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Coupled Coincidence Point Results for Mixed (G,S)-Monotone Mapping and Applications

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Abstract: We introduce the concept of mixed (G,S)-monotone mappings and prove coupled coincidence point theorems for such mappings satisfying a nonlinear contraction involving altering distance functions. Presented theorems extend, improve and generalize the recent results of Harjani, López and Sadarangani [J. Harjani, B. López and K. Sadarangani, Fixed point theorems for mixed monotone operators and applications to integral equations, Nonlinear Anal. 74 (2011) 1749-1760] and other existing results in the literature. As application, we present an existence theorem for solutions to a system of nonlinear integral equations.

Keywords: Coincidence point, (G,S)-monotone mapping, ordered set, altering distance, integral equations.

1 Introduction and preliminaries

Fixed point problems of contractive mappings in metric spaces endowed with a partially order have been studied by many authors (see [1]-[17]). Bhaskar and Lakshmikantham [3] introduced the concept of a coupled fixed point and studied the problems of a uniqueness of a coupled fixed point in partially ordered metric spaces and applied their theorems to problems of the existence of solution for a periodic boundary value problem. In [8], Lakshmikantham and Ćirić established some coincidence and common coupled fixed point theorems under nonlinear contractions in partially ordered metric spaces. Very recently, Harjani, López and Sadarangani [7] obtained some coupled fixed point theorems for a mixed monotone operator in a complete metric space endowed with a partial order by using altering distance functions. They applied their results to the study of the existence and uniqueness of a nonlinear integral equation. Now, we briefly recall various basic definitions and facts.

Definition 11(*see Bhaskar and Lakshmikantham [3]*). *Let $(X,\preceq)$ be a partially ordered set and $F : X\times X \to X$. Then the map F is said to have mixed monotone property if $F(x,y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is,*

$$x_1 \preceq x_2 \ \text{ implies } \ F(x_1,y) \preceq F(x_2,y) \text{ for all } y\in X$$

and

$$y_1 \preceq y_2 \ \text{ implies } \ F(x,y_2) \preceq F(x,y_1) \text{ for all } x\in X.$$

The main result obtained by Bhaskar and Lakshmikantham [3] is the following.

Theorem 11(*see Bhaskar and Lakshmikantham [3]*). *Let $(X,\preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Let $F : X\times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists $k\in[0,1)$ such that*

$$d(F(x,y),F(u,v)) \le \tfrac{k}{2}[d(x,u)+d(y,v)]$$

for each $u\preceq x$ and $y\preceq v$.

Suppose either F is continuous or X has the following properties:

(i) if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,
(ii) if a non-increasing sequence $x_n \to x$, then $x \preceq x_n$ for all n.

If there exist $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0,y_0) \text{ and } F(y_0,x_0) \preceq y_0,$$

then F has a coupled fixed point.

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Inspired by Definition 11, Lakshmikantham and Ćirić in [8] introduced the concept of a g-mixed monotone mapping.

Definition 12(*see Lakshmikantham and Ćirić [8]). Let $(X,\preceq)$ be a partially ordered set, $F: X\times X\to X$ and $g: X\to X$. Then the map F is said to have mixed g-monotone property if $F(x,y)$ is monotone g-non-decreasing in x and is monotone g-non-increasing in y, that is,*

$$gx_1 \preceq gx_2 \text{ implies } F(x_1,y)\preceq F(x_2,y) \text{ for all } y\in X$$

and

$$gy_1 \preceq gy_2 \text{ implies } F(x,y_2)\preceq F(x,y_1) \text{ for all } x\in X.$$

Definition 13(*Lakshmikantham and Ćirić [8]). Let X be a non-empty set, and let $F: X\times X\to X$, $g: X\to X$ be given mappings. An element $(x,y)\in X\times X$ is called a coupled common fixed point of the mappings F and g if $F(x,y)=gx=x$ and $F(y,x)=gy=y$.*
An element $(x,y)\in X\times X$ is called a coupled coincidence point of the mappings F and g if $F(x,y)=gx$ and $F(y,x)=gy$.

Definition 14(*Lakshmikantham and Ćirić [8]). Let X be a non-empty set. Then we say that the mappings $F: X\times X\to X$ and $g: X\to X$ are commutative if for all $x,y\in X$*

$$g(F(x,y))=F(gx,gy).$$

The main result of Lakshmikantham and Ćirić [8] is the following.

Theorem 12(*Lakshmikantham and Ćirić [8]). Let $(X,\preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Assume there is a function $\phi: [0,+\infty)\to[0,+\infty)$ with $\phi(t)<t$ and $\lim_{r\to t^+}\phi(r)<t$ for each $t>0$ and also suppose $F: X\times X\to X$ and $g: X\to X$ are such that F has the mixed g-monotone property and*

$$d(F(x,y),F(u,v))\leq \phi\left(\frac{d(gx,gu)+d(gy,gv)}{2}\right)$$

for all $x,y,u,v\in X$ with $gx\preceq gu$ and $gv\preceq gy$. Assume that $F(X\times X)\subseteq g(X)$, g is continuous and commutes with F and also suppose either F is continuous or X has the following properties:

(i)if a non-decreasing sequence $x_n\to x$, then $x_n\preceq x$ for all n,
(ii)if a non-increasing sequence $x_n\to x$, then $x\preceq x_n$ for all n.

If there exist $x_0,y_0\in X$ such that $gx_0\preceq F(x_0,y_0)$ and $F(y_0,x_0)\preceq gy_0$ then there exist $x,y\in X$ such that $gx=F(x,y)$ and $gy=F(y,x)$, that is, F and g have a coupled coincidence point.

Recently, Harjani, López and Sadarangani [7] established coupled fixed point theorems for a mixed monotone operator satisfying contraction involving altering distance functions in a complete partially ordered metric space.
Denote by $\mathscr{F}$ the set of functions $\varphi: [0,+\infty)\to[0,+\infty)$ satisfying the following properties:
(a) φ is continuous and non-decreasing,
(b) $\varphi(t)=0$ if and only if $t=0$.
The functions $\varphi\in\mathscr{F}$ satisfying these properties are called altering distance functions.

