Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/080541

Higher Order Close-to-Convex Functions related with Conic Domain

Khalida Inayat Noor∗ *and Muhammad Aslam Noor*

Mathematics Department, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan

Received: 13 Sep. 2013, Revised: 11 Dec. 2013, Accepted: 12 Dec. 2013
Published online: 1 Sep. 2014

Abstract: In this paper we define and study a class of analytic functions which map the open unit disk onto same conic regions and are related to Bazilevic and higher order close-to-convex functions. We investigate rate of growth of coefficients, Hankel determinant problem, inclusion result and establish univalence criterion. Some other interesting properties of this class are also studied.

Keywords: close-to-convex, univalent, conic regions, bounded boundary rotation, Caratheodory functions, convolution, subordination.
2010 AMS Subject Classification: 30C45, 30C50

1 Introduction

Let A be the class of functions analytic in the open unit disc $E=\{z:|z|<1\}$ and be given by

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n. \tag{1}$$

Let $S\subset A$ be the class of functions which are univalent and also K,S^*,C be the well known subclasses of S which, respectively, contain close-to-convex, starlike and convex functions.

Let $V_m(\rho), m\geq 2, 0\leq\rho<1$, be the class of functions f analytic and locally univalent in E and satisfying the condition

$$\int_0^{2\pi}\left|\Re\left(\frac{\frac{(zf'(z))'}{f'(z)}-\rho}{1-\rho}\right)\right|d\theta\leq m\pi. \tag{2}$$

When $\rho=0$, we obtain the class $V_m(m\geq 2)$ of functions with bounded boundary rotation, see [4]. The class $V_m(\rho)$ was introduced and discussed in some detail in [8]. It can easily be shown that $f\in V_m(\rho)$ if and only if there exists $f_1\in V_m$ such that

$$f'(z)=(f_1'(z))^{1-\rho}. \tag{3}$$

The convolution of two functions $f(z)$ given by (1) and $g(z)=z+\sum\limits_{n=2}^{\infty}b_nz^n$ is defined as

$$(f*g)(z)=(g*f)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n.$$

In [5], the domain $\Omega_k, k\in[0,\infty)$ is defined as follows:

$$\Omega_k=\{u+iv:u>k\sqrt{(u-1)^2+v^2}\}. \tag{4}$$

For fixed k, Ω_k represents the conic region bounded, successively, by the imaginary axis $(k=0)$, the right branch of a hyperbola $(0<k<1)$ and a parabola $(k=1)$ and an ellipse $(k>1)$. Also, we note that, for no choice of $k(k>1)$, Ω_k reduces to a disc, see [5,18].

In this paper we will choose $k\in[0,1]$. For $k\in[0,1]$, the following functions, denoted by $p_k(z)$, are univalent in E, continuous as regard to k, have real coefficients and map E onto Ω_k such that $p_k(0)=1, p_k'(0)>0$:

$$p_k(z)=\begin{cases}\frac{1+z}{1-z},\ (k=0),\\ 1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, \quad (k=1),\\ 1+\frac{2}{1-k^2}\sinh^2\left[\left(\frac{2}{\pi}\arccos k\right) arctan\sqrt(z)\right], \quad (0<k<1).\end{cases} \quad see[5]\ . \tag{5}$$

Let P denote the class of Caratheodory functions of positive real part. Then the class $P(p_k)\subset P$ is defined as follows

Definition 1. *Let $p(z)$ be analytic in E with $p(0)=1$. Then $p\in P(p_k)$, if $p(z)$ is subordinate to $p_k(z)$ given by (5). We write $p\in P_{p_k}$ implies $p(z)\prec p_k(z)$ in E, and $p(E)\in p_k(E)$.*

We note that $P(p_0)=P$. It is easy to verify that $P(p_k)$ is a convex set and

$$P(p_k)\subset P(\rho),\ \rho=\frac{k}{k+1},$$

∗ Corresponding author e-mail: khalidanoor@hotmail.com

where $P(\rho)$ is the class of functions with real part greater than ρ, see [6].
Also, for $p \in P(p_k)$, it is known [22] that

$$|\arg p(z)| < \frac{\sigma\pi}{2} = \begin{cases} \frac{\pi}{2}, & (k=0), \\ \arctan\frac{1}{k}, & (k \neq 0), \\ \frac{\pi}{4}, & (k=1). \end{cases} \quad (6)$$

We extend the class $P(p_k)$ as given below

Definition 2. *Let $p(z)$ be analytic in E with $p(0)=1$. Then $p \in P_m(p_k)$, if and only if, for $m \geq 2$, $k \in [0,1]$, we have*

$$p(z) = \left(\frac{m}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{m}{4}-\frac{1}{2}\right)p_2(z), \quad (7)$$

$p_1, p_2 \in P(p_k)$.

When $k=0$, we obtain the class P_m introduced and studied in [20]. Also $P_2(p_k) = P(p_k)$.

We now define the following

Definition 3. *Let $f \in A$. Then $f \in k-UV_m$, $k \in [0,1]$, $m \geq 2$ if and only if*

$$\left[1+\frac{z(f''(z))}{f'(z)}\right] \in P_m(p_k), \quad z \in E.$$

$k-UV_m$ is called the class of functions with k-uniform boundary rotation.

For $k=0$, $0-UV_m = V_m$, see [4,12,13,14].

The corresponding class $k-UR_m$ is defined as

$$k-UR_m = \{F \in A : F = zf', f \in k-UV_m\}.$$

We note that:

(i) $k-UV_2 = k-UCV$, is the class of uniformly convex functions.

(ii) $k-UR_2 = k-ST$ is the class of uniformly starlike functions.

For details of these special case, we refer to [22].

Definition 4. *Let $f \in A$. Then $f \in k-UT_m$ if there exists $g \in k-UV_m$ such that*

$$\frac{f'(z)}{g'(z)} \in P(p_k), z \in E.$$

For $k=0$, we have the class T_m, introduced and discussed in [10].

Also, for $k=0, m=2$, $k-UT_m$ reduces to well known class K of close-to-convex functions, see [7].

Let $\phi \in A$. Then $f \in k-UT_m(\phi)$ if and only if $(f*\phi) \in k-UT_m$ for $z \in E$.

