

## Extended Kappa Distribution in the Presence of Censored Data

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**Abstract:** In this study, we introduce a new extension of three-parameter Kappa distribution using the Topp-Leone family of distributions, called Topp-Leone Kappa distribution which offers distribution with more flexible and applicable study of the highly-skewed data. Some statistical properties of the proposed distribution are studied. Based on censoring schemes, the maximum likelihood estimation is considered for the new model parameters. A simulation study is carried out to evaluate the performance of the maximum likelihood estimates based on their biases and mean square errors. The new model is fitted to a real data set and it is shown that the distribution is better fits than some other competitive distributions.

**Keywords:** Kappa distribution, Topp-Leone family, Maximum likelihood estimates, Type I censored sample, Type II censored sample.

### 1 Introduction

The kappa (K) distribution is a family of positively skewed distributions which is an important probability distribution which can be used in modelling data with extreme values. It provides a good fit to several extreme events like heavy rains is necessary in the planning of water relevant setups, cultivation of crops in agriculture, floods basic control systems, climatic conditions and overseeing environmental changes. It is suitable model to explain rainfall data and weather modifications which received attention from the hydrologist as mentioned by [1-2]. Let X is a three-parameter kappa (K3) random variable with shape parameters and scale parameter, its probability density function (pdf) is given by:

$$g(x) = \frac{\alpha \theta}{\beta} (x/\beta)^{\theta-1} \left[ \alpha + (x/\beta)^{\alpha \theta} \right]^{\left( \frac{\alpha+1}{\alpha} \right)}, \quad x > 0. \quad (1)$$

The corresponding cumulative distribution function (cdf) is

$$G(x) = \left[ \frac{(x/\beta)^{\alpha \theta}}{\alpha + (x/\beta)^{\alpha \theta}} \right]^{\frac{1}{\alpha}}. \quad (2)$$

The unknown parameters of the (K3) distribution estimated by [3] using maximum likelihood estimation, moment estimation, and L-moment estimation and use the Monte Carlo simulation for performance evaluation of these estimators. The unknown parameters of the (K3) distribution estimated by [4] under Type II censored samples.

The kappa distribution families have closed algebraic expressions that can easily be analyzed than some common traditional distributions. In the literature, some extensions of the kappa distribution are available such as Exponentiated generalized kappa, Kumaraswamy generalized kappa, and McDonald generalized kappa distribution see [5], [6] studied the Marshall-Olkin kappa distribution, while [7] offered the Kumaraswamy generalized kappa and [8] introduced the odd kappa-G family.

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The paper is organized as follows. In section 2, we introduced a new generalization of Kappa distribution, namely, the Topp-Leone kappa (TLK3) distribution. In Section 3, useful expansions of (TLK3) pdf and cdf are obtained. Some statistical properties are derived in Section 4. Based on two censored samples, the population parameters are estimated in Section 5. Simulation study is carried out to illustrate theoretical results in Section 6. In Section 7, real data set is used to illustrate the usefulness of the (TLK3) distribution followed by conclusion.

## 2 The New Distribution

The new family of distributions called Topp Leone-G (TL-G) family of distributions suggested by [9], the corresponding pdf and cdf with  $b > 0$  shape parameter are given by, respectively

$$f_{TL-G}(x) = 2b g(x) \bar{G}(x) \left[1 - (\bar{G}(x))^2\right]^{b-1}, \quad x \in R. \quad (3)$$

and

$$F_{TL-G}(x) = \left[1 - (\bar{G}(x))^2\right]^b, \quad (4)$$

where  $\bar{G}(x) = 1 - G(x)$ . Inserting (2) in (4), the cdf of the (TLK3) distribution is

$$F_{TL-K3}(x) = \left[1 - \left(1 - \left[\frac{(x/\beta)^{\alpha\theta}}{\alpha + (x/\beta)^{\alpha\theta}}\right]^{\frac{1}{\alpha}}\right)^2\right]^b, \quad \alpha, \theta, \beta, b > 0, x > 0. \quad (5)$$

The corresponding pdf is obtained by differentiating (5) and is given as

$$f_{TL-K3}(x) = 2b \frac{\alpha\theta}{\beta} (x/\beta)^{\theta-1} \left[\alpha + (x/\beta)^{\alpha\theta}\right]^{-\left(\frac{\alpha+1}{\alpha}\right)} \left\{1 - \left[\frac{(x/\beta)^{\alpha\theta}}{\alpha + (x/\beta)^{\alpha\theta}}\right]^{\frac{1}{\alpha}}\right\} \times \left[1 - \left(1 - \left[\frac{(x/\beta)^{\alpha\theta}}{\alpha + (x/\beta)^{\alpha\theta}}\right]^{\frac{1}{\alpha}}\right)^2\right]^{b-1}, \quad \alpha, \theta, \beta, b > 0, x > 0. \quad (6)$$

The (TLK3) distribution has the following distributions as special sub-models: For  $\theta = 1$ , the (TLK3) distribution reduces to the Topp Leone-two parameter Kappa (TLK2) distribution with parameters  $\alpha$ ,  $\beta$ , and  $b$ . The (TLK3) distribution reduces to the Topp Leone-one parameter Kappa (TLK1) distribution with parameters  $\alpha$  and  $b$ , for  $\theta = \beta = 1$ .

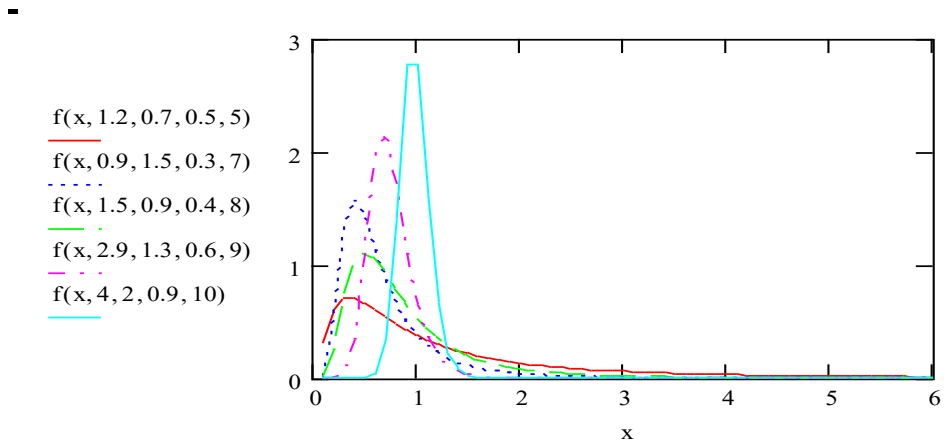
The survival (S) and hazard (h) functions corresponding to (6) are given by:

$$S_{TL-K3}(x) = 1 - \left[1 - \left(1 - \left[\frac{(x/\beta)^{\alpha\theta}}{\alpha + (x/\beta)^{\alpha\theta}}\right]^{\frac{1}{\alpha}}\right)^2\right]^b,$$