Theorem 13(*Harjani, López and Sadarangani [7]). Let $(X,\preceq)$ be a partially ordered set and d be a metric on X such that (X,d) is a complete metric space. Let $F: X\times X\to X$ be a mapping having the mixed monotone property on X and satisfying*

$$\varphi(d(F(x,y),F(u,v)))\leq\varphi(\max\{d(x,u),d(y,v)\}) \\ -\Phi(\max\{d(x,u),d(y,v)\})$$

for all $x,y,u,v\in X$ with $u\preceq x$ and $y\preceq v$, where $\varphi,\psi\in\mathscr{F}$. Suppose either F is continuous or X has the following properties:

(i)if a non-decreasing sequence $x_n\to x$, then $x_n\preceq x$ for all n,
(ii)if a non-increasing sequence $x_n\to x$, then $x\preceq x_n$ for all n.

If there exist $x_0,y_0\in X$ such that $x_0\preceq F(x_0,y_0)$ and $F(y_0,x_0)\preceq y_0$ then F has a coupled fixed point.

In this paper, we introduce the concept of mixed (G,S)-monotone mappings and prove coupled coincidence point theorems for such mappings satisfying a nonlinear contraction involving altering distance functions. Presented theorems extend, improve and generalize the results of Harjani, López and Sadarangani [7]. We end this paper by the study of the existence of solutions to a system of nonlinear integral equations.

2 Main Results

First, we introduce the concept of mixed (G,S)-monotone property.

Definition 21*Let X be a non-empty set endowed with a partial order $\preceq$. Consider the mappings $F: X\times X\to X$ and $G,S: X\to X$. We say that F has the mixed (G,S)-monotone property on X if for all $x,y\in X$,*

$$\begin{aligned} &x_1,x_2\in X, \quad G(x_1)\preceq S(x_2)\Rightarrow F(x_1,y)\preceq F(x_2,y),\\ &x_1,x_2\in X, \quad G(x_1)\succeq S(x_2)\Rightarrow F(x_1,y)\succeq F(x_2,y),\\ &y_1,y_2\in X, \quad G(y_1)\preceq S(y_2)\Rightarrow F(x,y_1)\succeq F(x,y_2),\\ &y_1,y_2\in X, \quad G(y_1)\succeq S(y_2)\Rightarrow F(x,y_1)\preceq F(x,y_2). \end{aligned}$$

Remark 1*If we take $G=S$, then F has the mixed (G,S)-monotone property implies that F has the mixed G-monotone property.*

Now, we state and prove our first result.

Theorem 21*Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let $G,S:X\to X$ and $F:X\times X\to X$ be a mapping having the mixed (G,S)-monotone property on X. Suppose that*

$$\varphi(d(F(x,y),F(u,v)))\leq\varphi(\max\{d(Gx,Su),d(Gy,Sv)\}) -\phi(\max\{d(Gx,Su),d(Gy,Sv)\}), \quad (1)$$

for all $x,y,u,v\in X$ with $G(x)\preceq S(u)$ or $G(x)\succeq S(u)$ and $S(y)\succeq G(v)$ or $S(y)\preceq G(v)$, where $\varphi,\phi\in\mathscr{F}$. Assume that $F(X\times X)\subseteq G(X)\cap S(X)$ and assume also that G,S and F satisfy the following hypotheses:

(I)F, G and S are continuous,
(II)F commutes respectively with G and S.

If there exist x_0, y_0, x_1 and y_1 such that

$$\begin{cases} G(x_0)\preceq S(x_1)\preceq F(x_0,y_0);\\ G(y_0)\succeq S(y_1)\succeq F(y_0,x_0),\end{cases}$$

then there exist $x,y\in X$ such that

$$G(x)=S(x)=F(x,y)\quad\text{and}\quad G(y)=S(y)=F(y,x),$$

that is, G,S and F have a coupled coincidence point $(x,y)\in X\times X$.

Proof. Let $x_0,y_0,x_1,y_1\in X$ such that

$$G(x_0)\preceq S(x_1)\preceq F(x_0,y_0)\ \text{ and }\ G(y_0)\succeq S(y_1)\succeq F(y_0,x_0).$$

Since $F(X\times X)\subseteq G(X)\cap S(X)$, we can choose $x_2,y_2,x_3,y_3\in X$ such that

$$\begin{cases} G(x_2)=F(x_0,y_0)\\ G(y_2)=F(y_0,x_0)\end{cases}$$

and

$$\begin{cases} S(x_3)=F(x_1,y_1)\\ S(y_3)=F(y_1,x_1)\end{cases}.$$

Continuing this process we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\begin{cases} G(x_{2n+2})=F(x_{2n},y_{2n})\\ G(y_{2n+2})=F(y_{2n},x_{2n})\end{cases};\ \begin{cases} S(x_{2n+3})=F(x_{2n+1},y_{2n+1})\\ S(y_{2n+3})=F(y_{2n+1},x_{2n+1})\end{cases} \quad (2)$$

for all $n\geq 0$.
We shall show that for all $n\geq 0$,

$$G(x_{2n})\preceq S(x_{2n+1})\preceq G(x_{2n+2}) \quad (3)$$

and

$$G(y_{2n})\succeq S(y_{2n+1})\succeq G(y_{2n+2}). \quad (4)$$

As $G(x_0)\preceq S(x_1)\preceq F(x_0,y_0)=G(x_2)$ and $G(y_0)\succeq S(y_1)\succeq F(y_0,x_0)=G(y_2)$, our claim is satisfied for $n=0$.