Definition 5. *Let $f \in A$. Then, for $a \geq 0, 0 \leq \gamma < 1$, $f \in k-UT_m(a,\gamma,\phi)$ if and only if there exists $g \in k-UT_m(\phi)$ such that*

$$zf'(z) + af(z) = (a+1)z(g'(z))^{\gamma}. \quad (8)$$

We note that

$$k-UT_m\left(0,1,\frac{z}{1-z}\right) = k-UT_m,$$
$$0-UT_m\left(0,1,\frac{z}{1-z}\right) = T_m.$$

Also

$$0-UT_2(0,1,-\log(1-z)) = C^*,$$

the class of quasi-convex functions, see [17].

Throughout this paper, we assume that $k \in [0,1]$, $\gamma \in (0,1]$, $m \geq 2$, $\Re(a) > -1$, $z \in E$, unless otherwise specified.

2 Preliminaries

Lemma 1([19]). *Let $q(z)$, be analytic in E with $q(0)=1$. If $\alpha \geq 1, \Re(c) \geq 0$, $0 \leq \theta_1 < \theta_2 \leq 2\pi, z = re^{i\theta}$, then*

$$\int_{\theta_1}^{\theta_2} \Re\left\{q(z) + \frac{\alpha z q'(z)}{c\alpha + q(z)}\right\} d\theta > -\pi$$

implies

$$\int_{\theta_1}^{\theta_2} \Re q(z) d\theta > -\pi.$$

Lemma 2([16]). *Let $f \in k-UR_m$. Then there exist $s_i \in k-ST, i=1,2$ such that*

$$f(z) = \frac{(s_1(z))^{\frac{m+2}{4}}}{(s_2(z))^{\frac{m-2}{4}}}.$$

Lemma 3([10]). *Let $g \in V_m(\rho)$. Then, for $0 \leq \rho < 1, \theta_1 < \theta_2$,*

(i) $g'(z) = (g_1'(z))^{1-\rho}, g_1 \in V_m$.

(ii) $\int_{\theta_1}^{\theta_2} \Re\left\{\frac{(zg'(z))'}{g'(z)} d\theta\right\} > -\left(\frac{m}{2}-1\right)(1-\rho)\pi.$

3 Main Results

Theorem 1. *Let $G \in k-UT_m$. Then, for $\theta_1 < \theta_2, z = re^{i\theta}$,*

$$\int_{\theta_1}^{\theta_2} \Re\left\{\frac{(zG'(z))'}{G'(z)} d\theta\right\} > -\left(\frac{m-2}{2(k+1)}+\sigma\right)\pi.$$

Proof. Since $G \in k-UT_m$, there exist $G_1 \in k-UV_m$ and $k-UV_m \subset V_m(\rho)$, $\rho = \frac{k}{k+1}$, such that

$$\frac{G'(z)}{G_1'(z)} = h^{\sigma}(z), \tag{9}$$

where σ is given by (6) and $h \in P$.
Also we observe that, for $h \in P$,

$$\begin{aligned}\frac{\partial}{\partial\theta}\arg h(re^{i\theta}) &= \frac{\partial}{\partial\theta}\Re\left\{-i\ln h(re^{i\theta})\right\}\\ &= \Re\left\{\frac{re^{i\theta}h'(re^{i\theta})}{h(re^{i\theta})}\right\},\end{aligned}$$

and so

$$\int_{\theta_1}^{\theta_2}\Re\left\{\frac{re^{i\theta}h'(re^{i\theta})}{h(re^{i\theta})}\right\}d\theta = \arg h(re^{i\theta_2}) - \arg h(re^{i\theta_1}).$$

Hence

$$\begin{aligned}&\max_{h\in P}\left|\int_{\theta_1}^{\theta_2}\Re\left\{\frac{r^ie^{\theta}h'(re^{i\theta})}{h(re^{i\theta})}\right\}d\theta\right|\\ &= \max_{h\in P}\left|\arg h(re^{i\theta_2}) - \arg h(re^{i\theta_1})\right|\\ &\le 2\sin^{-1}\frac{2r}{1-r^2}\\ &= \pi - 2\cos^{-1}\frac{2r}{1-r^2}.\end{aligned} \tag{10}$$

Now differentiating (9) logarithmically and using Lemma 3 together with (10), we obtain

$$\int_{\theta_1}^{\theta_2}\Re\left\{\frac{(zG'(z))'}{G'(z)}\right\}d\theta > -\left(\frac{m-2}{2(k+1)}+\sigma\right)\pi.$$

This completes the proof. □

Theorem 2. *Let $f \in k-UT_m(a,\gamma,\phi)$, $\Re(a) \ge 0, 0 < \gamma \le 1$, $\theta_1 < \theta_2$ and $z = re^{i\theta}$. Then*

$$\int_{\theta_1}^{\theta_2}\Re\left\{p(z)+\frac{zp'(z)}{a+p(z)}\right\}d\theta > -\gamma\left\{\frac{(m-2)}{2(k+1)}+\sigma\right\}\pi,$$

where

$$p(z) = \frac{zf'(z)}{f(z)}.$$

Proof. Let $G(z) = (g*\phi)(z)$. Then, by definition,

$$zf'(z) + af(z) = (a+1)z(G'(z))^{\gamma}, G \in k-UT_m.$$

Differentiating logarithmically, and with simple computations, we have

$$\frac{a+\frac{(zf'(z))'}{f'(z)}}{1+a\frac{f(z)}{zf'(z)}} = \gamma\frac{(zG'(z))'}{G'(z)} + (1-\gamma).$$

That is, with $p(z) = \frac{zf'(z)}{f(z)}$, we have

$$\Re\{p(z)+\frac{zp'(z)}{a+p(z)}\} \ge \gamma\Re\frac{(zG'(z))'}{G'(z)}.$$

Using Theorem 1, we obtain the required result. □

Corollary 1. *For $m \le \left[\frac{2(1-\gamma\sigma)(k+1)}{\gamma}+2\right]$, we use Lemma 1 to have from Theorem 2,*

$$\int_{\theta_1}^{\theta_2}\Re\left\{\frac{zf'(z)}{f(z)}\right\}d\theta > -\pi, \ f \in k-UT_m(a,\gamma,\phi).$$

Corollary 2. *Let $f \in k-UT_m(0,1,\phi)$. Then, for $m \le 2\{(1-\sigma)(k+1)+1\}$,*

$$\int_{\theta_1}^{\theta_2}\Re\left\{\frac{(zf'(z))'}{f'(z)}\right\}d\theta > -\pi,$$

and hence $f(z)$ is univalent in E, see [7].