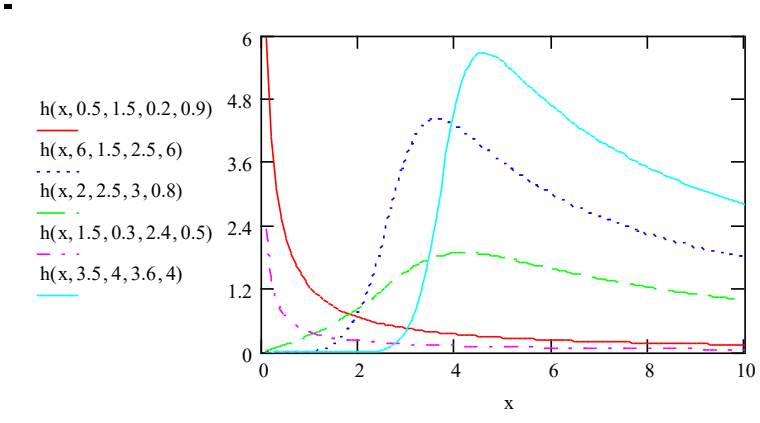
and

$$h_{TL-K3}(x) = 2b \frac{\alpha\theta}{\beta} (x/\beta)^{\theta-1} [\alpha + (x/\beta)^{\alpha\theta}]^{-\frac{(\alpha+1)}{\alpha}} \left\{ 1 - \left[ \frac{(x/\beta)^{\alpha\theta}}{\alpha + (x/\beta)^{\alpha\theta}} \right]^{\frac{1}{\alpha}} \right\} \\ \times \left[ 1 - \left( 1 - \left[ \frac{(x/\beta)^{\alpha\theta}}{\alpha + (x/\beta)^{\alpha\theta}} \right]^{\frac{1}{\alpha}} \right)^2 \right]^{b-1} \left[ 1 - \left[ 1 - \left( 1 - \left[ \frac{(x/\beta)^{\alpha\theta}}{\alpha + (x/\beta)^{\alpha\theta}} \right]^{\frac{1}{\alpha}} \right)^2 \right]^b \right]^{-1}.$$

Figures 1 and 2 illustrates the plots of the pdf and hazard function of (TLK3) distribution for various parameter values.



**Fig 1.** Pdf plot of (TLK3) distribution for various values of parameters



**Fig 2.** hf plot of (TLK3) distribution for various values of parameters

### 3 Useful Expansions

Here two important expansions are deduced for (TLK3) pdf and cdf distribution using infinite mixture forms of (K3) distribution, which are more efficient than computing those directly by numerical integration of its density function.

Since, if  $w > 0$  is real non-integer and  $|z| < 1$ , the generalized binomial theorem is written as follows:

$$(1 - z)^{w-1} = \sum_{i=0}^{\infty} (-1)^i \binom{w-1}{i} z^i \tag{7}$$

Then, by applying the binomial theorem (7) two times in (6), the (TLK3) pdf distribution becomes

$$f_{TL-K3}(x) = 2b \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{b-1}{i} \binom{2i+1}{j} \left(\frac{\alpha\theta}{\beta}\right) \left(\frac{x}{\beta}\right)^{\theta(j+1)-1} \left[\alpha + \left(\frac{x}{\beta}\right)^{\alpha\theta}\right]^{-\left(\frac{j+1}{\alpha}\right)-1}$$

Hence, the (TLK3) pdf can be expressed as infinite mixture of (K3) pdf as follows:

$$f_{TL-K3}(x) = \sum_{i,j=0}^{\infty} \psi_{i,j} \frac{\alpha\theta}{\beta} \left(\frac{x}{\beta}\right)^{\theta(j+1)-1} \left[\alpha + \left(\frac{x}{\beta}\right)^{\alpha\theta}\right]^{-\left(\frac{j+1}{\alpha}\right)-1}, \tag{8}$$

where

$$\psi_{i,j} = 2b(-1)^{i+j} \binom{b-1}{i} \binom{2i+1}{j}$$

Additionally, using the exponential expansion for  $[F_{TL-K3}(x)]^s$ , where s is positive integer, becomes

$$[F_{TL-K3}(x)]^s = \sum_{l,k=0}^{\infty} \omega_{l,k} \left[ \frac{(x/\beta)^{\alpha\theta}}{\alpha + (x/\beta)^{\alpha\theta}} \right]^{k\alpha^{-1}}, \tag{9}$$

where  $\omega_{l,k} = (-1)^{l+k} \binom{bs}{l} \binom{2l}{k}$ .

### 4 Mathematical Properties

In this section, we obtained some mathematical properties of the (TLK3) distribution.

#### 4.1 Quantile function

The quantile  $Q(q) = F^{-1}(q)$  of the (TLK3) distribution, if q has a Uniform distribution  $U(0,1)$ , is given as

$$Q(q) = \beta \alpha^{\frac{1}{\alpha\theta}} \left\{ 1 - \sqrt{1 - q^{\frac{1}{b}}}\right\}^{\frac{1}{\theta}} \left[ 1 - \left\{ 1 - \sqrt{1 - q^{\frac{1}{b}}}\right\}^{\alpha} \right]^{\frac{-1}{\alpha\theta}}, \quad 0 < q < 1 \tag{10}$$

Put  $q = 0.5$ , in (10) one gets the median of the (TL-K3) distribution.

#### 4.2 Moments and moment generating function

The  $r^{\text{th}}$  moment of the (TLk3) distribution is given by using the following relation:

$$\mu'_r = E(X^r) = \int_0^\infty x^r f_{TLk3}(x) dx.$$

Using (8), we have

$$\mu'_r = \sum_{i,j=0}^\infty \int_0^\infty x^r \psi_{i,j} \frac{\alpha\theta}{\beta} \left(\frac{x}{\beta}\right)^{\theta(j+1)-1} \left[\alpha + \left(\frac{x}{\beta}\right)^{\alpha\theta}\right]^{-\left(\frac{j+1}{\alpha}\right)-1} dx.$$

Let  $z = \left(\frac{x}{\beta}\right)^{\alpha\theta}$ , after simplification, we get

$$\mu'_r = \sum_{i,j=0}^\infty \psi_{i,j} \frac{\beta^r}{\alpha^{\frac{1}{\alpha}(j+1)+1}} \int_0^\infty z^{\frac{r}{\alpha\theta} + \frac{1}{\alpha}(j+1)-1} \left[1 - \left(1 - \left(1 + \frac{z}{\alpha}\right)^{-1}\right)\right]^{\frac{1}{\alpha}(j+1)+1} dz.$$

Let  $y = \left(1 - \left(1 + \frac{z}{\alpha}\right)^{-1}\right)$ , after simplification, we get

$$\mu'_r = \sum_{i,j=0}^\infty \psi_{i,j} \beta^r \alpha^{\frac{r}{\alpha\theta}-1} \int_0^1 y^{\frac{r}{\alpha\theta} + \frac{1}{\alpha}(j+1)-1} [1-y]^{1-\frac{r}{\alpha}-1} dy.$$

Then

$$\mu'_r = E(X^r) = \sum_{i,j=0}^\infty \psi_{i,j} \beta^r \alpha^{\frac{r}{\alpha\theta}-1} B\left(\frac{r}{\alpha\theta} + \frac{1}{\alpha}(j+1), 1 - \frac{r}{\alpha\theta}\right), \quad r = 1, 2, 3, 4. \tag{11}$$

Where  $B(a, b) = \int_0^1 w^{a-1} (1-w)^{b-1} dw$ , is the beta function. From (11), the mean and variance of the (TLk3) distribution can be obtained, respectively as follows:

$$E(X) = \sum_{i,j=0}^\infty \psi_{i,j} \beta \alpha^{\frac{1}{\alpha\theta}-1} B\left(\frac{1}{\alpha\theta} + \frac{1}{\alpha}(j+1), 1 - \frac{1}{\alpha\theta}\right),$$

and

$$\begin{aligned} Var(X) = & \sum_{i,j=0}^\infty \psi_{i,j} \beta^2 \alpha^{\frac{2}{\alpha\theta}-1} B\left(\frac{2}{\alpha\theta} + \frac{1}{\alpha}(j+1), 1 - \frac{2}{\alpha\theta}\right) \\ & - \left[ \sum_{i,j=0}^\infty \psi_{i,j} \beta \alpha^{\frac{1}{\alpha\theta}-1} B\left(\frac{1}{\alpha\theta} + \frac{1}{\alpha}(j+1), 1 - \frac{1}{\alpha\theta}\right) \right]^2, \end{aligned}$$

Where  $\psi_{i,j} = 2b(-1)^{i+j} \binom{b-1}{i} \binom{2i+1}{j}$ . Based on the first four moments of the (TLk3) distribution, the measures of skewness and kurtosis of the (TLk3) distribution can be obtained as:

$$S = \frac{\mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3}{(\mu_2' - \mu_1'^2)^{3/2}}, \text{ and } K = \frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4}{(\mu_2' - \mu_1'^2)^2}.$$