Suppose that (3) and (4) hold for some fixed $n>0$. Since $G(x_{2n})\preceq S(x_{2n+1})\preceq G(x_{2n+2})$ and $G(y_{2n})\succeq S(y_{2n+1})\succeq G(y_{2n+2})$, and as F has the mixed (G,S)-monotone property, we have

$$\begin{aligned}G(x_{2n+2})=F(x_{2n},y_{2n})&\preceq F(x_{2n+1},y_{2n})\\ &\preceq F(x_{2n+1},y_{2n+1})\preceq F(x_{2n+2},y_{2n+1})\\ &\preceq F(x_{2n+2},y_{2n+2}),\end{aligned}$$

then

$$G(x_{2n+2})\preceq S(x_{2n+3})\preceq G(x_{2n+4}).$$

On the other hand,

$$\begin{aligned}G(y_{2n+2})=F(y_{2n},x_{2n})&\succeq F(y_{2n+1},x_{2n})\\ &\succeq F(y_{2n+1},x_{2n+1})\succeq F(y_{2n+2},x_{2n+1})\\ &\succeq F(y_{2n+2},x_{2n+2}),\end{aligned}$$

then

$$G(y_{2n+2})\succeq S(y_{2n+3})\succeq G(y_{2n+4}).$$

Thus by induction, we proved that (3) and (4) hold for all $n\geq 0$.

We complete the proof in the following steps:

Step 1: We will prove that

$$\lim_{n\to+\infty}d(F(x_n,y_n),F(x_{n+1},y_{n+1}))= \lim_{n\to+\infty}d(F(y_n,x_n),F(y_{n+1},x_{n+1}))=0. \quad (5)$$

From (3), (4) and (1), we have

$$\begin{aligned}&\varphi(d(F(x_{2n},y_{2n}),F(x_{2n+1},y_{2n+1})))\\ &\leq\varphi(\max\{d(Gx_{2n},Sx_{2n+1}),d(Gy_{2n},Sy_{2n+1})\})\\ &-\phi(\max\{d(Gx_{2n},Sx_{2n+1}),d(Gy_{2n},Sy_{2n+1})\})\\ &\leq\varphi(\max\{d(Gx_{2n},Sx_{2n+1}),d(Gy_{2n},Sy_{2n+1})\}).\end{aligned} \quad (6)$$

Since φ is a non-decreasing function, we get that

$$d(F(x_{2n},y_{2n}),F(x_{2n+1},y_{2n+1}))\leq \max\{d(Gx_{2n},Sx_{2n+1}),d(Gy_{2n},Sy_{2n+1})\}.$$

Therefore

$$d(Gx_{2n+2},Sx_{2n+3})\leq\max\{d(Gx_{2n},Sx_{2n+1}),d(Gy_{2n},Sy_{2n+1})\}. \quad (7)$$

Again, using (3), (4) and (1), we have

$$\begin{aligned}&\varphi(d(F(y_{2n},x_{2n}),F(y_{2n+1},x_{2n+1})))\\ &\leq\varphi(\max\{d(Gy_{2n},Sy_{2n+1}),d(Gx_{2n},Sx_{2n+1})\})\\ &-\phi(\max\{d(Gy_{2n},Sy_{2n+1}),d(Gx_{2n},Sx_{2n+1})\})\\ &\leq\varphi(\max\{d(Gy_{2n},Sy_{2n+1}),d(Gx_{2n},Sx_{2n+1})\}).\end{aligned} \quad (8)$$

Since φ is non-decreasing, we have

$$d(F(y_{2n},x_{2n}),F(y_{2n+1},\ x_{2n+1}))\leq \max\{d(Gy_{2n},Sy_{2n+1}),d(Gx_{2n},Sx_{2n+1}\}.$$

Therefore

$$d(Gy_{2n+2}, Sy_{2n+3}) \leq \max\{d(Gy_{2n}, Sy_{2n+1}), d(Gx_{2n}, Sx_{2n+1})\}. \quad (9)$$

Combining (7) and (9), we obtain

$$\max\{d(Gx_{2n+2}, Sx_{2n+3}), d(Gy_{2n+2}, Sy_{2n+3})\} \leq \max\{d(Gx_{2n}, Sx_{2n+1}), d(Gy_{2n}, Sy_{2n+1})\}.$$

Then $\left\{\max\{d(Gx_{2n}, Sx_{2n+1}), d(Gy_{2n}, Sy_{2n+1})\}\right\}$ is a positive non-increasing sequence. Hence there exists $r \geq 0$ such that

$$\lim_{n\to+\infty} \max\{d(Gx_{2n}, Sx_{2n+1}), d(Gy_{2n}, Sy_{2n+1})\} = r.$$

Combining (6) and (8), we obtain

$$\begin{aligned}&\max\{\varphi(d(Gx_{2n+2}, Sx_{2n+3})), \varphi(d(Gy_{2n+2}, Sy_{2n+3}))\}\\ &\leq \varphi(\max\{d(Gx_{2n}, Sx_{2n+1}), d(Gy_{2n}, Sy_{2n+1})\})\\ &-\phi(\max\{d(Gx_{2n}, Sx_{2n+1}), d(Gy_{2n}, Sy_{2n+1})\}).\end{aligned}$$

Since φ is non-decreasing, we get

$$\begin{aligned}&\varphi(\max\{d(Gx_{2n+2}, Sx_{2n+3}), d(Gy_{2n+2}, Sy_{2n+3})\})\\ &\leq \varphi(\max\{d(Gx_{2n}, Sx_{2n+1}), d(Gy_{2n}, Sy_{2n+1})\})\\ &-\phi(\max\{d(Gx_{2n}, Sx_{2n+1}), d(Gy_{2n}, Sy_{2n+1})\}).\end{aligned}$$

Letting $n \to +\infty$ in the above inequality, we get

$$\varphi(r) \leq \varphi(r) - \phi(r),$$

which implies that $\phi(r) = 0$ and then, since ϕ is an altering distance function, $r = 0$. Consequently

$$\begin{aligned}\lim_{n\to+\infty}&\max\{d(F(x_{2n}, y_{2n}), F(x_{2n+1}, y_{2n+1})),\\ &d(F(y_{2n}, x_{2n}), F(y_{2n+1}, x_{2n+1}))\} = 0.\end{aligned} \quad (10)$$

By the same way, we obtain

$$\begin{aligned}&\lim_{n\to+\infty}\max\{d(F(x_{2n+1}, y_{2n+1}), F(x_{2n+2}, y_{2n+2})),\\ &d(F(y_{2n+1}, x_{2n+1}), F(y_{2n+2}, x_{2n+2}))\} = 0.\end{aligned} \quad (11)$$

Finally, (10) and (11) give the desired result, that is, (5) holds.