If $k = 1$, then $\sigma = \frac{1}{2}$ and in this case $f(z)$ is univalent in E for $2 \le m \le 4$.

Theorem 3. *For $0 < \gamma_1 < \gamma_2 \le 1$, $z \in E$,*

$$k-UT_2(a,\gamma_1,\phi) \subset k-UT_2(a,\gamma_2,\phi).$$

Proof. Let $f \in k-UT_2(a,\gamma_1,\phi)$. Then

$$\begin{aligned}zf'(z)+af(z) &= (a+1)z(G'(z))^{\gamma_1}, G(z) = (g*\phi)(z) \in k-UT_2,\\ &= (a+1)z(H'(z))^{\gamma_2},\end{aligned}$$

where

$$H'(z) = (G'(z))^{\frac{\gamma_1}{\gamma_2}}, \quad H = h*\phi.$$

We now show that $H \in k-UT_2$ and this will prove that $f \in k-UT_2(a,\gamma_2,\phi)$.
Now

$$H'(z) = (G'(z))^{\frac{\gamma_1}{\gamma_2}}, \quad G \in k-UT_2, \frac{\gamma_1}{\gamma_2} < 1.$$

Since $G \in k-UT_2$, there exists a function $G_1 = (g_1*\phi) \in k-UV_2$ such that $\frac{G'(z)}{G_1'(z)} \in P(p_k)$ in E.
Let $G_*'(z) = (G_1'(z))^{\frac{\gamma_1}{\gamma_2}}$, $\frac{\gamma_1}{\gamma_2} < 1$.
It is easy to verify that $G_* \in k-UV_2$ in E. Thus

$$\frac{H'(z)}{G_*'(z)} = \left(\frac{G'(z)}{G_1'(z)}\right)^{\frac{\gamma_1}{\gamma_2}} \in P(p_k),$$

since $\frac{\gamma_1}{\gamma_2} < 1$. This completes the proof. □

Remark. From definition 5, the following integral representation for the class $k-UT_m(a,\gamma,\phi)$ can easily be obtained.

A function $f \in k-UT_m(a,\phi,\gamma)$ if and only if there exists a function $G \in k-UT_m(\infty,\gamma,\phi)$, such that

$$f(z)=\frac{a+1}{z^a}\int_0^a z^{a-1}G(t)dt. \tag{11}$$

Theorem 4. *Let $f \in 0-T_m(a,1,\phi)=T_m(a,1,\phi)$. Then f is a Bazilevic function and hence univalent in $|z|<r_m$, where r_m is given by*

$$r_m=\frac{1}{2}\left\{m-\sqrt{m^2-4}\right\}. \tag{12}$$

Proof. We can write, for $f \in T_m(a,1,\phi)$,

$$f(z)=\frac{a+1}{z^a}\int_0^z t^{a-1}F(t)dt,\, F \in T_m(\infty,1,\phi).$$

Let $a=c+id$, $c>0$. Then we have

$$f(z)=\frac{(c+1)+id}{z^{c+id}}\int_0^z t^c p(t)g(t)t^{id-1}dt, \tag{13}$$

where $p \in P$, $g \in 0-UR_m=R_m$.
We define

$$G(z)=z\left(\frac{g(z)}{z}\right)^{\frac{1}{c+1}}.$$

Then

$$\frac{zG'(z)}{G(z)}=\left(1-\frac{1}{c+1}\right)+\frac{1}{c+1}\frac{zg'(z)}{g(z)}.$$

Now $\frac{zg'(z)}{g(z)} \in P_m$ and P_m is a convex set, so $G \in R_m$ and, it is known [20] that $G \in R_m$ is starlike for $|z|<r_m$ where r_m is given by (12).
Further we define $f_1(z)$ as

$$f_1(z)=\left[(c+1+id)\int_0^z G^{c+1}(t)p(t)t^{id-1}dt\right]^{\frac{1}{c+1+id}}.$$

$f_1(z)$ is Bazilevic function, see [1], and hence univalent in $|z|<r_m$. Therefore $\frac{f_1(z)}{z}\neq 0$, $|z|<r_m$.
We note that

$$f_1(z)=z\left(\frac{f(z)}{z}\right)^{\frac{1}{a+1}}, \quad a=c+id.$$

This means that $f(z)$, given by (13), is analytic and for $\left(\frac{f(z)}{z}\right)^{\frac{1}{a+1}}$, it is possible to select uniform branch which takes the value one for $z=0$ and which is analytic for $|z|<r_m$ and also allows us to compute the derivative in $|z|<r_m$. Thus we conclude that $f(z)$ is univalent in $|z|<r_m$, where r_m is given by (12). This completes the proof. □

Theorem 5. *Let $f \in 0-UT_m(\infty,\gamma,\phi)=T_m(\infty,\gamma,\phi)$. Then the radius r_{m_1} of the disc which f maps onto a starlike domain is given by*

$$r_{m_1}=\begin{cases}\frac{1}{2\gamma_1}\left\{m_1-\sqrt{m_1^2-4\gamma_1}\right\}, \gamma\neq\frac{1}{2}, \\ \frac{1}{m_1}, \quad \gamma=\frac{1}{2}.\end{cases}, \tag{14}$$

where $m_1=(m+2)\gamma$ and $\gamma_1=(2\gamma-1)$.

