). Therefore, most of the algebras related to the t-norm based logic such as MTL [5], BL, hoop, MV [4] (i.e lattice implication algebra) and Boolean algebras etc., are extensions of BCK-algebras which have a lot of applications in computer science (see [19]). Cosequetly, BCK/BCI-algebras are considerably general structures.

Throughout the present paper $X$ will denote a $BCI$-algebra. Jun and Xin [11] applied the notion of derivation in ring and near-ring theory to $BCI$-algebras in the year 2004 and introduced a new concept called a (regular) derivation in $BCI$-algebras, and investigated some of its properties. Using the notion of a regular derivation, they also established characterizations of a $p$-semisimple $BCI$-algebra. For a self map $d$ of a $BCI$-algebra, they defined a $d$-invariant ideal, and gave conditions for an ideal to be $d$-invariant. During the last 10 years, a greater interest has been devoted to the study on derivations in $BCI$-algebras and a number of research articles have been published in this direction on various aspects (see [1,8,15,16,17,18,20]).

Motivated by notions of *left derivations* [1] and *generalized derivations* [18] in the theory of $BCI$-algebras, in this paper, we introduced the notion of *generalized left derivations* on $BCI$-algebras and investigate related properties. The concept of *generalized left derivations* covers the concept of *left derivations* on $BCI$-algebras. Further, we obtain some results on regular *generalized left derivations*. Also, for a generalized left derivation $H$, we introduce the concept of a $H$-invariant *generalized left derivations* and give some examples. Using this concept we provide a condition for a *generalized left derivation* to be regular. Finally, we characterize the notion of p-semisimple BCI-algebra $X$ by using the concept of *generalized left derivation* and show that let $H$ be a self map in a p-semisimple $BCI$-algebra $X$. Then $H$ is a *generalized left derivation* if and only if it is a derivation on $X$.

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## 2 Preliminaries

in this section, we collect the following definitions and properties from the existing literature that will be needed in the sequel.

A nonempty set $X$ with a constant $0$ and a binary operation $*$ is called a *BCI-algebra* if for all $x,y,z \in X$ the following conditions hold:

(I)$((x*y)*(x*z))*(z*y)=0$,
(II)$(x*(x*y))*y=0$,
(III)$x*x=0$,
(IV)$x*y=0$ and $y*x=0$ imply $x=y$.

Define a binary relation $\leq$ on $X$ by letting $x*y=0$ if and only if $x\leq y$. Then $(X,\leq)$ is a partially ordered set. A BCI-algebra $X$ satisfying $0\leq x$ for all $x\in X$, is called BCK-algebra.

A *BCI*-algebra $X$ has the following properties: for all $x,y,z\in X$

(a1)$x*0=x$.
(a2)$(x*y)*z=(x*z)*y$.
(a3)$x\leq y$ implies $x*z\leq y*z$ and $z*y\leq z*x$.
(a4)$(x*z)*(y*z)\leq x*y$.
(a5)$x*(x*(x*y))=x*y$.
(a6)$0*(x*y)=(0*x)*(0*y)$.
(a7)$x*0=0$ implies $x=0$.

For a *BCI*-algebra $X$, denote by $X_+$ (resp. $G(X)$) the *BCK*-part (resp. the *BCI*-G part) of $X$, i.e., $X_+$ is the set of all $x\in X$ such that $0\leq x$ (resp. $G(X):=\{x\in X\mid 0*x=x\}$). Note that $G(X)\cap X_+=\{0\}$ (see [13]). If $X_+=\{0\}$, then $X$ is called a *p-semisimple BCI-algebra*. In a *p*-semisimple *BCI*-algebra $X$, the following hold:

(a8)$(x*z)*(y*z)=x*y$.
(a9)$0*(0*x)=x$ for all $x\in X$.
(a10)$x*(0*y)=y*(0*x)$.
(a11)$x*y=0$ implies $x=y$.
(a12)$x*a=x*b$ implies $a=b$.
(a13)$a*x=b*x$ implies $a=b$.
(a14)$a*(a*x)=x$.
(a15)$(x*y)*(w*z)=(x*w)*(y*z)$.

Let $X$ be a *p*-semisimple *BCI*-algebra. We define addition "+" as $x+y=x*(0*y)$ for all $x,y\in X$. Then $(X,+)$ is an abelian group with identity $0$ and $x-y=x*y$. Conversely let $(X,+)$ be an abelian group with identity $0$ and let $x*y=x-y$. Then $X$ is a *p*-semisimple *BCI*-algebra and $x+y=x*(0*y)$ for all $x,y\in X$ (see [14]).