Step 2: We will prove that $F(x_n, y_n)$ and $F(y_n, x_n)$ are Cauchy sequences.
From (5), it is sufficient to show that $F(x_{2n}, y_{2n})$ and $F(y_{2n}, x_{2n})$ are Cauchy sequences.
We proceed by negation and suppose that at least one of the sequences $F(x_{2n}, y_{2n})$ or $F(y_{2n}, x_{2n})$ is not a Cauchy sequence.
This implies that $d(F(x_{2n}, y_{2n}), F(x_{2m}, y_{2m})) \nrightarrow 0$ or $d(F(y_{2n}, x_{2n}), F(y_{2m}, x_{2m})) \nrightarrow 0$ as $n, m \to +\infty$. Consequently

$$\begin{aligned}&\max\{d(F(x_{2n}, y_{2n}), F(x_{2m}, y_{2m})),\\ &d(F(y_{2n}, x_{2n}), F(y_{2m}, x_{2m}))\} \nrightarrow 0, \quad \text{as } n, m \to +\infty.\end{aligned}$$

Then there exists $\varepsilon > 0$ for which we can find two subsequences of positive integers $\{m(i)\}$ and $\{n(i)\}$ such that $n(i)$ is the smallest index for which $n(i) > m(i) > i$,

$$\begin{aligned}&\max\{d(F(x_{2m(i)}, y_{2m(i)}), F(x_{2n(i)}, y_{2n(i)})),\\ &d(F(y_{2m(i)}, x_{2m(i)}), F(y_{2n(i)}, x_{2n(i)}))\} \geq \varepsilon.\end{aligned} \quad (12)$$

This means that

$$\begin{aligned}&\max\{d(F(x_{2m(i)}, y_{2m(i)}), F(x_{2n(i)-2}, y_{2n(i)-2})),\\ &d(F(y_{2m(i)}, x_{2m(i)}), F(y_{2n(i)-2}, x_{2n(i)-2}))\} < \varepsilon.\end{aligned} \quad (13)$$

From (12), (13) and using the triangular inequality, we get

$$\begin{aligned}\varepsilon \leq &\max\{d(F(x_{2m(i)}, y_{2m(i)}), F(x_{2n(i)}, y_{2n(i)})),\\ &d(F(y_{2m(i)}, x_{2m(i)}), F(y_{2n(i)}, x_{2n(i)}))\}\\ \leq &\max\{d(F(x_{2m(i)}, y_{2m(i)}), F(x_{2n(i)-2}, y_{2n(i)-2})),\\ &d(F(y_{2m(i)}, x_{2m(i)}), F(y_{2n(i)-2}, x_{2n(i)-2}))\}\\ &+\max\{d(F(x_{2n(i)-2}, y_{2n(i)-2}), F(x_{2n(i)-1}, y_{2n(i)-1})),\\ &d(F(y_{2n(i)-2}, x_{2n(i)-2}), F(y_{2n(i)-1}, x_{2n(i)-1}))\}\\ &+\max\{d(F(x_{2n(i)-1}, y_{2n(i)-1}), F(x_{2n(i)}, y_{2n(i)})),\\ &d((F(y_{2n(i)-1}, x_{2n(i)-1}), F(y_{2n(i)}, x_{2n(i)})))\}\\ < \varepsilon + &\max\{d(F(x_{2n(i)-2}, y_{2n(i)-2}), F(x_{2n(i)-1}, y_{2n(i)-1})),\\ &d(F(y_{2n(i)-2}, x_{2n(i)-2}), F(y_{2n(i)-1}, x_{2n(i)-1}))\}\\ &+\max\{d(F(x_{2n(i)-1}, y_{2n(i)-1}), F(x_{2n(i)}, y_{2n(i)})),\\ &d(F(y_{2n(i)-1}, x_{2n(i)-1}), F(y_{2n(i)}, x_{2n(i)}))\}.\end{aligned}$$

Letting $i \to +\infty$ in above inequality and using (5), we obtain that

$$\begin{aligned}&\lim_{i\to+\infty}\max\{d(F(x_{2m(i)}, y_{2m(i)}), F(x_{2n(i)}, y_{2n(i)})),\\ &d(F(y_{2m(i)}, x_{2m(i)}), F(y_{2n(i)}, x_{2n(i)}))\} = \varepsilon.\end{aligned} \quad (14)$$

Also, we have

$$\begin{aligned}\varepsilon \le \max\{&d(F(x_{2m(i)},y_{2m(i)}),F(x_{2n(i)},y_{2n(i)})),\\ &d(F(y_{2m(i)},x_{2m(i)}),F(y_{2n(i)},x_{2n(i)}))\}\\ \le \max\{&d(F(x_{2m(i)},y_{2m(i)}),F(x_{2n(i)-1},y_{2n(i)-1})),\\ &d(F(y_{2m(i)},x_{2m(i)}),F(y_{2n(i)-1},x_{2n(i)-1}))\}\\ +\max\{&d(F(x_{2n(i)-1},y_{2n(i)-1}),F(x_{2n(i)},y_{2n(i)})),\\ &d(F(y_{2n(i)-1},x_{2n(i)-1}),F(y_{2n(i)},x_{2n(i)}))\}\\ \le \max\{&d(F(x_{2m(i)},y_{2m(i)}),F(x_{2n(i)},y_{2n(i)})),\\ &d(F(y_{2m(i)},x_{2m(i)}),F(y_{2n(i)},x_{2n(i)}))\}\\ +\max\{&d(F(x_{2n(i)},y_{2n(i)}),F(x_{2n(i)-1},y_{2n(i)-1})),\\ &d(F(y_{2n(i)},x_{2n(i)}),F(y_{2n(i)-1},x_{2n(i)-1}))\}\\ +\max\{&d(F(x_{2n(i)-1},y_{2n(i)-1}),F(x_{2n(i)},y_{2n(i)})),\\ &d(F(y_{2m(i)},x_{2m(i)}),F(y_{2n(i)},x_{2n(i)}))\}.\end{aligned}$$