Proof. $f \in T_m(\infty,\gamma,\phi)$ implies that

$$f(z)=z(G'(z))^\gamma, \quad G=(g*\phi)\in T_m. \tag{15}$$

Logarithmic differentiation of (15) gives us

$$\frac{zf'(z)}{f(z)}=\frac{\gamma(zG'(z))'}{G'(z)}+(1-\gamma).$$

Therefore, using a result [11] for $G \in T_m$, we obtain

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\}\geq\frac{(2\gamma-1)r^2-\gamma(m+2)r+1}{1-r^2},$$

and right hand side is positive for $|z|<r_{m_1}$. This proves the result. □

We now investigate the rate of growth of coefficients of $f \in k-UT_m(a,\gamma,\phi)$. Let $f(z)$ be given by (1) and let $g(z)=z+\sum_{n=2}^{\infty} b_n z^n$, $\phi(z)=z+\sum_{n=2}^{\infty} c_n z^n$.
We have:

Theorem 6. *Let $f \in k-UT(a,\gamma,\phi)$. Then, for $m>\left\{\frac{(2-\sigma\gamma)(k+1)}{\sigma}-2\right\}$,*

$$|a_n|\leq C(m,\gamma,k)\left|\frac{a+1}{n+a}\right|n^{\left\{\frac{\gamma}{k+1}\left(\frac{m}{2}+1\right)+\gamma\sigma-1\right\}}, n\to\infty,$$

where $C(m,\gamma,k)$ is constant depending only on m,γ and k.

Proof. We can write

$$zf'(z)+af(z)=(a+1)z(G'(z))^\gamma, \tag{16}$$

where

$$G(z)=(g*\phi)(z)\in k-UT_m.$$

This implies there exists $G_1 \in k-UV_m$ such that

$$G'(z)=(G_1'(z))(h(z))^\sigma. \tag{17}$$

Then, with $z=re^{i\theta}$, we have

$$\begin{aligned}&(n+a)|a_n| \\ &=\frac{1}{2\pi r^n}\left|\int_0^{2\pi}\{zf'(z)+af(z)\}e^{-in\theta}d\theta\right| \\ &=\frac{1}{2\pi r^{n-1}}\left|\int_0^{2\pi}(a+1)(G'(t))^\gamma d\theta\right|, \; G\in k-UT_m.\end{aligned} \tag{18}$$

Now, since $G \in k-UT_m$, there exists $G_1 \in k-UV_m$, such that

$$G'(z) = G_1'(z)h^{\sigma}(z),\, h \in P,$$

and σ is given by (6).
Using Lemma 2 together with the known result [22] that $k-ST \subset S^*(\rho)$, $\rho = \frac{k}{k+1}$, we have

$$G_1'(z) = \frac{\left(\frac{t_1(z)}{z}\right)^{(1-\rho)\left(\frac{m}{4}+\frac{1}{2}\right)}}{(t_2(z)z)^{(1-\rho)\left(\frac{m}{4}-\frac{1}{2}\right)}}, \quad t_1,t_2 \in S^*. \tag{19}$$

Define, for $m > \left\{\frac{2-\sigma\gamma}{\sigma(1-\rho)} - 2\right\}$, $z = re^{i\theta}$.

$$I_\gamma(r) = \int_0^{2\pi} |G'(z)|^\gamma d\theta, \quad G \in k-UT_m.$$

Then, from (19)

$$\begin{aligned} I_\gamma(r) &= \frac{1}{r^{\gamma(1-\rho)}} \int_0^{2\pi} \frac{|t_1(z)|^{\gamma(1-\rho)\left(\frac{m}{4}+\frac{1}{2}\right)}}{|t_2(z)|^{\gamma(1-\rho)\left(\frac{m}{4}-\frac{1}{2}\right)}} |h(z)|^{\gamma\sigma} d\theta \\ &\le \frac{1}{r^{\gamma(1-\rho)}} \left(\frac{4}{r}\right)^{\gamma(1-\rho)\left(\frac{m}{4}-\frac{1}{2}\right)} \\ &\quad \times \int_0^{2\pi} |t_1(z)|^{\gamma(1-\rho)\left(\frac{m}{4}+\frac{1}{2}\right)} |h(z)|^{\gamma\sigma} d\theta, \end{aligned}$$

where we have used well-known distortion result for the starlike function $t_2(z)$. We now apply Holder's inequality, use subordination for starlike functions and a result due to Pommerenke [21] for $h \in P$ to have, for $\sigma(1-\rho)(m+2) > 2-\sigma\gamma$,

$$\begin{aligned} I_\gamma(r) &\le \frac{1}{r^{\gamma(1-\rho)}} \left(\frac{4}{r}\right)^{\gamma(1-\rho)\left(\frac{m}{4}-\frac{1}{2}\right)} \\ &\quad \times \left(\int_0^{2\pi} |t_1(z)|^{\gamma(1-\rho)\left(\frac{m}{4}+\frac{1}{2}\right)}\right)^{\frac{2-\sigma\gamma}{2}} \left(\int_0^{2\pi} |h(z)|^2 d\theta\right)^{\frac{\gamma\sigma}{2}} \\ &\le C(m,\gamma,k)\left(\frac{1}{1-r}\right)^{\frac{\gamma\sigma}{2}} \left(\frac{1}{1-r}\right)^{\gamma(1-\rho)\left(\frac{m}{2}+1\right)+\frac{\gamma\sigma}{2}-1}, \end{aligned}$$

where $C(m,\gamma,k)$ is a constant depending only on m,γ,k. That is

$$I_\gamma(r) = O(1)\left(\frac{1}{1-r}\right)^{\gamma(1-\rho)\left(\frac{m}{2}+1\right)+\frac{\gamma\sigma}{2}-1},$$

Now, with $r = 1-\frac{1}{n}$, we have from (18),

$$|a_n| \le C(m,\gamma,k)\left|\frac{a+1}{n+a}\right| n^{\frac{\gamma(m+2)}{2(k+1)}+\gamma\sigma-1}, \quad (n\to\infty).$$

This completes the proof. □

We note the following special cases.

(i) Let $f \in 1-UT_m\left(\infty, 1, \frac{z}{1-z}\right) = UT_m(\infty,1)$. Then $\rho = \frac{1}{2}$, $\gamma = \frac{1}{2}$ and

$$a_n = O(1)n^{\frac{m}{4}} \;\; for\; m > 4,$$

and, for $f \in 1-UT_m\left(\infty,1,\frac{z}{1-z}\right) = UT_m(\infty,1)$, we get $a_n = O(1).n^{\frac{m}{4}-1}, m > 4$.