For a *BCI*-algebra $X$ we denote $x\wedge y=y*(y*x)$, in particular $0*(0*x)=a_x$, and $L_p(X):=\{a\in X\mid x*a=0 \Rightarrow x=a, \forall x\in X\}$. We call the elements of $L_p(X)$ the *p-atoms* of $X$. For any $a\in X$, let $V(a):=\{x\in X\mid a*x=0\}$, which is called the *branch* of $X$ with respect to $a$. It follows that $x*y\in V(a*b)$ whenever $x\in V(a)$ and $y\in V(b)$ for all $x,y\in X$ and all $a,b\in L_p(X)$. Note that $L_p(X)=\{x\in X\mid a_x=x\}$, which is the *p*-semisimple part of $X$, and $X$ is a *p*-semisimple *BCI*-algebra if and only if $L_p(X)=X$ (see [12],[Proposition 3.2]). Note also that $a_x\in L_p(X)$, i.e., $0*(0*a_x)=a_x$, which implies that $a_x*y\in L_p(X)$ for all $y\in X$. It is clear that $G(X)\subset L_p(X)$, and $x*(x*a)=a$ and $a*x\in L_p(X)$ for all $a\in L_p(X)$ and all $x\in X$. For more details, refer to [2,3,11,12,13,14].

## 3 Generalized Left Derivations

We introduce the notion of *generalized left derivation* of a BCI-algebra $X$ as follows:

**Definition 1.***Let $X$ be a BCI-algebra. Then a self map $H: X\rightarrow X$ is called a generalized left derivation of $X$ if there exists a left derivation $D: X\rightarrow X$ such that*

$$D(x*y)=x*H(y)\wedge y*D(x) \quad for\ all\ x,y\in X.$$

Note that if $H=D$, then the *generalized left derivation* of a *BCI*-algebra $X$ is a left derivation of a *BCI*-algebra $X$.

*Example 1.*Let be $X=\{0,1,2\}$ a *BCI*-algebra with the following Cayley table:

| $*$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 2 |
| 1 | 1 | 0 | 2 |
| 2 | 2 | 2 | 0 |

(1) We define a map

$$D: X\rightarrow X,\ x\mapsto \begin{cases} 2 \text{ if } x\in\{0,1\}, \\ 0 \text{ if } x=2, \end{cases}$$

It can be easily verified that $D$ is a left derivation of $X$. Again, define a map

$$H: X\rightarrow X,\ x\mapsto \begin{cases} 0 \text{ if } x\in\{0,1\}, \\ 2 \text{ if } x=2, \end{cases}$$

It is easy to check that $H$ is a *generalized left derivation* of $X$.

(2) Define a map

$$D: X\rightarrow X,\ x\mapsto \begin{cases} 0 \text{ if } x\in\{0,2\}, \\ 1 \text{ if } x=1, \end{cases}$$

It is easy to check that $D$ is a left derivation of $X$.

(2.1) Define a map

$$H: X\rightarrow X,\ x\mapsto \begin{cases} 2 \text{ if } x\in\{0,1\}, \\ 0 \text{ if } x=2, \end{cases}$$

It is esay to see that $H$ is a *generalized left derivation* of $X$.

(2.2) If we define a map $H : X \to X$ by $H(x) = 2$ for all $x \in X$, then we can easily verify that $H$ is *generalized left derivation*of $X$.

(2.3) If we define a map $H : X \to X$ by $H(x) = 0$ for all $x \in X$, then we can easily verify that $H$ is *generalized left derivation*of $X$.

**Theorem 1.***Let $H$ be a generalized left derivation of a BCI-algebra $X$. Then*

(1)$x \in L_p(X) \Rightarrow H(x) \in L_p(X)$ *for all* $x \in X$.
(2)$H(x) = 0 + H(x)$ *for all* $x \in X$.
(3)$H(x+y) = x + H(y)$ *for all* $x \in L_p(X)$.
(4)$x \in G(X) \Rightarrow H(x) \in G(X)$ *for all* $x \in X$.