Using (5), (14) and letting $i\to+\infty$ in the above inequality, we obtain

$$\lim_{i\to+\infty}\max\{d(F(x_{2m(i)},y_{2m(i)}),F(x_{2n(i)-1},y_{2n(i)-1})),\; d(F(y_{2m(i)},x_{2m(i)}),F(y_{2n(i)-1},x_{2n(i)-1}))\}=\varepsilon. \quad (15)$$

On other hand, we have

$$\begin{aligned}\max\{&d(F(x_{2m(i)},y_{2m(i)}),F(x_{2n(i)},y_{2n(i)})),\\ &d(F(y_{2m(i)},x_{2m(i)}),F(y_{2n(i)},x_{2n(i)}))\}\\ \le \max\{&d(F(x_{2m(i)},y_{2m(i)}),F(x_{2m(i)+1},y_{2m(i)+1})),\\ &d(F(y_{2m(i)},x_{2m(i)}),F(y_{2m(i)+1},x_{2m(i)+1}))\}\\ +\max\{&d(F(x_{2m(i)+1},y_{2m(i)+1}),F(x_{2m(i)+2},y_{2m(i)+2})),\\ &d(F(y_{2m(i)+1},x_{2m(i)+1}),F(y_{2m(i)+2},x_{2m(i)+2}))\}\\ +\max\{&d(F(x_{2m(i)+2},y_{2n(i)+1}),F(x_{2n(i)+1},y_{2n(i)+1})),\\ &d(F(y_{2m(i)+2},x_{2m(i)+2}),F(y_{2n(i)+1},x_{2n(i)+1}))\}\\ +\max\{&d(F(x_{2n(i)+1},y_{2n(i)+1}),F(x_{2n(i)},y_{2n(i)})),\\ &d(F(y_{2n(i)+1},x_{2n(i)+1}),F(y_{2n(i)},x_{2n(i)}))\}.\end{aligned}$$

Since φ is a continuous non-decreasing function, using (5) in the above inequality, we get taking the upper limit

$$\begin{aligned}\varphi(\varepsilon)\le\;&\varphi(\limsup_{i\to+\infty}\max\{d(F(x_{2m(i)+2},y_{2m(i)+2}),\\ &F(x_{2n(i)+1},y_{2n(i)+1})),d(F(y_{2m(i)+2},x_{2m(i)+2}),\\ &F(y_{2n(i)+1},x_{2n(i)+1}))\}).\end{aligned} \quad (16)$$

Using the contractive condition (1), on one hand we have

$$\begin{aligned}&\varphi(d(F(x_{2m(i)+2},y_{2m(i)+2}),F(x_{2n(i)+1},y_{2n(i)+1}))\\ &\le\varphi(\max\{d(Gx_{2m(i)+2},Sx_{2n(i)+1}),d(Gy_{2m(i)+2},Sy_{2n(i)+1})\})\\ &-\phi(\max\{d(Gx_{2m(i)+2},Sx_{2n(i)+1}),d(Gy_{2m(i)+2},Sy_{2n(i)+1})\})\\ &=\varphi(\max\{d(F(x_{2m(i)},y_{2m(i)}),F(x_{2n(i)-1},y_{2n(i)-1})),\\ &d(F(y_{2m(i)},x_{2m(i)}),F(y_{2n(i)-1},x_{2n(i)-1}))\})\\ &-\phi(\max\{d(F(x_{2m(i)},y_{2m(i)}),F(x_{2n(i)-1},y_{2n(i)-1})),\\ &d(F(y_{2m(i)},x_{2m(i)}),F(y_{2n(i)-1},x_{2n(i)-1}))\}).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\varphi(d(F(y_{2m(i)+2},x_{2m(i)+2}),F(y_{2n(i)+1},x_{2n(i)+1})))\\ &\le\varphi(\max\{d(Gy_{2m(i)+2},Sy_{2n(i)+1}),\\ &d(Gx_{2m(i)+2},Sx_{2n(i)+1})\})\\ &-\phi(\max\{d(Gy_{2m(i)+2},Sy_{2n(i)+1}),\\ &d(Gx_{2m(i)+2},Sx_{2n(i)+1})\})\\ &=\varphi(\max\{d(F(y_{2m(i)},x_{2m(i)}),F(y_{2n(i)-1},x_{2n(i)-1})),\\ &d(F(x_{2m(i)},y_{2m(i)}),F(x_{2n(i)-1},y_{2n(i)-1}))\})\\ &-\phi(\max\{d(F(y_{2m(i)},x_{2m(i)})),F(y_{2n(i)-1},x_{2n(i)-1}),\\ &d(F(x_{2m(i)},y_{2m(i)})),F(x_{2n(i)-1},y_{2n(i)-1})\}).\end{aligned}$$

Therefore

$$\begin{aligned}\varphi(\max\{&d(F(x_{2m(i)+2},y_{2m(i)+2}),F(x_{2n(i)+1},y_{2n(i)+1})),\\ &d(F(y_{2m(i)+2},x_{2m(i)+2}),F(y_{2n(i)+1},x_{2n(i)+1}))\})\\ \le\max\{&\varphi(d(F(x_{2m(i)+2},y_{2m(i)+2}),F(x_{2n(i)+1},y_{2n(i)+1})),\\ &\varphi(d(F(y_{2m(i)+2},x_{2m(i)+2}),F(y_{2n(i)+1},x_{2n(i)+1}))\}\\ \le\varphi(\max\{&d(F(x_{2m(i)},y_{2m(i)}),F(x_{2n(i)-1},y_{2n(i)-1})),\\ &d(F(y_{2m(i)},x_{2m(i)}),F(y_{2n(i)-1},x_{2n(i)-1}))\}\\ -\phi(\max\{&d(F(x_{2m(i)},y_{2m(i)}),F(x_{2n(i)-1},y_{2n(i)-1})),\\ &d(F(y_{2m(i)},x_{2m(i)}),F(y_{2n(i)-1},x_{2n(i)-1}))\}).\end{aligned} \quad (17)$$

Finally, taking the lim sup as $i\to+\infty$ in (17), using (15), (16) and the continuity of φ and ϕ, we get

$$\varphi(\varepsilon)\le\varphi(\varepsilon)-\phi(\varepsilon),$$

which implies that $\phi(\varepsilon)=0$, that is, $\varepsilon=0$, a contradiction. Thus $\{F(x_{2n},y_{2n})\}$ and $\{F(y_{2n},x_{2n})\}$ are Cauchy sequences in X, which give us that $\{F(x_n,y_n)\}$ and $\{F(y_n,x_n)\}$ are also Cauchy sequences.