The moment generating function and characteristic function of the (TLk3) distribution can be obtained, respectively as follows:

$$M_X(t) = E(e^{tx}) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r' = \sum_{r,i,j=0}^{\infty} \frac{t^r}{r!} \psi_{i,j} \beta^r \alpha^{\frac{r}{\alpha\theta}-1} B\left(\frac{r}{\alpha\theta} + \frac{1}{\alpha}(j+1), 1 - \frac{r}{\alpha\theta}\right),$$

and

$$\varphi_X(t) = E(e^{itx}) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \mu_r' = \sum_{r,i,j=0}^{\infty} \frac{(it)^r}{r!} \psi_{i,j} \beta^r \alpha^{\frac{r}{\alpha\theta}-1} B\left(\frac{r}{\alpha\theta} + \frac{1}{\alpha}(j+1), 1 - \frac{r}{\alpha\theta}\right).$$

### 4.3 Mean deviation

The amount of scattering in a population can be measured by the totality of deviations from the mean and the median. Let  $\mu, M$  are the mean and median respectively of the (TLk3) distribution. The mean deviation about the mean and the mean deviation about the median are, respectively, defined by

$$\delta_1(\mu) = 2[\mu F(\mu) - J(\mu)] \text{ and } \delta_2(M) = \mu - 2J(M), \quad (12)$$

Where  $J(t) = \int_0^t x f_{TL-K3}(x) dx$  is the first incomplete moment of the (TL-k3) distribution. From (8) we have

$$J(t) = \sum_{i,j=0}^{\infty} \psi_{i,j} \frac{\alpha\theta}{\beta} \int_0^t x \left(\frac{x}{\beta}\right)^{\theta(j+1)-1} \left[\alpha + \left(\frac{x}{\beta}\right)^{\alpha\theta}\right]^{-\left(\frac{j+1}{\alpha}\right)-1} dx.$$

Let  $z = \left(\frac{x}{\beta}\right)^{\alpha\theta}$ , after simplification, we get

$$J(t) = \sum_{i,j=0}^{\infty} \psi_{i,j} \beta \alpha^{\frac{1}{\alpha\theta}-1} B_y\left(\frac{1}{\alpha\theta} + \frac{1}{\alpha}(j+1), 1 - \frac{1}{\alpha\theta}\right), \quad y = \left(1 - \left(1 + \frac{z}{\alpha}\right)^{-1}\right),$$

Where  $B_y(a, b) = \int_0^y w^{a-1} (1-w)^{b-1} dw$ , is the incomplete beta function. Using (9) with  $s = 1$ , one can easily find  $\delta_1(\mu)$

and  $\delta_2(M)$  from (12).

Another application of the first incomplete moment, it can be used to determine some inequality measures, namely Lorenz and Bonferroni curves. These curves are very useful in a wide variety of fields such as economics, reliability, demography, and medicine.

The Lorenz  $L(p)$  and Bonferroni  $B(p)$  curves of the (TLk3) distribution are, respectively, given by using the following relation:

$$L(p) = J(p) / \mu = \frac{1}{\mu} \int_0^p x f_{TL-K3}(x) dx = \frac{\sum_{i,j=0}^{\infty} \psi_{i,j} B_y \left( \frac{1}{\alpha\theta} + \frac{1}{\alpha}(j+1), 1 - \frac{1}{\alpha\theta} \right)}{\sum_{i,j=0}^{\infty} \psi_{i,j} B \left( \frac{1}{\alpha\theta} + \frac{1}{\alpha}(j+1), 1 - \frac{1}{\alpha\theta} \right)},$$

And

$$B(P) = \frac{L(P)}{F_{TL-K3}(x)} = \left[ \sum_{l,k=0}^{\infty} \omega_{l,k} \left[ \frac{\left(\frac{x}{\beta}\right)^{\alpha\theta}}{\alpha + \left(\frac{x}{\beta}\right)^{\alpha\theta}} \right]^{k\alpha^{-1}} \right]^{-1} \left[ \frac{\sum_{i,j=0}^{\infty} \psi_{i,j} B_y \left( \frac{1}{\alpha\theta} + \frac{1}{\alpha}(j+1), 1 - \frac{1}{\alpha\theta} \right)}{\sum_{i,j=0}^{\infty} \psi_{i,j} B \left( \frac{1}{\alpha\theta} + \frac{1}{\alpha}(j+1), 1 - \frac{1}{\alpha\theta} \right)} \right],$$

where  $\omega_{l,k} = (-1)^{l+k} \binom{b}{l} \binom{2l}{k}$ .

#### 4.4 Probability weighted moments

The probability weighted moments (PWMs) can be obtained using the following relation:

$$\tau_{r,s} = E[X^r F(x)^s] = \int_{-\infty}^{\infty} x^r F(x)^s f(x) dx \tag{13}$$

The PWMs of the (TLk3) distribution is obtained by substituting (8) and (9) into (13), as follows:

$$\begin{aligned} \tau_{r,s} &= \int_0^{\infty} \sum_{i,j,l,k=0}^{\infty} \psi_{i,j} \omega_{l,k} x^r \frac{\alpha\theta}{\beta} \left(\frac{x}{\beta}\right)^{\theta(j+k+1)-1} \left[ \alpha + \left(\frac{x}{\beta}\right)^{\alpha\theta} \right]^{-\left(\frac{j+k+1}{\alpha}\right)-1} dx, \\ &= \sum_{i,j,l,k=0}^{\infty} \psi_{i,j} \omega_{l,k} \beta^r \alpha^{\frac{r}{\alpha\theta}-1} B \left( \frac{r}{\alpha\theta} + \frac{1}{\alpha}(j+k+1), 1 - \frac{r}{\alpha\theta} \right), \end{aligned}$$

Where

$$\psi_{i,j} = 2b(-1)^{i+j} \binom{b-1}{i} \binom{2i+1}{j}, \quad \omega_{l,k} = (-1)^{l+k} \binom{bs}{l} \binom{2l}{k}.$$

#### 4.5 Order Statistics

Let  $X_{(1)}, X_{(2)}, \dots, X_{(n)}$  be the order statistics corresponding to a simple random sample of size n from the (TLk3) distribution. The pdf of  $X_{(k)}$ ,  $k = 1, 2, \dots, n$ , the kth order statistics, is given by

$$f_{X_{(k)}}(x) = \sum_{t=0}^{n-k} \frac{(-1)^t}{B(k, n-k+1)} \binom{n-k}{t} [F_{TL-K3}(x)]^{k+t-1} f_{TL-K3}(x). \tag{14}$$

Inserting pdf and cdf of (TLk3) distribution into (14) and simplifying we get:

$$f_{X_{(k)}}(x) = \frac{1}{B(k, n-k+1)} \sum_{t=0}^{n-k} \sum_{i,j=0}^{\infty} (-1)^t \binom{n-k}{t} \psi_{i,j} \frac{\alpha\theta}{\beta} (x/\beta)^{\theta(j+1)-1} \left[ \alpha + (x/\beta)^{\alpha\theta} \right]^{-\left(\frac{j+1}{\alpha}\right)-1} \times \left[ 1 - \left( 1 - \left[ \frac{(x/\beta)^{\alpha\theta}}{\alpha + (x/\beta)^{\alpha\theta}} \right]^{\frac{1}{\alpha}} \right)^2 \right]^{b+k+t-1}.$$

The pdf of the first order statistics can be obtained by using (k=1) and the pdf of the nth order statistics can be defined by replacing (k=n).

### 5 Parameter Estimation

In this section, the estimators of the (TLk3) distribution parameters using maximum likelihood (ML) method are obtained under two different types of censoring schemes.