*Proof.*(1) For any $x \in L_p(X)$, we have

$$\begin{aligned} H(x) &= H(0*(0*x)) \\ &= (0*H(0*x)) \wedge ((0*x)*D(0)) \\ &= ((0*x)*D(0))*(((0*x)*D(0))*(0*H(0*x))) \\ &= 0*H(0*x) \in L_p(X). \end{aligned}$$

(2) By (1), we have $H(x) \in L_p(X)$. Then

$$H(x) = 0*(0*H(x)) = 0 + H(x).$$

(3) For any $x, y \in L_p(X)$, we have

$$\begin{aligned} H(x+y) &= H(x*(0*y)) \\ &= (x*H(0*y)) \wedge ((0*y)*D(x)) \\ &= ((0*y)*D(x))*(((0*y)*D(x))*(x*H(0*y))) \\ &= x*H(0*y) \\ &= x*((0*H(y)) \wedge (y*D(0))) \\ &= x*(0*H(y)) \\ &= x + H(y). \end{aligned}$$

(4)Let $x \in G(X)$. Then $0*x = x$, and so

$$\begin{aligned} H(x) &= H(0*x) \\ &= (0*H(x)) \wedge (x*D(0)) \\ &= (x*D(0))*((x*D(0))*(0*H(x))) \\ &= 0*H(x) \end{aligned}$$

since $0*H(x) \in L_p(X)$. Hence $H(x) \in G(X)$. This completes the proof.

If we take $H = D$ in Theorem 1, then we have the following corollary.

**Corollary 1([1]).** *Let $D$ be a left derivation of a BCI-algebra $X$. Then*

(1)$x \in L_p(X) \Rightarrow D(x) \in L_p(X)$ *for all* $x \in X$.
(2)$D(x) = 0 + D(x)$ *for all* $x \in X$.
(3)$D(x+y) = x + D(y)$ *for all* $x, y \in L_p(X)$.
(4)$x \in G(X) \Rightarrow D(x) \in G(X)$ *for all* $x \in X$.

**Theorem 2.***Let $H$ be a generalized left derivation of a BCI-algebra $X$. Then*

(1)$x \in L_p(X) \Rightarrow H(x) = x*H(0) = x + H(0)$ *for all* $x \in X$.
(2)$H(x+y) = H(x) + H(y) - H(0)$ *for all* $x, y \in L_p(X)$.
(3)*$H$ is identity on $L_p(X)$ if and only if $H(0) = 0$.*
(4)$H(x*y) \le x*H(y)$ *for all* $x, y \in X$.

*Proof.*(1) For any $x \in L_p(X)$, we have

$$\begin{aligned} H(x) = H(x*0) &= (x*H(0)) \wedge (0*D(x)) \\ &= (0*D(x))*((0*D(x))*(x*H(0))) \\ &= (0*D(x))*((0*(x*H(0)))*D(x)) \\ &= 0*(0*(x*H(0))) \\ &= x*H(0) = x*(0*H(0)) \\ &= x + H(0) \end{aligned}$$

since $x*H(0) \in L_p(X)$ and $H(0) \in G(X)$.

(2) If $x, y \in L_p(X)$, then $x + y \in L_p(X)$. Using (1), we have

$$\begin{aligned} H(x+y) &= (x+y) + H(0) \\ &= x + H(0) + y + H(0) - H(0) \\ &= H(x) + H(y) - H(0). \end{aligned}$$

(3) It follows from (1).

(4) For any $x, y \in X$, we have

$$\begin{aligned} H(x*y) &= (x*H(y)) \wedge (y*D(x)) \\ &= (y*D(x))*((y*D(x))*(x*H(y))) \\ &\le x*H(y). \end{aligned}$$

This completes the proof.

**Definition 2.***A generalized left derivation $H$ of a BCI-algebra $X$ is said to be regular if $H(0) = 0$.*

*Example 2.*(1) The *generalized left derivation* $H$ of $X$ in Examples 1 (1) and 1 (2.3) are regular.
(2) The *generalized left derivation* $H$ of $X$ in Examples 1 (2.1) and 1 (2.2) are not regular.

**Theorem 3.***If $X$ is a BCK-algebra, then every generalized left derivation of $X$ is regular.*

*Proof.*Let $H$ be a *generalized left derivation* of a *BCK*-algebra $X$. Then

$$\begin{aligned} H(0) &= H(0*x) \\ &= (0*H(x)) \wedge (x*D(0)) \\ &= 0 \wedge (x*D(0)) = 0. \end{aligned}$$

Hence $H$ is regular.

In a $BCI-$algebra, Theorem 3 is not true as seen in the following example:

*Example 3.*In Example 1 (2.1), $H$ is a *generalized left derivation* of a *BCI*-algebra $X$ which is not regular.