Step 3: Existence of a coupled coincidence point.
Since$\{F(x_n,y_n)\}$ and $\{F(y_n,x_n)\}$ are Cauchy sequences in

the complete metric space (X,d), there exist α, $\alpha' \in X$ such that

$$\lim_{n\to+\infty} F(x_n,y_n)=\alpha \quad \text{and} \quad \lim_{n\to+\infty} F(y_n,x_n)=\alpha'.$$

Therefore, $\lim_{n\to+\infty} G(x_{2n+2}) = \alpha$, $\lim_{n\to+\infty} G(y_{2n+2}) = \alpha'$, $\lim_{n\to+\infty} S(x_{2n+3}) = \alpha$ and $\lim_{n\to+\infty} S(y_{2n+3}) = \alpha'$.
Using the continuity and the commutativity of F and G, we have

$$\begin{aligned} G(G(x_{2n+2})) &= G(F(x_{2n},y_{2n})) \\ &= F(Gx_{2n},Gy_{2n}) \end{aligned}$$

and

$$\begin{aligned} G(G(y_{2n+2})) &= G(F(y_{2n},x_{2n})) \\ &= F(Gy_{2n},Gx_{2n}). \end{aligned}$$

Letting $n \to +\infty$, we get $G(\alpha) = F(\alpha,\alpha')$ and $G(\alpha') = F(\alpha',\alpha)$.
Using also the continuity and the commutativity of F and S, by the same way, we obtain $S(\alpha) = F(\alpha,\alpha')$ and $S(\alpha') = F(\alpha',\alpha)$.
Therefore,,

$$G(\alpha)=F(\alpha,\alpha')=S(\alpha) \quad \text{and} \quad G(\alpha')=F(\alpha',\alpha)=S(\alpha').$$

Thus we proved that (α,α') is a coupled coincidence point of G,S and F. ■

In the next result, we prove that the previous theorem is still valid if we replace the continuity of F by some conditions.

Theorem 22*If we replace the continuity hypothesis of F in Theorem 21 by the following conditions:*

(i)if (x_n) is a non-decreasing sequences with $x_n \to x$ then $x_n \preceq x$ for each $n\in\mathbb{N}$,
(ii)if (y_n) is a non-increasing sequences with $y_n \to y$ then $y \preceq y_n$ for each $n\in\mathbb{N}$,
(iii)$x,y\in X$, $\quad x\preceq y \Rightarrow Gx \preceq Sy$,
(iv)$x,y\in X$, $\quad x\succeq y \Rightarrow Gx \succeq Sy$.

Then G,S and F have a coupled coincidence point.

Proof. Following the proof of Theorem 21, we have that $F(x_n,y_n)$ and $F(y_n,x_n)$ are Cauchy sequences in the complete metric space (X,d), there exist α, $\alpha' \in X$ such that

$$\lim_{n\to+\infty} F(x_n,y_n)=\alpha \quad \text{and} \quad \lim_{n\to+\infty} F(y_n,x_n)=\alpha'.$$

Therefore, $\lim_{n\to+\infty} F(x_{2n},y_{2n}) = \alpha$ and $\lim_{n\to+\infty} F(y_{2n},x_{2n}) = \alpha'$. Hence, $\lim_{n\to+\infty} G(x_{2n+2}) = \alpha$, $\lim_{n\to+\infty} G(y_{2n+2}) = \alpha'$, $\lim_{n\to+\infty} S(x_{2n+3}) = \alpha$ and $\lim_{n\to+\infty} S(y_{2n+3}) = \alpha'$. Using the commutativity of $\{F,G\}$ and $\{F,S\}$ and the contractive condition (1), it follows from the conditions (iii) and (iv) that

$$\begin{aligned}
&\varphi(d(G(F(x_{2n},y_{2n})),S(F(x_{2n+1},y_{2n+1}))))\\
&=\varphi(d(F(Gx_{2n},Gy_{2n}),F(Sx_{2n+1},Sy_{2n+1})))\\
&\le \varphi(\max\{d(G(Gx_{2n}),S(Sx_{2n+1})), \qquad (18)\\
&d(G(Gy_{2n}),S(Sy_{2n+1}))\})\\
&-\phi(\max\{d(G(Gx_{2n}),S(Sx_{2n+1})),d(G(Gy_{2n}),S(Sy_{2n+1}))\}).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\varphi(d(G(F(y_{2n},x_{2n})),S(F(y_{2n+1},x_{2n+1}))))\\
&=\varphi(d(F(Gy_{2n},Gx_{2n}),F(Sy_{2n+1},Sx_{2n+1})))\\
&\le \varphi(\max\{d(G(Gy_{2n}),S(Sy_{2n+1})), \qquad (19)\\
&d(G(Gx_{2n}),S(Sx_{2n+1}))\})\\
&-\phi(\max\{d(G(Gy_{2n}),S(Sy_{2n+1})),\\
&d(G(Gx_{2n}),S(Sx_{2n+1}))\}).
\end{aligned}$$

Combining (18), (19) and the fact that $\max\{\varphi(a),\varphi(b)\} = \varphi(\max\{a,b\})$ for $a,b\in[0,+\infty)$, from (iii) and (iv), we obtain

$$\begin{aligned}
&\varphi(\max\{d(G(F(x_{2n},y_{2n})),S(F(x_{2n+1},y_{2n+1}))),\\
&d(G(F(y_{2n},x_{2n})),S(F(y_{2n+1},x_{2n+1})))\})\\
&\le \varphi(\max\{d(G(Gx_{2n}),S(Sx_{2n+1})),\\
&d(G(Gy_{2n}),S(Sy_{2n+1}))\})\\
&-\phi(\max\{d(G(Gx_{2n}),S(Sx_{2n+1})),\\
&d(G(Gy_{2n}),S(Sy_{2n+1}))\}).
\end{aligned}$$