(ii) Let $k = 0$ and $f \in T_m\left(\infty,\gamma,\frac{z}{1-z}\right)$. Then, for $m \ge 2$

$$a_n = O(1)n^{\gamma\left(\frac{m}{2}+1\right)+\gamma-1} = O(1)n^{\frac{\gamma m}{2}+2\gamma-1}, (n\to\infty).$$

When we take $\gamma = 1$, then

$$a_n = O(1)n^{\frac{m}{2}+1}.$$

(iii) Let $f \in T_m\left(0,\gamma,\frac{z}{1-z}\right)$. Then, for $m \ge 2$

$$a_n = O(1)n^{\frac{\gamma m}{2}+2\gamma-2},$$

and with $\gamma = 1$, we obtain, for $m \ge 2$

$$a_n = O(1)n^{\frac{m}{2}}, \quad (n\to\infty).$$

This result is proved in [11]. See also [15].

(iv) Let $f \in k-UT_m(\infty,\gamma,\log(1-z))$. Then, for $m > 2\left\{\frac{2-\gamma\sigma}{\gamma(1-\rho)} - 1\right\}$,

$$a_n = O(1)n^{\frac{\gamma(m+2)}{2(k+1)}+\gamma\sigma-2}.$$

With $k = 1$, $\gamma = 1$, we have $\sigma = \frac{1}{2}$ and so, for $m > 4$

$$a_n = O(1)n^{\frac{m}{4}-1}, \quad (n\to\infty).$$

Also, if we take $k = 0, \gamma = 1$. Then $\sigma = 1$ and so

$$a_n = O(1)n^{\frac{m}{2}}, \quad (n\to\infty).$$

Theorem 7. *Let $f \in k-UT_m(0,1,\phi)$. Denote by $L(r,f)$, the length of the image of the circle $|z| = r$ under f, by $A(r,f)$, the area of $f(|z|<r)$ and $M(r,f) = \max_\theta |f(re^{i\theta})|$. Then*

$$L(r,f) = O(1)M(r,f)\log\frac{1}{1-r},$$

where $O(1)$ is a constant.

Proof. Since $f \in k-UT_m(0,1,\phi)$, we have

$$zf'(z) = zG'(z),\; G(z) = (g*\phi)(z) \in k-UT_m.$$

This implies that $f \in k-UT_m$. So there exists $G_1 \in k-UV_m \subset V_m(\rho)$ such that

$$\frac{G'}{G_1'} = p \in P(p_k) \subset P(\rho).$$

Now, with $z = re^{i\theta}$,

$$\begin{aligned} L(r,f) &= \int_0^{2\pi} |zf'(z)|d\theta \\ &= \int_0^{2\pi} |zG'(z)|d\theta \\ &= \int_0^{2\pi} |zG_1'(z)p(z)|d\theta, \quad G_1 V_m(\rho), p \in P(\rho) \\ &\le \int_0^{2\pi}\int_0^{r} |G_1'(z)p(z)|H(z) + G_1'(z)(zp'(z))d\xi d\theta, \\ &\qquad \left(H(z) = \frac{(zG_1'(z))'}{G_1'(z)}\right) \\ &\le \int_0^{r}\int_0^{2\pi} |f'(z)H(z)|d\theta d\xi + \int_0^{r}\int_0^{2\pi} |zp'(z)G_1'(z)|d\theta d\xi \\ &= I_1(r) + I_2(r). \end{aligned} \tag{20}$$

Now

$$I_1(r) = \int_0^{r}\int_0^{2\pi} |f'(z)H(z)|d\theta d\xi,$$

where

$$H(z) = \frac{(zG_1'(z))'}{G_1'(z)} = 1 + \sum_{n=1}^{\infty} d_n z^n,$$

$f(z)$ given by (1), $|d_n| \le m\left(1 - \frac{k}{k+1}\right) = \frac{m}{k+1}$, and for $n \ge 1$, we have

$$\begin{aligned} &I_1(r) \\ &\le \int_0^{r} \left[\left(\int_0^{2\pi} |f'(z)|^2 d\theta\right)^{\frac{1}{2}} \left(\int_0^{2\pi} |H(z)|^2 d\theta\right)^{\frac{1}{2}}\right] d\xi \\ &= 2\pi \int_0^{r} \left(\sum_{n=1}^{\infty} n^2 |a_n|^2 \xi^{2n-2}\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} |\alpha_n|^2 \xi^{2n}\right)^{\frac{1}{2}} d\xi \\ &\le \sqrt{2}\left(\frac{m}{k+1}\right)\pi \left(\sum_{n=1}^{\infty} \frac{n^2}{2n-1}|a_n|^2 r^{2n-1}\right)^{\frac{1}{2}} \left(\log\frac{1+r}{1-r}\right)^{\frac{1}{2}} \\ &\le \sqrt{2}\left(\frac{m}{k+1}\right)\pi \left(\sum_{n=1}^{\infty} n|a_n|^2 r^{2n-1}\right)^{\frac{1}{2}} \left(\log\frac{1+r}{1-r}\right)^{\frac{1}{2}}. \end{aligned}$$

But $A(r,f) = \pi \sum_{n=1}^{\infty} n|a_n|^2 r^{2n}$ is the area of the image of $|z| < r$ by $w = f(z)$. Therefore

$$I_1(r) \le \sqrt{2}\left(\frac{m}{k+1}\right)\pi \left(\frac{A(r,f)}{\pi r}\right)^{\frac{1}{2}} \left(\log\frac{1+r}{1-r}\right)^{\frac{1}{2}}.$$

Also, since $A(r,f) \le \pi M^2(r,f)$, we have

$$I_1(r) \le \sqrt{2}\left(\frac{m}{k+1}\right) M(r,f) \left(\frac{1}{r}\log\frac{1+r}{1-r}\right)^{\frac{1}{2}}. \tag{21}$$

We now estimate $I_2(r)$.