#### 5.1 ML Estimators under Type I Censoring

In a typical life test, n times whose lifetimes follow the (TLk3) distribution (6) with parameter vector  $\Theta = \{b, \alpha, \beta, \theta\}$  are placed on test and each failure occurs is noted. At a predetermined fixed time T, the test is terminated. The data collected consists of items  $x_{(1)} < x_{(2)} < \dots < x_{(r)}$  plus (n-r) items survived beyond the time of termination, T. clearly the number of failures r is a random variable. [10] gave the log-likelihood function, under Type I censored, as follows:

$$\ln L_1(\Theta) = \ln \left( \frac{n!}{(n-r)!} \right) + r \ln \left( \frac{2b\alpha\theta}{\beta^\theta} \right) + (\theta-1) \sum_{i=1}^r \ln(x_{(i)}) - \left( 1 + \frac{1}{\alpha} \right) \sum_{i=1}^r \ln \left[ \alpha + \left( \frac{x_{(i)}}{\beta} \right)^{\alpha\theta} \right] + \sum_{i=1}^r \ln(1 - Z_i) + (b-1) \sum_{i=1}^r \ln \left[ 1 - (1 - Z_i)^2 \right] + (n-r) \ln \left[ 1 - \left\{ 1 - (1 - W)^2 \right\}^b \right],$$

Where

$$Z_i = \left[ \frac{\left( \frac{x_{(i)}}{\beta} \right)^{\alpha\theta}}{\alpha + \left( \frac{x_{(i)}}{\beta} \right)^{\alpha\theta}} \right]^{\frac{1}{\alpha}}, \quad W = \left[ \frac{\left( \frac{T}{\beta} \right)^{\alpha\theta}}{\alpha + \left( \frac{T}{\beta} \right)^{\alpha\theta}} \right]^{\frac{1}{\alpha}}.$$

The components of the score vector  $U(\Theta)$  are given by:



$$U(b) = \frac{r}{b} + \sum_{i=1}^r \ln[1 - (1 - Z_i)^2] - (n - r) \left(1 - \{1 - (1 - W)^2\}^b\right)^{-1} \left[\{1 - (1 - W)^2\}^b \ln\{1 - (1 - W)^2\}\right], \tag{15}$$

$$U(\alpha) = \frac{r}{\alpha} + \frac{1}{\alpha^2} \sum_{i=1}^r \ln\left(\alpha + \left(\frac{x_{(i)}}{\beta}\right)^{\alpha\theta}\right) - \left(1 + \frac{1}{\alpha}\right) \sum_{i=1}^r \left(\alpha + \left(\frac{x_{(i)}}{\beta}\right)^{\alpha\theta}\right)^{-1} - \theta \left(1 + \frac{1}{\alpha}\right) \sum_{i=1}^r \left[Z_i^\alpha \ln\left(\frac{x_{(i)}}{\beta}\right)\right] \\ - \sum_{i=1}^r [Z'_{i\alpha} (1 - Z_i)^{-1}] + 2(b - 1) \sum_{i=1}^r [Z'_{i\alpha} (1 - Z_i) [1 - (1 - Z_i)^2]^{-1}] - 2b(n - r) W'_\alpha (1 - W) \times \\ \{1 - (1 - W)^2\}^{b-1} \left(1 - \{1 - (1 - W)^2\}^b\right)^{-1}, \tag{16}$$

$$U(\beta) = \frac{-r\theta}{\beta} + \left(1 + \frac{1}{\alpha}\right) \left(\frac{\alpha\theta}{\beta}\right) \sum_{i=1}^r (Z_i^\alpha) - \sum_{i=1}^r [Z'_{i\beta} (1 - Z_i)^{-1}] + 2(b - 1) \sum_{i=1}^r [Z'_{i\beta} (1 - Z_i) [1 - (1 - Z_i)^2]^{-1}] \\ - 2b(n - r) W'_\beta (1 - W) \{1 - (1 - W)^2\}^{b-1} \left(1 - \{1 - (1 - W)^2\}^b\right)^{-1}, \tag{17}$$

and

$$U(\theta) = \frac{r}{\theta} (1 - \theta \ln \beta) + \sum_{i=1}^r \ln(x_{(i)}) - (1 + \alpha) \sum_{i=1}^r \left[Z_i^\alpha \ln\left(\frac{x_{(i)}}{\beta}\right)\right] - \sum_{i=1}^r [Z'_{i\theta} (1 - Z_i)^{-1}] \\ + 2(b - 1) \sum_{i=1}^r [Z'_{i\theta} (1 - Z_i) [1 - (1 - Z_i)^2]^{-1}] - 2b(n - r) W'_\theta (1 - W) \times \\ \{1 - (1 - W)^2\}^{b-1} \left(1 - \{1 - (1 - W)^2\}^b\right)^{-1}. \tag{18}$$

This system of non-linear equations (15-18) cannot be solved analytically, therefore we apply statistical software to solve them numerically to obtain the ML estimates (MLEs) of the four unknown parameters  $b, \alpha, \beta$ , and  $\theta$  of the (TLk3) distribution under type I censored, called  $(\hat{b}, \hat{\alpha}, \hat{\beta}, \hat{\theta})$ .

### 5.2 ML Estimators under Type II Censoring

Suppose n times whose lifetimes follow the (TLk3) distribution (6) are placed on test. The life test is terminated after predetermined number of failures r has occurred. The log-likelihood function, under Type II censored, is given by [10] as follows:

$$\ln L_2(\Theta) = \ln\left(\frac{n!}{(n - r)!}\right) + r \ln\left(\frac{2b\alpha\theta}{\beta^\theta}\right) + (\theta - 1) \sum_{i=1}^r \ln(x_{(i)}) - \left(1 + \frac{1}{\alpha}\right) \sum_{i=1}^r \ln\left[\alpha + \left(\frac{x_{(i)}}{\beta}\right)^{\alpha\theta}\right] \\ + \sum_{i=1}^r \ln(1 - Z_i) + (b - 1) \sum_{i=1}^r \ln[1 - (1 - Z_i)^2] + (n - r) \ln\left[1 - \{1 - (1 - Z_r)^2\}^b\right].$$

The components of the score vector  $U^*(\Theta)$  are given by:

$$U^*(b) = \frac{r}{b} + \sum_{i=1}^r \ln[1 - (1 - Z_i)^2] - (n - r) \left(1 - \{1 - (1 - Z_r)^2\}^b\right)^{-1} \left[\{1 - (1 - Z_r)^2\}^b \ln\{1 - (1 - Z_r)^2\}\right], \tag{19}$$

$$\begin{aligned}
 U^*(\alpha) = & \frac{r}{\alpha} + \frac{1}{\alpha^2} \sum_{i=1}^r \ln \left( \alpha + \left( \frac{x_{(i)}}{\beta} \right)^{\alpha\theta} \right) - \left( 1 + \frac{1}{\alpha} \right) \sum_{i=1}^r \left( \alpha + \left( \frac{x_{(i)}}{\beta} \right)^{\alpha\theta} \right)^{-1} - \theta \left( 1 + \frac{1}{\alpha} \right) \sum_{i=1}^r \left[ Z_i^\alpha \ln \left( \frac{x_{(i)}}{\beta} \right) \right] \\
 & - \sum_{i=1}^r \left[ Z_{i\alpha}' (1 - Z_i)^{-1} \right] + 2(b-1) \sum_{i=1}^r \left[ Z_{i\alpha}' (1 - Z_i) \left[ 1 - (1 - Z_i)^2 \right]^{-1} \right] - 2b(n-r) Z_{r\alpha}' (1 - Z_r) \times \\
 & \left\{ 1 - (1 - Z_r)^2 \right\}^{b-1} \left( 1 - \left\{ 1 - (1 - Z_r)^2 \right\}^b \right)^{-1}, \tag{20}
 \end{aligned}$$

$$\begin{aligned}
 U^*(\beta) = & \frac{-r\theta}{\beta} + \left( 1 + \frac{1}{\alpha} \right) \left( \frac{\alpha\theta}{\beta} \right) \sum_{i=1}^r (Z_i^\alpha) - \sum_{i=1}^r \left[ Z_{i\beta}' (1 - Z_i)^{-1} \right] + 2(b-1) \sum_{i=1}^r \left[ Z_{i\beta}' (1 - Z_i) \left[ 1 - (1 - Z_i)^2 \right]^{-1} \right] \\
 & - 2b(n-r) Z_{r\beta}' (1 - Z_r) \left\{ 1 - (1 - Z_r)^2 \right\}^{b-1} \left( 1 - \left\{ 1 - (1 - Z_r)^2 \right\}^b \right)^{-1}, \tag{21}
 \end{aligned}$$