Letting $n\to+\infty$ in the last expression, using the continuity of G and S, we get

$$\begin{aligned}
&\varphi(\max\{d(G(\alpha),S(\alpha)),d(G(\alpha'),S(\alpha'))\})\\
&\le \varphi(\max\{d(G(\alpha),S(\alpha)),d(G(\alpha'),S(\alpha'))\})\\
&-\phi(\max\{d(G(\alpha),S(\alpha)),d(G(\alpha'),S(\alpha'))\}).
\end{aligned}$$

This implies that $\phi(\max\{d(G(\alpha),S(\alpha)),d(G(\alpha'),S(\alpha'))\}) = 0$ and, since ϕ is an altering distance function, then

$$\max\{d(G(\alpha),S(\alpha)),d(G(\alpha'),S(\alpha'))\}=0.$$

Consequently

$$G(\alpha)=S(\alpha) \quad \text{and} \quad G(\alpha')=S(\alpha'). \qquad (20)$$

To finish the proof, we claim that $F(\alpha,\alpha')=G(\alpha)=S(\alpha)$ and $F(\alpha',\alpha)=G(\alpha')=S(\alpha')$.

Indeed, using the contractive condition (1), (3) and (4), it follows from (i)-(iv) that

$$\begin{aligned}&\varphi(d(F(Gx_{2n},Gy_{2n}),F(\alpha,\alpha')))\\&\leq\varphi(\max\{d(G(Gx_{2n}),S(\alpha)),d(G(Gy_{2n}),S(\alpha'))\})\\&-\phi(\max\{d(G(Gx_{2n}),S(\alpha)),d(G(Gy_{2n}),S(\alpha'))\})\\&\leq\varphi(\max\{d(G(Gx_{2n}),S(\alpha)),d(G(Gy_{2n}),S(\alpha'))\}).\end{aligned}$$

Using the fact that φ is non-decreasing, we get

$$\begin{aligned}&d(F(Gx_{2n},Gy_{2n}),F(\alpha,\alpha'))\leq\\&\max\{d(G(Gx_{2n}),S(\alpha)),d(G(Gy_{2n}),S(\alpha'))\}.\end{aligned}\tag{21}$$

Similarly, we have

$$\begin{aligned}&\varphi(d(F(Gy_{2n},Gx_{2n}),F(\alpha',\alpha)))\\&\leq\varphi(\max\{d(G(Gy_{2n}),S(\alpha')),d(G(Gx_{2n}),S(\alpha))\})\\&-\phi(\max\{d(G(Gy_{2n}),S(\alpha')),d(G(Gx_{2n}),S(\alpha))\}\\&\leq\varphi(\max\{d(G(Gy_{2n}),S(\alpha')),d(G(Gx_{2n}),S(\alpha))\}).\end{aligned}$$

Using the fact that φ is non-decreasing, we see that

$$\begin{aligned}&d(F(Gy_{2n},Gx_{2n}),F(\alpha',\alpha))\leq\\&\max\{d(G(Gy_{2n}),S(\alpha')),d(G(Gx_{2n}),S(\alpha))\}.\end{aligned}\tag{22}$$

Combining (21) and (22), we get

$$\begin{aligned}&\max\{d(F(Gx_{2n},Gy_{2n}),F(\alpha,\alpha')),\\&d(F(Gy_{2n},Gx_{2n}),F(\alpha',\alpha)))\\&\leq\max\{d(G(Gx_{2n}),S(\alpha)),d(G(Gy_{2n}),S(\alpha'))\}.\end{aligned}$$

Using the commutativity of F and G, we write

$$\begin{aligned}&\max\{d(G(F(x_{2n},y_{2n}))),F(\alpha,\alpha')),\\&d(G(F(y_{2n},x_{2n})),F(\alpha',\alpha))\}\\&\leq\max\{d(G(Gx_{2n}),S(\alpha)),d(G(Gy_{2n}),S(\alpha'))\}.\end{aligned}$$

Letting $n\to+\infty$, using the continuity of G, we obtain

$$\begin{aligned}&\max\{d(G(\alpha),F(\alpha,\alpha')),d(G(\alpha'),F(\alpha',\alpha))\}\leq\\&\max\{d(G(\alpha),S(\alpha)),d(G(\alpha'),S(\alpha'))\}.\end{aligned}$$

Looking at (20), we deduce that

$$\max\{d(G(\alpha),F(\alpha,\alpha')),d(G(\alpha'),F(\alpha',\alpha))\}=0.$$

Therefore,

$$d(G(\alpha),F(\alpha,\alpha'))=0\quad\text{and}\quad d(G(\alpha'),F(\alpha',\alpha))=0.$$

Consequently

$$G(\alpha)=F(\alpha,\alpha')\quad\text{and}\quad G(\alpha')=F(\alpha',\alpha).\tag{23}$$

By the same way, we get

$$S(\alpha)=F(\alpha,\alpha')\quad\text{and}\quad S(\alpha')=F(\alpha',\alpha).\tag{24}$$

Finally, combining (20), (23) and (24), we deduce that (α,α') is a coupled coincidence point of F, G and S. ■

Remark 2
Taking $G=S=I_X$ (the identity mapping of X) in Theorem 21, we obtain [7, Theorem 2].
Taking $G=S=I_X$ in Theorem 22, we obtain [7, Theorem 3].

Taking $S=G$, we get the following:

Corollary 21 *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let $G:X\to X$ be a continuous mapping and $F:X\times X\to X$ be a mapping having the mixed G-monotone property on X. Suppose that*

$$\begin{aligned}&\varphi(d(F(x,y),F(u,v)))\leq\varphi(\max\{d(Gx,Gu),d(Gy,Gv)\})\\&-\phi(\max\{d(Gx,Gu),d(Gy,Gv)\}),\end{aligned}\tag{25}$$

for all $x,y,u,v\in X$ with $G(x)\preceq G(u)$ or $G(x)\succeq G(u)$ and $G(y)\succeq G(v)$ or $G(y)\preceq G(v)$, where $\varphi,\phi\in\mathscr{F}$. Assume that $F(X\times X)\subseteq G(X)\cap G(X)$ and assume that

(I)F is continuous or assumptions $(i)-(ii)$ of Theorem 22 hold with G non-decreasing.
(II)F commutes with G.