$p \in P(\rho), \rho = \frac{k}{k+1}$, implies that we can write

$$p(z) = \frac{1-\rho}{2\pi}\int_0^{2\pi} \frac{1+ze^{it}}{1-ze^{it}} d\mu(t), \quad \int_0^{2\pi} d\mu(t) = 2\pi.$$

So

$$p'(z) = \frac{1-\rho}{\pi}\int_0^{2\pi} \frac{e^{it}}{(1-ze^{it})^2} d\mu(t).$$

Therefore

$$I_2(r) \le \frac{1-\rho}{\pi}\int_0^{r}\int_0^{2\pi}\int_0^{2\pi} \frac{|zG_1'(z)|}{|1-ze^{it}|^2} d\mu(t)d\theta d\xi.$$

Also

$$\Re p(z) = \frac{1-\rho}{\pi}\int_0^{2\pi} \frac{1-\xi^2}{|1-ze^{it}|^2} d\mu(t),$$

and hence

$$\begin{aligned} I_2(r) &\le 2(1-\rho)\int_0^{r}\int_0^{2\pi} |zG'(z)|\Re H(z) d\theta \frac{d\xi}{1-\xi^2} \\ &= 2(1-\rho)\int_0^{2\pi} \Re\{zG'(z)e^{-i\arg zG_1'}\}d\theta \frac{d\xi}{1-\xi^2}. \end{aligned}$$

Integrating by parts gives us

$$I_2(r) \le [2m(1-\rho) + 2\rho]\pi \int_0^{r} \frac{M(r,f)}{1-\xi^2} d\xi. \tag{22}$$

From (20), (21) and (22), we obtain the desired result. □

We study arc-length problem with a different technique as follows.

Theorem 8. *Let $f \in k - UT_m(0,\gamma,\phi)$. Then, for $m > \left\{\frac{(2-\sigma\gamma)(k+1)}{\gamma} - 2\right\}$,*

$$L(r,f) = O(1)\left(\frac{1}{1-r}\right)^{\frac{\gamma}{k+1}\left(\frac{m}{2}+1\right)+\sigma\gamma-1}, (r \to 1),$$

where σ is given by (6) and $O(1)$ is a constant.

Proof. We can write

$$\begin{aligned} zf'(z) &= z(G'(z))^{\gamma},\ G=(g*\phi)\in k-UT_m \\ &= z(G'(z)h^{\sigma}(z))^{\gamma},\ G=(g*\phi)\in k-UT_m, h\in P \\ &= \frac{z\left(\frac{s_1(z)}{z}\right)^{\gamma\left(\frac{m}{4}+\frac{1}{2}\right)}}{\left(\frac{s_2(z)}{z}\right)^{\gamma\left(\frac{m}{4}-\frac{1}{2}\right)}} h^{\sigma\gamma}(z), s_1,s_2\in k-ST, \quad (23) \end{aligned}$$

by using Lemma 2.
Also $s_i \in k-ST$ implies that $s_i \in S^*(\rho), \rho = \frac{k}{k+1}, i=1,2.$
Therefore, for $z = re^{i\theta}$

$$L(r,f) = \int_0^{2\pi} |zf'(z)|d\theta = \int_0^{2\pi} \frac{|s_1(z)|^{\frac{\gamma}{k+1}\left(\frac{m}{4}+\frac{1}{2}\right)}}{|s_2(z)|^{\frac{\gamma}{k+1}\left(\frac{m}{4}-\frac{1}{2}\right)}} |h(z)|^{\sigma\gamma} d\theta.$$

Since $s_2(z)$ is starlike and hence univalent, so we have

$$L(r,f) \le \left(\frac{4}{r}\right)^{\frac{\gamma}{k+1}\left(\frac{m}{4}-\frac{1}{2}\right)} \left(\int_0^{2\pi} |s_1(z)|^{\frac{\gamma}{k+1}\left(\frac{m}{4}+\frac{1}{2}\right)} |h(z)|^{\sigma\gamma} d\theta\right).$$

Holder's inequality together with subordination for starlike functions, we have

$$\begin{aligned} &L(r,f) \\ &\le \left(\frac{4}{r}\right)^{\frac{\gamma}{k+1}\left(\frac{m}{4}-\frac{1}{2}\right)} \left(\int_0^{2\pi} |h(z)|^2 d\theta\right)^{\frac{\sigma\gamma}{2}} \\ &\left(\int_0^{2\pi} \left(\frac{r}{|1-re^{i\theta}|}\right)^{\frac{2\gamma}{k+1}\left(\frac{m}{4}+\frac{1}{2}\right)\frac{2}{2-\sigma\gamma}} d\theta\right)^{\frac{2-\sigma\gamma}{2}} \\ &\le O(1)\left(\frac{1}{1-r}\right)^{\frac{\gamma}{k+1}\left(\frac{m}{2}+1\right)+\sigma\gamma-1}, \end{aligned}$$

for $\frac{\gamma(m+2)}{k+1} > (2-\sigma\gamma)$. This completes the proof. □

As special cases, we note the following

(i) For $\gamma = 1, m = 2$ and $k = 1$, which gives us $\sigma = \frac{1}{2}$. This gives us

$$L(r,f) = O(1)\left(\frac{1}{1-r}\right)^{\frac{1}{2}}.$$

(ii) We take $k = 0$ and $\gamma = 1$. Then $\sigma = 1$ and $f \in T_m$. This gives us

$$L(r,f) = O(1)\left(\frac{1}{1-\gamma}\right)^{\frac{m}{2}+1}, \quad (r\to 1).$$

We shall estimate the growth rate of $H_q(n)$ for the functions in the class $UT_m(0,\gamma,\phi)$. This is the main motivation of next result.

Let $f \in A$ and be given by (1). Suppose that the qth Hankel determinant of f is defined for $q \ge 1, n \ge 1$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & \vdots \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & \dots & \dots & a_{n+2q-2} \end{vmatrix} \quad (24)$$

Theorem 9. *Let $f \in UT_m(0,\gamma,\phi)$ and let the qth Hankel determinant of $f(z)$ for $q \ge 1, n \ge 1$, be defined by (24). The, for $m \ge \left\{\frac{8q}{\gamma} - 2\right\}$,*

$$H_q(n) = O(1)n^{\left(\frac{m\gamma}{4}+\gamma-1\right)q-q^2}, (n\to\infty),$$

where $O(1)$ is a constant depending upon γ, m and q only.

To prove Theorem 9, we need the following results and for these we refer to [9].