And

$$\begin{aligned}
 U^*(\theta) = & \frac{r}{\theta} (1 - \theta \ln \beta) + \sum_{i=1}^r \ln(x_{(i)}) - (1 + \alpha) \sum_{i=1}^r \left[ Z_i^\alpha \ln \left( \frac{x_{(i)}}{\beta} \right) \right] - \sum_{i=1}^r \left[ Z_{i\theta}' (1 - Z_i)^{-1} \right] \\
 & + 2(b-1) \sum_{i=1}^r \left[ Z_{i\theta}' (1 - Z_i) \left[ 1 - (1 - Z_i)^2 \right]^{-1} \right] - 2b(n-r) Z_{r\theta}' (1 - Z_r) \times \\
 & \left\{ 1 - (1 - Z_r)^2 \right\}^{b-1} \left( 1 - \left\{ 1 - (1 - Z_r)^2 \right\}^b \right)^{-1}. \tag{22}
 \end{aligned}$$

Setting equations (19-22) to zero and solving the resulting system of non-linear equations simultaneously to obtain the MLs of the unknown parameters of the (TLk3) distribution under type II censored, called  $(\tilde{b}, \tilde{\alpha}, \tilde{\beta}, \tilde{\theta})$ . For interval estimation of the unknown parameters, we require the two observed  $4 \times 4$  information matrix under type I censored or type II censored are given in Appendix. The  $100(1-\gamma)\%$  confidence intervals for the parameters  $b, \alpha, \beta$ , and  $\theta$  of the (TLk3) distribution of two types of censored are respectively given by:

$$\begin{aligned}
 & \hat{b} \pm z_{\frac{\gamma}{2}} \sqrt{\text{var}(\hat{b})}, \hat{\alpha} \pm z_{\frac{\gamma}{2}} \sqrt{\text{var}(\hat{\alpha})}, \hat{\beta} \pm z_{\frac{\gamma}{2}} \sqrt{\text{var}(\hat{\beta})}, \hat{\theta} \pm z_{\frac{\gamma}{2}} \sqrt{\text{var}(\hat{\theta})} \\
 & , \text{ and} \\
 & \tilde{b} \pm z_{\frac{\gamma}{2}} \sqrt{\text{var}(\tilde{b})}, \tilde{\alpha} \pm z_{\frac{\gamma}{2}} \sqrt{\text{var}(\tilde{\alpha})}, \tilde{\beta} \pm z_{\frac{\gamma}{2}} \sqrt{\text{var}(\tilde{\beta})}, \tilde{\theta} \pm z_{\frac{\gamma}{2}} \sqrt{\text{var}(\tilde{\theta})} \\
 & ,
 \end{aligned}$$

where  $Z_{\gamma/2}$  is the upper  $(\gamma/2)^{th}$  percentile of the standard normal distribution and the  $Var(\cdot)$ 's denote the diagonal elements of  $I(\hat{\Theta})^{-1}$  and  $I(\tilde{\Theta})^{-1}$  corresponding to parameters  $(b, \alpha, \beta, \theta)$ .

## 6 Simulation Study

In this section, a simulation study is performed to evaluate the performance of the MLE for the (TLk3) distribution parameters based on two types of censoring schemes (Type I and Type II). The simulation study is designed as follows:

**Step 1:** Generate 1000 random samples of size 80, 100, 120 and 150 from the (TLk3) distribution under Type I and Type II censored samples.

**Step 2:** Two set of parameters values are selected as: Case I  $(b = 0.5, \alpha = 0.5, \beta = 0.5, \theta = 0.5)$  and Case II  $(b = 0.5, \alpha = 0.5, \beta = 1.5, \theta = 0.5)$ .

**Step 3:** In Type I censoring, two terminated times are selected as  $T=5$  and  $T=10$ . Also, for Type II censoring, the number of failure items;  $r$ , is selected based on two levels of censoring as 70% and 90%.

**Step 4:** The MLEs of the four unknown parameters are obtained by solving the non-linear equations (15-18) based on Type I censoring. Also, the non-linear equations (19-22) are solved, to obtain the MLEs of the unknown parameters under Type II censoring.

**Step 5:** Compute the biases, mean square errors (MSEs), and confidence interval length (CL) of MLEs of the unknown parameters.

The numerical results for two types of censoring schemes are summarized in Tables 1 and 2. For different choices of sample sizes, the estimates are work well and MSE and bias decreases as the sample size increases. Generally the results corresponding to type II censoring scheme are better than those corresponding to type I censoring scheme in the sense of MSE, Bias, and CL.

### 7 Application to Real Data

In this section, one data set will be analyzed to show the applicability of the (TLk3) distribution. The data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, analyzed by [11]. The data are:

In this section, one data set will be analyzed to show the applicability of the (TLk3) distribution. The data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, analyzed by [11]. The data are:

0.1	0.33	0.44	0.56	0.59	0.72	0.74	0.77	0.92	0.93	0.96	1
1	1.02	1.05	1.07	0.07	0.08	1.08	1.08	1.09	1.12	1.13	1.15
1.16	1.2	1.21	1.22	1.22	1.24	1.3	1.34	1.36	1.39	1.44	1.46
1.53	1.59	1.6	1.63	1.63	1.68	1.71	1.72	1.76	1.83	1.95	1.96
1.97	2.02	2.13	2.15	2.16	2.22	2.3	2.31	2.4	2.45	2.51	2.53
2.54	2.54	2.78	2.93	3.27	3.42	3.47	3.61	4.02	4.32	4.58	5.55

Here, these data will be fitted to three distributions namely: Kumaraswamy generalized Kappa (KGK), three parameter Kappa (K3), and Two parameter Kappa (K2), The MLEs of the unknown parameters will be calculated for each competitive distribution. To compare (TLK3) distribution with these distributions that represent this data the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (CAIC), and Kolmogorov-Smirnov (KS) distances between the fitted distribution and the empirical distribution function with corresponding p-value are calculated as shown in table (3).



**Table (1D):** Biases, MSEs and confidence interval length of TLK3 distribution parameters based on Type I censored data