**Theorem 4.***Let $H$ be a regular generalized left derivation of a BCI-algebra $X$. Then*

(1)*Both $x$ and $H(x)$ belong to the same branch for all $x \in X$.*
(2)$H(x) \le x$ *for all* $x \in X$.
(3)$H(x) * y \le x * H(y)$ *for all* $x, y \in X$.

*Proof.*(1) Let $x \in X$. Then we have

$$\begin{aligned} 0 = H(0) &= H(a_x * x) \\ &= (a_x * H(x)) \wedge (x * D(a_x)) \\ &= (x * D(a_x)) * ((x * D(a_x)) * (a_x * H(x))) \\ &= a_x * H(x) \end{aligned}$$

since $a_x * H(x) \in L_p(X)$. Hence $a_x \le H(x)$, and so $H(x) \in V(a_x)$. Obviously, $x \in V(a_x)$.

(2) Since $H$ is regular, $H(0) = 0$. Then

$$\begin{aligned} H(x) &= H(x * 0) \\ &= (x * H(0)) \wedge (0 * D(x)) \\ &= (x * 0) \wedge (0 * D(x)) \\ &= (0 * D(x)) * ((0 * D(x)) * x) \\ &\le x. \end{aligned}$$

(3) Since $H(x) \le x$ for all $x \in X$ by (2). Using (a3), we have

$$H(x) * y \le x * y \le x * H(y).$$

This completes the proof.

**Theorem 5.***For any generalized left derivation $H$ of a BCI-algebra $X$, the set*

$$H^{-1}(0) := \{x \in X \mid H(x) = 0\}$$

*is a subalgebra of $X$ if $x = 0$ for all $x \in X$. Moreover, $H^{-1}(0) \subseteq X_+$.*

*Proof.*Assume that $x = 0$ for all $x \in X$. Let $x, y \in H^{-1}(0)$. Then $H(x) = 0 = H(y)$, and so

$$H(x * y) \le x * H(y) = 0 * 0 = 0$$

by Theorem 2(4). Hence $H(x * y) = 0$ by (a7), that is, $x * y \in H^{-1}(0)$. Hence $H^{-1}(0)$ is a subalgebra of $X$. Further, let $x \in H^{-1}(0)$. Then $0 = H(x) \le x$ by Theorem 4(2), which implies that $x \in X_+$, showing that $H^{-1}(0) \subseteq X_+$. This completes the proof.

**Definition 3.***For a generalized left derivation $H$ of a BCI-algebra $X$, we say that an ideal $I$ of $X$ is $H$-invariant if $H(I) \subseteq I$.*

*Example 4.*(1) Let $H$ be a *generalized left derivation* of $X$ which is described in Example 1 (2.1). We know that $I := \{0, 1\}$ is an ideal of $X$ which is not $H$-invariant.

(2) Let $H$ be a *generalized left derivation* of $X$ which is described in Example 1 (1). We know that $I := \{0, 1\}$ is a $H$-invariant ideal of $X$.

**Theorem 6.***Let $H$ be a generalized left derivation of a BCI-algebra $X$. Then $H$ is regular if and only if every ideal of $X$ is $H$-invariant.*

*Proof.*Let $I$ be an ideal of $X$. Suppose $H$ is regular, then it follows from Theorem 4 (2) that $H(x) \le x$ for all $x \in X$ implying thereby $H(x) * x = 0$. Let $y \in X$ be such that $y \in H(I)$. Then $y = H(x)$ for some $x \in I$. Thus

$$y * x = H(x) * x = 0 \in I.$$

Since $I$ is an ideal of $X$, it follows that $y \in A$ so that $H(I) \subseteq I$. Therefore $I$ is $H$-invariant.

Conversely, suppose that every ideal of $X$ is $H$-invariant. Since the zero ideal $\{0\}$ is clearly $H$-invariant, we have $H(\{0\}) \subseteq \{0\}$, and so $H(0) = 0$. Hence $H$ is regular.

If we take $H = D$ in Theorem 6, then we have the following corollary.

**Corollary 2([1]).** *Let $D$ be a left derivation of a BCI-algebra $X$. Then $D$ is regular if and only if every ideal of $X$ is $D$-invariant.*

Next, we prove some results in a p-semisimple *BCI*-algebra.

**Theorem 7.***Let $H$ be a generalized left derivation of a p-semisimple BCI-algebra $X$, we have the following assertions:*

(1)$x * H(x) = y * H(y)$ *for all* $x, y \in X$.
(2)$H(x * y) = x * H(y)$ *for all* $x, y \in X$.
(3)$H(x) * x = H(y) * y$ *for all* $x, y \in X$.
(4)$H(x) * x = y * H(y)$ *for all* $x, y \in X$.