If there exist x_0,y_0 such that

$$\begin{cases}G(x_0)\preceq F(x_0,y_0);\\G(y_0)\succeq F(y_0,x_0),\end{cases}$$

then there exist $x,y\in X$ such that

$$G(x)=F(x,y)\quad\text{and}\quad G(y)=F(y,x),$$

3 Applications to nonlinear integral equations

Let $X=C([0,T],\mathbb{R})$ be the set of all continuous functions $u:[0,T]\to\mathbb{R}$, $T>0$, and $G:X\to X$ is a given mapping. We endow X with the metric $d(u,v)=\max\limits_{t\in[0,T]}|u(t)-v(t)|$ for $u,v\in X$.
This space can be equipped with a partial order given by

$$x,y\in X,\quad x\preceq y\Leftrightarrow x(t)\leq y(t),\quad\text{for any } t\in[0,T].$$

In $X\times X$ we define the following partial order

$$(x,y),\ (u,v)\in X\times X,\quad (x,y)\preceq(u,v)\Leftrightarrow x\preceq u\text{ and }y\succeq v.$$

In [10] it is proved that $(X,\preceq)$ satisfies assumptions (i) and (ii) of Theorem 22.

Consider the system of integral equations:

$$\begin{cases}Gu(t)=\displaystyle\int_0^T k(t,s)f(s,u(s),v(s))ds\\Gv(t)=\displaystyle\int_0^T k(t,s)f(s,v(s),u(s))ds\end{cases}\tag{26}$$

where the functions $k : [0,T] \times [0,T] \to [0,+\infty[$ and $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to [0,+\infty[$ are two continuous functions satisfying the following conditions:
$(H1)$

$$\sup_{t\in[0,T]} \int_0^T k(t,s)ds \leq 1.$$

$(H2)$ For all $s, b \in [0,T]$, $u, v \in X$

$$Gu \preceq Gv, \Rightarrow f(s,u(s),b) \leq f(s,v(s),b)$$
$$Gu \preceq Gv, \Rightarrow f(s,b,u(s)) \geq f(s,b,v(s)).$$

$(H3)$ For all $x, y, u, v \in X$ such that $Gx \preceq Gu$ and $Gy \succeq Gv$ we have

$$|f(s,x(s),y(s)) - f(s,u(s),v(s))| \leq$$
$$\ln\left[1 + (\max\{|Gx(s) - Gu(s)|, |Gy(s) - Gv(s)|\})^2\right].$$

(H4) There exist α, $\beta \in X$ such that for all $t \in [0,T]$ we have

$$\begin{cases} G\alpha(t) \leq \int_0^T k(t,s) f(s,\alpha(s),\beta(s))ds \\ G\beta(t) \leq \int_0^T k(t,s) f(s,\beta(s),\alpha(s))ds. \end{cases}$$

Now, we shall prove the following result.

Theorem 31 *Suppose that $G : X \to X$ is a non-decreasing continuous mapping. Suppose also that (H1)-(H4) hold. Then (26) has a solution.*

Proof. We introduce the operator $F : X \times X \to X$ defined by

$$F(u,v)(t) = \int_0^T k(t,s)[f(s,u(s),v(s))\, ds$$

for all $u, v \in X$ and $t \in [0,T]$.

From (H2) it follows directly that F has the mixed G-monotone property.
Let $u, v \in X$ such that $G(x) \preceq G(u)$ and $G(y) \succeq G(v)$. We have

$$d(F(x,y),F(u,v)) = \max_{t\in[0,T]} |F(x,y)(t) - F(u,v)(t)|$$
$$\leq \max_{t\in[0,T]} \int_0^T k(t,s)|f(s,x(s),y(s)) - f(s,u(s),v(s)|ds.$$

Using (H3) we get

$$d(F(x,y),F(u,v)) \leq \max_{t\in[0,T]} \int_0^T k(t,s)$$
$$\ln[1 + (\max\{|Gx(s) - Gu(s)|, |Gy(s) - Gv(s)|\})^2]ds \leq$$
$$\max_{t\in[0,T]} \int_0^T k(t,s) \ln[1 + (\max\{d(Gx,Gu), d(Gy,Gv)\})^2]ds$$
$$\leq \ln[1 + (\max\{d(Gx,Gu), d(Gy,Gv)\})^2] \times$$
$$\max_{t\in[0,T]} \int_0^T k(t,s)ds.$$

From (H1), we obtain

$$d(F(x,y),F(u,v)) \leq$$
$$\ln[(\max\{d(Gx,Gu),d(Gy,Gv)\})^2 + 1]$$

which implies that

$$(d(F(x,y),F(u,v)))^2 \leq$$
$$(\ln[(\max\{d(Gx,Gu),d(Gy,Gv)\})^2 + 1])^2.$$

Then,

$$(d(F(x,y),F(u,v)))^2 \leq (\max\{d(Gx,Gu),d(Gy,Gv)\})^2$$
$$- \left[(\max\{d(Gx,Gu),d(Gy,Gv)\})^2\right.$$
$$\left. - (\ln[(\max\{d(Gx,Gu),d(Gy,Gv)\})^2 + 1])^2\right].$$

Set $\varphi(t) = t^2$ and $\phi(t) = t^2 - \ln(t^2 + 1)$. Clearly φ and ϕ are altering distance functions and from the above inequality, we obtain

$$\varphi(d(F(x,y),F(u,v))) \leq \varphi(\max\{d(Gx,Gu),d(Gy,Gv)\})$$
$$-\phi((\max\{d(Gx,Gu),d(Gy,Gv)\}))$$

for all $x,y,u,v \in X$ such that $G(x) \preceq G(u)$ and $G(y) \succeq G(v)$. Now, let $\alpha, \beta \in X$ be the functions given by (H4), then we have

$$G(\alpha) \preceq F(\alpha,\beta) \quad \text{and} \quad F(\beta,\alpha) \preceq G(\beta).$$

Thus, we proved that all the required hypotheses of Corollary 21 are satisfied. Hence, G and F have a coupled coincidence point $(u,v) \in X \times X$, that is, (u,v) is a solution of (26). ■

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