n	CaseI	T	Bias	MSE	CL	CaseII	T	Bias	MSE	CL	
80	b=0.5	5	-0.032	0.017	0.034	b=0.5	5	-0.029	0.065	0.031	
		10	-0.047	0.038	0.026	α=0.5	10	0.049	0.035	0.238	
		5	0.082	0.008	0.131		5	-0.026	0.076	0.138	
		10	0.07	0.006	0.126	β=1.5	10	1.075	0.231	0.033	
	5	-0.016	0.021	0.082	5		-0.123	0.096	0.034		
	10	0.028	0.006	0.041	10		0.042	0.029	0.042		
	5	0.548	0.061	0.102	5		0.126	0.085	0.107		
	100	b=0.5	10	0.004	0.012	0.083	b=0.5	10	0.004	0.012	0.101
			5	-0.076	0.009	0.024	α=0.5	5	0.068	0.052	0.027
			10	-0.146	0.028	0.019		10	0.042	0.025	0.132
			5	0.063	0.005	0.112	β=1.5	5	0.035	0.031	0.122
		10	0.053	0.005	0.106	10		0.842	0.027	0.023	
5		-0.042	0.011	0.052	5	0.024		0.042	0.032		
10		-0.014	0.012	0.031	10	0.027		0.016	0.032		
120		b=0.5	5	-0.014	0.036	0.051	θ=0.5	5	0.019	0.026	0.089
			10	-0.015	0.009	0.031	α=0.5	10	0.013	0.009	0.073
			5	-0.082	0.007	0.021		b=0.5	5	-0.124	0.044
			10	-0.162	0.026	0.016	β=1.5	10	0.007	0.019	0.061
		5	0.055	0.004	0.128	5		-0.013	0.023	0.089	
	10	0.022	0.004	0.072	10	-0.047		0.018	0.019		
	5	-0.061	0.009	0.041	5	-0.185		0.022	0.021		
	150	b=0.5	10	-0.032	0.008	0.031	θ=0.5	10	0.005	0.009	0.023
			5	-0.024	0.036	0.032	α=0.5	5	-0.162	0.016	0.087
			10	-0.027	0.008	0.023		b=0.5	10	-0.039	0.006
			5	-0.119	0.006	0.071	β=1.5	5	-0.082	0.005	0.018
		10	-0.194	0.002	0.014	10		-0.015	0.009	0.029	
5		0.038	0.002	0.122	5	-0.123		0.001	0.076		
10		0.002	0.006	0.031	10	-0.057		0.003	0.018		
150		β=0.5	5	-0.109	0.007	0.015	5	-0.053	0.011	0.012	
			10	-0.079	0.005	0.015	10	0.004	0.008	0.019	
		θ=0.5	5	-0.142	0.023	0.026	5	-0.183	0.006	0.041	
			10	-0.097	0.001	0.013	10	-0.093	0.001	0.014	

**Table (2):** Biases, MSEs and confidence interval length of TLK3 distribution parameters based on Type II censored data

n	CaseII	xr	Bias	MSE	CL	CaseII	xr	Bias	MSE	CL		
80	b=0.5	60	-0.048	0.014	0.016	b=0.5	60	-0.051	0.032	0.081		
		80	-0.047	0.038	0.015		80	-0.161	0.003	0.019		
		60	0.063	0.004	0.122	$\alpha=0.5$	60	-0.138	0.045	0.012		
	80	b=0.5	60	0.061	0.004	0.112		80	-0.227	0.026	0.003	
			60	-0.217	0.014	0.011	$\beta=1.5$	60	-0.175	0.052	0.022	
			80	0.028	0.012	0.016		80	-0.204	0.019	0.021	
	60	$\theta=0.5$	60	-0.153	0.025	0.012	$\theta=0.5$	60	-0.188	0.034	0.014	
			80	0.004	0.006	0.013		80	-0.209	0.009	0.012	
			60	-0.093	0.008	0.014	b=0.5	60	-0.081	0.015	0.017	
	100	b=0.5	60	-0.161	0.021	0.014		80	0.031	0.002	0.013	
			80	0.047	0.004	0.102	$\alpha=0.5$	60	-0.081	0.017	0.009	
			60	0.023	0.004	0.091		80	0.003	0.023	0.002	
80		b=0.5	60	-0.267	0.008	0.003	$\beta=1.5$	60	-0.149	0.034	0.016	
			80	-0.087	0.007	0.011		80	0.015	0.003	0.012	
			60	-0.189	0.008	0.005	$\theta=0.5$	60	-0.035	0.026	0.011	
60		$\theta=0.5$	80	-0.075	0.006	0.003		80	-0.016	0.006	0.005	
			60	-0.096	0.004	0.012	b=0.5	60	-0.292	0.007	0.041	
			80	-0.168	0.021	0.013		80	-0.108	0.006	0.011	
120		$\alpha=0.5$	60	0.046	0.002	0.109	$\alpha=0.5$	60	-0.399	0.008	0.007	
			80	0.021	0.003	0.009		80	-0.165	0.017	0.002	
			60	-0.276	0.006	0.007	$\beta=1.5$	60	-0.677	0.012	0.013	
	80	b=0.5	60	-0.109	0.007	0.009		80	-0.126	0.001	0.005	
			80	-0.189	0.006	0.006	$\theta=0.5$	60	-1.241	0.008	0.005	
			60	-0.091	0.003	0.005		80	-0.566	0.003	0.004	
	150	b=0.5	60	-0.132	0.002	0.021	b=0.5	60	-0.172	0.001	0.007	
			80	-0.194	0.002	0.012		80	-0.046	0.004	0.006	
			60	0.035	0.002	0.091	$\alpha=0.5$	60	-0.274	0.001	0.003	
		80	b=0.5	60	0.001	0.003	0.002		80	-0.134	0.002	0.001
				60	-0.376	0.003	0.006	$\beta=1.5$	60	-0.227	0.006	0.011
				80	-0.388	0.003	0.006		80	0.003	0.002	0.007
60		$\theta=0.5$	60	-0.247	0.002	0.003	$\theta=0.5$	60	-0.255	0.002	0.005	
			80	-0.263	0.001	0.002		80	-0.139	0.002	0.004	
			60	-0.132	0.002	0.021	b=0.5	60	-0.172	0.001	0.007	

The (TLK3) distribution provides a better fit than the other tested distributions, because it has the smallest value among AIC, BIC, CAIC and KS and largest p-value.

**Table 3:** The MLEs and the values of AIC, BIC, CAIC and KS statistics.

Model	Parameters					AIC	BIC	CAIC	KS	p-value
	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$					
TLK3	-	9.864	0.043	0.288	18.811	334.32	343.427	334.917	0.014	1
KGK	1.707	0.982	3.312	1.108	4.004	607.426	618.809	608.335	0.108	0.37
K3	-	-	0.168	2.834	3.917	353.048	359.878	353.401	0.097	0.507
K2	-	-	3.371	15.97	-	466.732	471.285	466.906	0.128	0.269

### 8 Conclusions

In this paper, we introduce a new four- parameter distribution called the Topp-Leone kappa distribution. Some mathematical properties of the proposed distribution such as quantile function, moments, mean deviation, and order statistics are derived. The maximum likelihood estimation is employed based on two censoring schemes to estimate the model parameters. Finally, a simulation study and real data set are considered.

**Conflicts of Interest Statement:** The authors declare no conflict of interest.

### Appendix

The elements of the 4×4 observed information matrix under type I or type II censored for the (TLK3) distribution with parameters  $(b, \alpha, \beta, \theta)$  are given by:

$$\begin{aligned}
 U(bb) &= \frac{-r}{b^2} - (n-r) \{1-(1-A)^2\}^b \left[ \ln\{1-(1-A)^2\} \right]^2 \left(1-\{1-(1-A)^2\}^b\right)^{-1} \\
 &\quad + \left[1-\{1-(1-A)^2\}^b\right]^2 \left[ \{1-(1-A)^2\}^b \ln\{1-(1-A)^2\} \right]^2, \\
 U(b\alpha) &= 2 \sum_{i=1}^r \left[ \frac{Z'_{i\alpha} (1-Z_i)}{\{1-(1-Z_i)^2\}} \right] - 2(n-r)(1-A) A'_\alpha \{1-(1-A)^2\}^{b-1} \left(1-\{1-(1-A)^2\}^b\right)^{-1} \\
 &\quad \times \left[ 1+b \ln\{1-(1-A)^2\} + b \left[1-\{1-(1-A)^2\}^b\right]^{-1} \left[ \{1-(1-A)^2\}^b \ln\{1-(1-A)^2\} \right] \right],
 \end{aligned}$$

$$U(b\beta) = 2 \sum_{i=1}^r \left[ \frac{Z'_{i\beta} (1 - Z_i)}{\{1 - (1 - Z_i)^2\}} \right] - 2(n-r)(1-A) A'_\beta \{1 - (1-A)^2\}^{b-1} \left(1 - \{1 - (1-A)^2\}^b\right)^{-1} \\ \times \left[ 1 + b \ln\{1 - (1-A)^2\} + b \left[1 - \{1 - (1-A)^2\}^b\right]^1 \left[ \{1 - (1-A)^2\}^b \ln\{1 - (1-A)^2\} \right] \right],$$