*Proof.*(1) Let $X$ be a p-semisimple *BCI*-algebra. Then for any $x, y \in X$, we have

$$H(0) = H(x * x) = (x * H(x)) \wedge (x * D(x)) = x * H(x).$$

Also,

$$H(0) = H(y * y) = (y * H(y)) \wedge (y * D(y)) = y * H(y).$$

Henceforth, we get $x * H(x) = y * H(y)$.

(2) Let $X$ be a p-semisimple *BCI*-algebra. Then for any $x, y \in X$, we have

$$H(x * y) = (x * H(y)) \wedge (y * D(x)) = x * H(y).$$

(3) Using (I), we have

$$(x*y)*(x*H(y)) \leq H(y)*y$$

and

$$(y*x)*(y*H(x)) \leq H(x)*x$$

these above inequalities can be rewritten as

$$((x*y)*(x*H(y)))*(H(y)*y) = 0$$

and

$$((y*x)*(y*H(x)))*(H(x)*x) = 0$$

Consequently, we get

$$((x*y)*(x*H(y)))*(H(y)*y) = ((y*x)*(y*H(x)))*(H(x)*x) \quad (3.1)$$

Also, using (1) and (2), we obtain

$$(x*y)*H(x*y) = (y*x)*H(y*x)$$

$$\implies (x*y)*(x*H(y)) = (y*x)*(y*H(x)) \quad (3.2)$$

Since, $X$ is a p-semisimple *BCI*-algebra. Hence, by using equation (3.2) and (a12), the above equation (3.1) yields $H(x)*x = H(y)*y$.

(4) We know that $H(0) = x*H(x)$. Using (3), we get $H(0)*0 = H(y)*y$ implies $H(0) = H(y)*y$. Therefore $H(y)*y = x*H(x)$ implying thereby $H(x)*x = y*H(y)$. This completes the proof.

If we take $H = D$ in Theorem 7, then we have the following corollary.

**Corollary 3([1]).** *Let $D$ be a left derivation of a p-semisimple BCI-algebra $X$, we have the following assertions:*

(1)$D(x*y) = x*D(y)$ *for all* $x,y \in X$.
(2)$D(x)*x = D(y)*y$ *for all* $x,y \in X$.
(3)$D(x)*x = y*D(y)$ *for all* $x,y \in X$.

**Theorem 8.***Let $H$ be a self map in a p-semisimple BCI-algebra $X$. Then $H$ is a generalized left derivation if and only if it is a derivation on $X$.*

*Proof.*Suppose that $H$ is a *generalized left derivation* on $X$. First , we show that $H$ is a *(r,l)-derivation* on $X$. Let $x,y \in X$. Using (a14) , we have

$$\begin{aligned} H(x*y) &= x*H(y) \\ &= (H(x)*y)*((H(x)*y)*(x*H(y))) \\ &= (x*H(y)) \wedge (H(x)*y). \end{aligned}$$

Hence $H$ is a *(r,l)-derivation* on $X$.

Again, we show that $H$ is a *(l,r)-derivation* on $X$. Let $x,y \in X$. Using Theorem 7(4) and (a15), we have

$$\begin{aligned} H(x*y) &= x*H(y) \\ &= (x*0)*H(y) \\ &= (x*(H(0)*H(0)))*H(y) \\ &= (x*((x*H(x))*(H(y)*y)))*H(y) \\ &= (x*H(y))*((x*H(x))*(H(y)*y)) \\ &= (x*H(y))*((x*H(y))*(H(x)*y)) \\ &= (H(x)*y) \wedge (x*H(y)). \end{aligned}$$

Conversely, suppose that $H$ is a derivation of $X$. As $H$ is a *(r,l)-derivation* on $X$. Then for any $x,y \in X$, we have

$$\begin{aligned} H(x*y) &= (x*H(y)) \wedge (H(x)*y) \\ &= (H(x)*y)*((H(x)*y)*(x*H(y))) \\ &= x*H(y) \\ &= (y*D(x))*((y*D(x))*(x*H(y))) \\ &= (x*H(y)) \wedge (y*D(x)). \end{aligned}$$

Hence $H$ is a *generalized left derivation*. This completes the proof.

If we take $H = D$ in Theorem 8, then we have the following corollary.

**Corollary 4([1]).** *Let $D$ be a self map in a p-semisimple BCI-algebra $X$. Then $D$ is a left derivation if and only if it is a derivation on $X$.*

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