Lemma 4. *Let $f \in A$ and be given by (1) and let the qth Hankel determinant of f be defined by (24). Then, writing $\Delta_j(n) = \Delta_j(n,z_1,f)$. We have*

$$\begin{aligned} &H_q(n) \\ &= \begin{vmatrix} \Delta_{2q-1}(n) & \Delta_{2q-3}(n+1) & \dots & \Delta_{q-1}(n+q-1) \\ \Delta_{2q-3}(n+1) & \Delta_{2q-4}(n+2) & \dots & \Delta_{q-2}(n+q) \\ \vdots & \vdots & & \vdots \\ \Delta_{q-1}(n+q-1) & \dots & \dots & \Delta_q(n+2q-2) \end{vmatrix}, \quad (25) \end{aligned}$$

where, with $\Delta_0(n,z_1,f) = a_n$, we define for $j \ge 1$,

$$\Delta_j(n,z,f) = \Delta_{j-1}(n,z_1,f) - z_1\Delta_{j-1}(n+1,z_1,f). \quad (26)$$

Lemma 5. *With $x = \left(\frac{n}{n+1}y\right), v \ge 0$ and integer*

$$\begin{aligned} &\Delta_j(n+v,v,x,zf'(z)) \\ &= \sum_{l=0}^{j} \binom{j}{l} \frac{y^l(v-(l-1)n)}{(n+1)^l} \Delta_{j-l}(n+v,v+l,y,f). \end{aligned}$$

Proof(Theorem 9). Since $f \in UT_m(0,\gamma,\phi)$, we can write

$$zf'(z) = z(G'(z))^{\gamma}, g\in UT_m, G=(g*\phi). \quad (27)$$

Now, for $G \in UT_m$, there exists $G_1 \in UV_m \subset V_m\left(\frac{1}{2}\right)$ such that $\frac{G'}{G_1'} \in P(p_1)$. Also, for $p \in P(p_1)$, we have $|\arg p(z)| < \frac{\pi}{4}$ which gives us $\sigma = \frac{1}{2}$.
Thus we can write (27) as

$$\begin{aligned} f'(z) &= [(G_2'(z))^{\frac{1}{2}} p^{\frac{1}{2}}(z)]^{\gamma} \\ &= (G_1'(z)p(z))^{\frac{\gamma}{2}}, G_2 \in V_m, p \in P \\ &= \left[\frac{\left(\frac{s_1(z)}{2}\right)^{\frac{\gamma}{2}\left(\frac{m}{4}+\frac{1}{2}\right)}}{\left(\frac{s_2(z)}{2}\right)^{\frac{\gamma}{2}\left(\frac{m}{4}-\frac{1}{2}\right)}}\right](p(z))^{\frac{\gamma}{2}}, \end{aligned}$$

where $s_1, s_2 \in S^*$, where we have used a result due to Brannan [2]. Also we can choose a z_1 with $|z_1| = r$ such that for any univalent functions $s(z)$

$$\max_{|z|=r} |(z - z_1)s(z)| \le \frac{2r^2}{1-r^2}, \tag{28}$$

see [3].

Now, for $j \ge 0, z_1$ any nonzero complex number, consider

$$\Delta_j(n, z_1, f'(z)) = \frac{1}{2\pi r^{n+j}} \left| \frac{(z-z_1)^j \left(\frac{s_1(z)}{z}\right)^{\frac{\gamma}{2}\left(\frac{m}{4}+\frac{1}{2}\right)}}{\left(\frac{s_2(z)}{z}\right)^{\frac{\gamma}{2}\left(\frac{m}{4}-\frac{1}{2}\right)}} (p(z))^{\frac{\gamma}{2}} d\theta \right|.$$

Thus, for $\gamma(m+2) \ge 8(j+1)$,

$$\begin{aligned}
&\Delta_j(n, z_1, f') \\
&\le \frac{1}{2\pi r^{n+j-1}} \int_0^{2\pi} |z - z_1|^j \frac{|s_1(z)|^{\frac{\gamma}{2}\left(\frac{m}{4}+\frac{1}{2}\right)}}{|s_2(z)|^{\frac{\gamma}{2}\left(\frac{m}{4}-\frac{1}{2}\right)} |p(z)|^{\frac{\gamma}{2}} d\theta} \\
&\le \frac{1}{r^{n+j-1}} \left(\frac{2r^2}{1-r^2}\right)^j \left(\frac{4}{r}\right)^{\frac{\gamma}{2}\left(\frac{m}{4}-\frac{1}{2}\right)} \\
&\quad \times \left[\frac{1}{2\pi} \int_0^{2\pi} |s_1(z)|^{\frac{\gamma}{2}\left(\frac{m}{4}+\frac{1}{2}\right)-j} |p(z)|^{\frac{\gamma}{2}} d\theta\right] \\
&\le \frac{1}{r^{n+j-1}} \left(\frac{2r^2}{1-r^2}\right)^j \left(\frac{4}{r}\right)^{\frac{\gamma}{2}\left(\frac{m}{4}-\frac{1}{2}\right)} \left[\frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 d\theta\right]^{\frac{\gamma}{4}} \\
&\quad \times \left[\frac{1}{2\pi} \int_0^{2\pi} |s_1(z)|^{\frac{\gamma}{2}\left(\frac{m}{4}+\frac{1}{2}\right)-j} \frac{4}{4-\gamma}\right]^{\frac{4-\gamma}{4}} \\
&\le C(m, \gamma, j) \left(\frac{2r^2}{1-r^2}\right)^j \left(\frac{1+3r^2}{1-r^2}\right)^{\frac{\gamma}{4}} \\
&\quad \times \left(\frac{1}{1-r}\right)^{\left\{\left[\gamma\left(\frac{m}{4}+\frac{1}{2}-2j\right)\right]\frac{4}{4-\gamma}-1\right\}\frac{4-\gamma}{4}} \\
&= O(1) \left(\frac{1}{1-r}\right)^{\frac{m\gamma}{4}-j+\gamma-1},
\end{aligned}$$

$O(1)$ is a constant and we have used (28), distortion results for starlike functions, Holder's inequality and a result for $h \in P$, see [21].