$$U(b\theta) = 2 \sum_{i=1}^r \left[ \frac{Z'_{i\theta} (1 - Z_i)}{\{1 - (1 - Z_i)^2\}} \right] - 2(n-r)(1-A) A'_\theta \{1 - (1-A)^2\}^{b-1} \left(1 - \{1 - (1-A)^2\}^b\right)^{-1} \\ \times \left[ 1 + b \ln\{1 - (1-A)^2\} + b \left[1 - \{1 - (1-A)^2\}^b\right]^1 \left[ \{1 - (1-A)^2\}^b \ln\{1 - (1-A)^2\} \right] \right],$$

$$U(\alpha\alpha) = \frac{-r}{\alpha^2} - \frac{2}{\alpha^3} \sum_{i=1}^r \ln \left[ \alpha + \left(\frac{x_{(i)}}{\beta}\right)^{\alpha\theta} \right] + \frac{2}{\alpha^2} \sum_{i=1}^r \left[ \alpha + \left(\frac{x_{(i)}}{\beta}\right)^{\alpha\theta} \right]^{-1} + \frac{2\theta}{\alpha^2} \sum_{i=1}^r \left[ Z_i^\alpha \ln \left(\frac{x_{(i)}}{\beta}\right) \right] \\ + \left(1 + \frac{1}{\alpha}\right) \sum_{i=1}^r \left\{ \left[ \alpha + \left(\frac{x_{(i)}}{\beta}\right)^{\alpha\theta} \right]^{-2} + \theta Z_i^{2\alpha} \left(\frac{x_{(i)}}{\beta}\right)^{-\alpha\theta} \ln \left(\frac{x_{(i)}}{\beta}\right) \right\} - \sum_{i=1}^r \left[ \frac{Z''_{i\alpha} (1 - Z_i) + (Z'_{i\alpha})^2}{(1 - Z_i)^2} \right] \\ - \theta \left(1 + \frac{1}{\alpha}\right) \sum_{i=1}^r \ln \left(\frac{x_{(i)}}{\beta}\right) \left[ Z_i^\alpha \ln Z_i + \alpha Z_i^{\alpha-1} Z'_{i\alpha} \right] + 2(b-1) \sum_{i=1}^r \left[ \frac{Z''_{i\alpha} (1 - Z_i) - (Z'_{i\alpha})^2}{[1 - (1 - Z_i)^2]} \right] \\ - 4(b-1) \sum_{i=1}^r \left( \frac{Z'_{i\alpha} (1 - Z_i)}{[1 - (1 - Z_i)^2]} \right)^2 - 2b(n-r) \{1 - (1-A)^2\}^{b-1} \left(1 - \{1 - (1-A)^2\}^b\right)^{-1} \times \\ \left[ A_\alpha'^2 \left( \frac{2(b-1)(1-A)^2}{\{1 - (1-A)^2\}} - 1 \right) + A_\alpha''(1-A) + \frac{2b A_\alpha'^2 (1-A)^2 \{1 - (1-A)^2\}^{b-1}}{[1 - \{1 - (1-A)^2\}^b]} \right],$$

$$U(\alpha\beta) = \frac{-\theta}{\alpha\beta} \sum_{i=1}^r Z_i^\alpha - \frac{(\alpha+1)\theta}{\beta} \sum_{i=1}^r \left[ Z_i^{2\alpha} \left(\frac{x_{(i)}}{\beta}\right)^{-\alpha\theta} \right] - \sum_{i=1}^r \left[ \frac{Z''_{i\alpha\beta}}{(1 - Z_i)} \right] - \sum_{i=1}^r \left[ \frac{Z'_{i\alpha} Z'_{i\beta}}{(1 - Z_i)^2} \right] \\ - \theta \left(1 + \frac{1}{\alpha}\right) \sum_{i=1}^r \left\{ \alpha Z_i^{\alpha-1} Z'_{i\beta} \ln \left(\frac{x_{(i)}}{\beta}\right) - \left(\frac{Z_i^\alpha}{\beta}\right) \right\} + 2(b-1) \sum_{i=1}^r \left[ \frac{Z''_{i\alpha\beta} (1 - Z_i) - (Z'_{i\alpha} Z'_{i\beta})}{[1 - (1 - Z_i)^2]} \right] \\ - 4(b-1) \sum_{i=1}^r \left( \frac{Z'_{i\alpha} Z'_{i\beta} (1 - Z_i)^2}{[1 - (1 - Z_i)^2]^2} \right) - 2b(n-r) \{1 - (1-A)^2\}^{b-1} \left(1 - \{1 - (1-A)^2\}^b\right)^{-1} \times \\ \left[ A'_\alpha A'_\beta \left( \frac{2(b-1)(1-A)^2}{\{1 - (1-A)^2\}} - 1 \right) + A''_{\alpha\beta} (1-A) + \frac{2b A'_\alpha A'_\beta (1-A)^2 \{1 - (1-A)^2\}^{b-1}}{[1 - \{1 - (1-A)^2\}^b]} \right],$$

$$\begin{aligned}
 U(\alpha\theta) &= (\alpha + 1) \sum_{i=1}^r \ln \left( \frac{x_{(i)}}{\beta} \right) Z_i^{2\alpha} \left( \frac{x_{(i)}}{\beta} \right)^{-\alpha\theta} - \sum_{i=1}^r \left[ Z_i^\alpha \ln \left( \frac{x_{(i)}}{\beta} \right) \right] - \sum_{i=1}^r \left[ \frac{Z_{i\alpha\theta}''}{(1-Z_i)} \right] \\
 &- \sum_{i=1}^r \left[ \frac{Z_{i\alpha} Z_{i\theta}'}{(1-Z_i)^2} \right] - \theta(\alpha + 1) \sum_{i=1}^r \left\{ Z_i^{\alpha-1} Z_{i\theta}' \ln \left( \frac{x_{(i)}}{\beta} \right) \right\} + 2(b-1) \sum_{i=1}^r \left[ \frac{Z_{i\alpha\theta}''(1-Z_i) - (Z_{i\alpha}' Z_{i\theta}')}{[1-(1-Z_i)^2]} \right] \\
 &- 4(b-1) \sum_{i=1}^r \left( \frac{Z_{i\alpha}' Z_{i\theta}' (1-Z_i)^2}{[1-(1-Z_i)^2]^2} \right) - 2b(n-r) \left\{ 1 - (1-A)^2 \right\}^{b-1} \left( 1 - \left\{ 1 - (1-A)^2 \right\}^b \right)^{-1} \times \\
 &\left[ A_\alpha' A_\theta' \left( \frac{2(b-1)(1-A)^2}{\{1-(1-A)^2\}} - 1 \right) + A_{\alpha\theta}''(1-A) + \frac{2b A_\alpha' A_\theta' (1-A)^2 \{1-(1-A)^2\}^{b-1}}{[1-\{1-(1-A)^2\}^b]} \right],
 \end{aligned}$$

$$\begin{aligned}
 U(\beta\beta) &= \frac{r\theta}{\beta^2} + \frac{(\alpha+1)\theta}{\beta^2} \sum_{i=1}^r Z_i^\alpha - \frac{(\alpha+1)\theta\alpha}{\beta} \sum_{i=1}^r [Z_i^{\alpha-1} Z_{i\beta}'] - \sum_{i=1}^r \left[ \frac{Z_{i\beta}''(1-Z_i) + (Z_{i\beta}')^2}{(1-Z_i)^2} \right] \\
 &+ 2(b-1) \sum_{i=1}^r \left[ \frac{Z_{i\beta}''(1-Z_i) - (Z_{i\beta}')^2}{[1-(1-Z_i)^2]} \right] - 4(b-1) \sum_{i=1}^r \left( \frac{Z_{i\beta}'(1-Z_i)}{[1-(1-Z_i)^2]} \right)^2 \\
 &- 2b(n-r) \left\{ 1 - (1-A)^2 \right\}^{b-1} \left( 1 - \left\{ 1 - (1-A)^2 \right\}^b \right)^{-1} \times \\
 &\left[ A_\beta'^2 \left( \frac{2(b-1)(1-A)^2}{\{1-(1-A)^2\}} - 1 \right) + A_\beta''(1-A) + \frac{2b A_\beta'^2 (1-A)^2 \{1-(1-A)^2\}^{b-1}}{[1-\{1-(1-A)^2\}^b]} \right],
 \end{aligned}$$