Choosing $r = 1 - \frac{1}{n}$, we have, for $\gamma(m+2) \ge 8(j+1)$,

$$\Delta_j(n, z_1, f') = O(1).n^{\frac{m\gamma}{4}+\gamma-j-1},$$

and using Lemma 5, we obtain

$$\Delta_j(n, e^{i\theta_n}, f) = O(1)n^{\frac{m\gamma}{4}+\gamma-j-2}, \quad (n \to \infty). \tag{29}$$

We use Lemma 4 and follow the similar argument given in [9], to have

$$H_q(n) = O(1)n^{\left(\frac{m\gamma}{4}+\gamma-1\right)q-q^2}, \quad (n \to \infty)$$

for $\gamma(m+2) \ge 8q$.

This completes the proof. □

Special Case.

When $\gamma = 1$, $m \ge 6$, we have $a_n = O(1)n^{\frac{m}{4}-1}$ and

$$H_q(n) = O(1)n^{\left(\frac{m}{4}\right)q-q^2}, \ n \to \infty.$$

For this case we note that

$$H_2(n) = O(1)n^{\frac{m}{2}-4}, \quad m \ge 14.$$

Acknowledgement

The authors are grateful to Dr. S. M. Junaid Zaidi, Rector, COMSATS Institute of Information Technology, Pakistan for providing excellent research and academic environment.

References

[1] I. E. Bazilevic, On a class of integrability in quadratures of the Loewner-Kufarev equation, Math. Sb., **37**, 471-476 (1955).

[2] D. A Brannan, On functions of bounded boundary rotations, Proc. Edin. Math. Soc., **2**, 339-347 (1968-1969).

[3] G. Golusin, On distortion theorems and coefficients of univalent functions, Mat. Sb., **19**, 183-203 (1946).

[4] A. W. Goodman, Univalent Functions, Vol. I,II, Polygonal Publishing House, Washington, New Jersey, (1983).

[5] S. Kanas, Techniques of differential subordination for domains bounded by conic sections, Internat. J. Math. Math. Sci., **38**, 2389-2400 (2003).

[6] S. Kanas, A. Wisniowska, Conic regions and k-uniform convexity, J. Comput. Appl. Math., **105**, 327-336 (1999).

[7] W. Kaplan, Close-to-convex Schlicht functions, Michigan Math. J., **1**, 169-185 (1952).

[8] E. J. Moulis, Generalizations of the Robertson functions, Pacific J. Math., **81**, 169-174 (1979).

[9] J. W. Noonan, D. K. Thomas, On Hankel determinant of areally mean p-valent functions, Proc. Lond. Math. Soc., **25**, 503-524 (1972).

[10] K. I. Noor, On subclasses of close-to-convex functions of higher order, Internat. J. Math. Math. Sci., **6**, 327-334 (1983).

[11] K. I. Noor, On generalizations of close-to-convexity, Mathematica, **23**, 217-219 (1981).

[12] K. I. Noor, Some recdent developments in the theory of functions with bounded boundary rotation, Inst. Math. Comput. Sci., **2**, 25-37 (2007).

[13] K. I. Noor, W. Ul-Haq, M. Arif, S. Mustafa, On bounded boundary and bounded radius rotation, J. Inequal. Appl., **2009**, Article ID 813687, 12 pages (2009).

[14] K. I. Noor, Some properties of analytic functions with bounded radius rotation, Complex Var. Elliptic, Equat., **54**, 865-877 (2009).

[15] K. I. Noor, Higher order close-to-convex functions, Math. Japonica, **37**, 1-8 (1992).

[16] K. I. Noor, On a generalization of uniformaly convex and related functions, Comput. Math. Appl., **61**, 117-125 (2011).

[17] K. I. Noor, On quasi-convex univalent functions and related topics, Internat. J. Math. Sci., **2**, 241-258 (1987).

[18] K. I. Noor, R. Fayyaz, M. A. Noor Some classes of *k*-uniformly functions with bounded radius rotation, Appl. Math. Inf. Sci., **8**, No. 2, 527-533 (2014).

[19] R. Parvatham, S. Radha, On certain classes of analytic functions, Ann. Polon. Math., **49**, 31-34 (1980).

[20] B. Pinchuk, Functions of bounded boundary rotation, Isr. J. Math., **10**, 7-16 (1971).

[21] Ch. Pommerenke, On starlike and close-to-convex functions, Proc. London Math. Soc., **3**, 290-304 (1963).

[22] F. Ronning, Uniform convex functions and a corresponding class of starlike functions, Proc. Amer. Math. Soc., **118**, 189-196 (1993).

Khalida Inayat Noor is a leading world-known figure in mathematics. She obtained her PhD from University of Wales (Swansea)(UK). She has a vast experience of teaching and research at university levels in various countries including Iran, Pakistan, Saudi Arabia, Canada and United Arab Emirates. She was awarded HEC best research paper award in 2009 and CIIT Medal for innovation in 2009. She has been awarded by the President of Pakistan: President's Award for pride of performance on August 14, 2010 for her outstanding contributions in mathematical sciences and other fields. Her field of interest and specialization is Complex analysis, Geometric function theory, Functional and Convex analysis. She introduced a new technique, now called as Noor Integral Operator which proved to be an innovation in the field of geometric function theory and has brought new dimensions in the realm of research in this area. She has been personally instrumental in establishing PhD/MS programs at COMSATS Institute of Information Technology, Pakistan. Dr. Khalida Inayat Noor has supervised successfully several Ph.D and MS/M.Phil students. She has been an invited speaker of number of conferences and has published more than 400 (Four hundred) research articles in reputed international journals of mathematical and engineering sciences. She is member of editorial boards of several international journals of mathematical and engineering sciences.

M. Aslam Noor earned his PhD degree from Brunel University, London, UK (1975) in the field of Applied Mathematics (Numerical Analysis and Optimization). He has vast experience of teaching and research at university levels in various countries including Pakistan, Iran, Canada, Saudi Arabia and United Arab Emirates. His field of interest and specialization is versatile in nature. It covers many areas of Mathematical and Engineering sciences such as Variational Inequalities, Operations Research and Numerical Analysis. He has been awarded by the President of Pakistan: President's Award for pride of performance on August 14, 2008, in recognition of his contributions in the field of Mathematical Sciences. He was awarded HEC Best Research paper award in 2009. He has supervised successfully several Ph.D and MS/M.Phil students. He is currently member of the Editorial Board of several reputed international journals of Mathematics and Engineering sciences. He has more than 750 research papers to his credit which were published in leading world class journals.