$$\begin{aligned}
 U(\beta\theta) &= \frac{-r}{\beta} - \frac{(\alpha+1)}{\beta} \sum_{i=1}^r Z_i^\alpha - \frac{(\alpha+1)\theta\alpha}{\beta} \sum_{i=1}^r Z_i^{\alpha-1} Z_{i\theta}' - \sum_{i=1}^r \left[ \frac{Z_{i\beta\theta}''(1-Z_i) + (Z_{i\beta}' Z_{i\theta}')}{(1-Z_i)^2} \right] \\
 &+ 2(b-1) \sum_{i=1}^r \left[ \frac{Z_{i\beta\theta}''(1-Z_i) - (Z_{i\beta}' Z_{i\theta}')}{[1-(1-Z_i)^2]} \right] - 4(b-1) \sum_{i=1}^r \left( \frac{Z_{i\beta}' Z_{i\theta}' (1-Z_i)^2}{[1-(1-Z_i)^2]^2} \right) \\
 &- 2b(n-r) \left\{ 1 - (1-A)^2 \right\}^{b-1} \left( 1 - \left\{ 1 - (1-A)^2 \right\}^b \right)^{-1} \times \\
 &\left[ A_\beta' A_\theta' \left( \frac{2(b-1)(1-A)^2}{\{1-(1-A)^2\}} - 1 \right) + A_{\beta\theta}''(1-A) + \frac{2b A_\beta' A_\theta' (1-A)^2 \{1-(1-A)^2\}^{b-1}}{[1-\{1-(1-A)^2\}^b]} \right],
 \end{aligned}$$

and



$$\begin{aligned}
 U(\theta\theta) &= \frac{-r}{\theta^2} - \alpha(\alpha + 1) \sum_{i=1}^r \left[ Z_i^{\alpha-1} Z'_{i\theta} \ln\left(\frac{x_{(i)}}{\beta}\right) \right] - \sum_{i=1}^r \left[ \frac{Z''_{i\theta}(1-Z_i) + (Z'_{i\theta})^2}{(1-Z_i)^2} \right] \\
 &+ 2(b-1) \sum_{i=1}^r \left[ \frac{Z''_{i\theta}(1-Z_i) - (Z'_{i\theta})^2}{[1-(1-Z_i)^2]} \right] - 4(b-1) \sum_{i=1}^r \left( \frac{Z'_{i\theta}(1-Z_i)}{[1-(1-Z_i)^2]} \right)^2 \\
 &- 2b(n-r) \left\{ 1 - (1-A)^2 \right\}^{b-1} \left( 1 - \left\{ 1 - (1-A)^2 \right\}^b \right)^{-1} \times \\
 &\left[ A_{\theta}^{\prime 2} \left( \frac{2(b-1)(1-A)^2}{\left\{ 1 - (1-A)^2 \right\}} - 1 \right) + A_{\theta}''(1-A) + \frac{2b A_{\theta}^{\prime 2}(1-A)^2 \left\{ 1 - (1-A)^2 \right\}^{b-1}}{\left[ 1 - \left\{ 1 - (1-A)^2 \right\}^b \right]} \right],
 \end{aligned}$$

where  $A = W$ , for type I censored and for type II censored  $A = Z_r$  and

$$Z'_{i\alpha} = \left(\frac{1}{\alpha}\right) Z_i \ln\left(\alpha + \left(\frac{x_{(i)}}{\beta}\right)^{\alpha\theta}\right) - \left(\frac{1}{\alpha}\right) Z_i^{\alpha+1} \left[ \left(\frac{x_{(i)}}{\beta}\right)^{-\alpha\theta} + \theta \ln\left(\frac{x_{(i)}}{\beta}\right) \right],$$

$$Z'_{i\beta} = \frac{\theta}{\beta} (Z_i^{\alpha+1} - Z_i),$$

$$Z'_{i\theta} = \ln\left(\frac{x_{(i)}}{\beta}\right) (Z_i - Z_i^{\alpha+1}),$$

$$\begin{aligned}
 Z''_{i\alpha} &= \left(\frac{1}{\alpha^2}\right) \ln\left(\alpha + \left(\frac{x_{(i)}}{\beta}\right)^{\alpha\theta}\right) \left[ Z'_{i\alpha} - \left(\frac{2}{\alpha}\right) Z_i \right] + \left(\frac{\theta}{\alpha}\right) Z_i^{\alpha+1} \left(\frac{x_{(i)}}{\beta}\right)^{-\alpha\theta} \ln\left(\frac{x_{(i)}}{\beta}\right) \\
 &+ \left\{ \left(\frac{x_{(i)}}{\beta}\right)^{-\alpha\theta} + \theta \ln\left(\frac{x_{(i)}}{\beta}\right) \right\} \left[ \left(\frac{2}{\alpha^2}\right) Z_i^{\alpha+1} - \left(1 + \frac{1}{\alpha}\right) Z_i^{\alpha} Z'_{i\alpha} - \left(\frac{1}{\alpha}\right) Z_i^{\alpha+1} \ln Z_i \right],
 \end{aligned}$$

$$\begin{aligned}
 Z''_{i\alpha\beta} &= \left(\frac{1}{\alpha^2}\right) Z'_{i\beta} \ln\left(\alpha + \left(\frac{x_{(i)}}{\beta}\right)^{\alpha\theta}\right) - \left(1 + \frac{1}{\alpha}\right) Z_i^{\alpha} Z'_{i\beta} \left[ \left(\frac{x_{(i)}}{\beta}\right)^{-\alpha\theta} + \theta \ln\left(\frac{x_{(i)}}{\beta}\right) \right] \\
 &+ \left(\frac{\theta}{\beta}\right) Z_i^{\alpha+1} \left(\frac{x_{(i)}}{\beta}\right)^{-\alpha\theta},
 \end{aligned}$$

$$\begin{aligned}
 Z''_{i\alpha\theta} &= \left(\frac{1}{\alpha^2}\right) Z'_{i\theta} \ln\left(\alpha + \left(\frac{x_{(i)}}{\beta}\right)^{\alpha\theta}\right) - \left(1 + \frac{1}{\alpha}\right) Z_i^{\alpha} Z'_{i\theta} \left[ \left(\frac{x_{(i)}}{\beta}\right)^{-\alpha\theta} + \theta \ln\left(\frac{x_{(i)}}{\beta}\right) \right] \\
 &+ Z_i^{\alpha+1} \left(\frac{x_{(i)}}{\beta}\right)^{-\alpha\theta} \ln\left(\frac{x_{(i)}}{\beta}\right),
 \end{aligned}$$

$$Z''_{i\beta} = \left(\frac{\theta}{\beta^2}\right) Z_i [1 - Z_i^{\alpha}] + \left(\frac{\theta}{\beta}\right) Z'_{i\beta} [(\alpha + 1) Z_i^{\alpha} - 1],$$

$$Z''_{i\beta\theta} = \left(\frac{1}{\beta}\right) \left[ Z_i^{\alpha+1} + \theta(\alpha+1) Z'_{i\theta} Z_i^\alpha - \theta Z'_{i\theta} - Z_i \right],$$

$$Z''_{i\theta} = \ln\left(\frac{x^{(i)}}{\beta}\right) Z'_{i\theta} \left[ 1 - (\alpha+1) Z_i^\alpha \right],$$

**Conflicts of Interest Statement:** The authors declare no conflict of interest.

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