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# Bell-Touchard-G Family of Distributions: Applications to Quality Control and Actuarial Data

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**Abstract:** In this article, we developed a new statistical model named as the generalized complementary exponentiated Bell-Touchard model. The exponential model is taken as a special baseline model with a configurable failure rate function. The proposed model is based on several features of zero-truncated Bell numbers and Touchard polynomials that can address the complementary risk matters. The linear representation of the density of the proposed model is provided that can be used to obtain numerous important properties of the special model. The well-known actuarial metrics namely value at risk and expected shortfall are formulated, computed and graphically illustrated for the sub model. The maximum likelihood approach is used to estimate the parameters. Furthermore, we designed the group acceptance sampling plan for the proposed model by using the median as a quality parameter for truncated life tests. Three real data applications are offered for the complementary exponentiated Bell Touchard exponential model. The analysis of the two failure times data and comparative study yielded optimized results of the group acceptance sampling plan under the proposed model. The application to insurance claim data also provided the best results and showed that the proposed model had heavier tail.

**Keywords:** Actuarial measures; Complementary exponentiated Bell-Touchard; GASP; Maximum-likelihood estimation; Quality parameter

## 1 Introduction

Statistical models are the mathematical representations of real-world phenomena to evaluate and interpret data. These models are developed based on statistical techniques with the goals of describing the connection between variables, making predictions, and drawing inferences from data. Therefore, the need for novel, flexible, adaptable statistical models that can handle complex events is common among researchers and practitioners. A wide variety of new generators, families of distributions, and techniques for parameter induction are proposed in the statistical literature. These developments also give applied researchers a wider range of model options resulting in better results that are ultimately more accurate. References [1,2,3,4] provide a comprehensive review of the numerous distinct statistical techniques to develop novel statistical models. On recently proposed extended flexible families of distribution, there is some valuable literature available from [5,6,7]. The concept of compounding is a widely used approach to develop new models as a combination or composition of two or more similar or different models (for detailed illustration, readers are referred to [2] and [3]). Discretization is another statistical technique to develop statistical models by which continuous variables are converted into discrete variables. This technique is useful in a variety of circumstances such as lowering noise, addressing missing data, streamlining the analysis and fulfilling the needs of certain categorical variable-based algorithms. A discrete Bell distribution that outperforms the Poisson distribution, while maintaining several key Bell numbers features were recently, proposed by Castellares *et al.* [8]. Further Fayomi *et al.* [9] introduced its G class and presented the compounded EBellE model. Castellares *et al.* [10] extended the Bell distribution into the Bell-Touchard, which outperforms other well-known discrete models. The probability mass function (PMF) of the two-parameter

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Bell-Touchard distribution is given by Castellares *et al.* [10]

$$\mathbb{P}(M=m)=\frac{\exp\left\{\theta\left[1-e^{\zeta}\right]\right\}\zeta^{m}T_{m}(\theta)}{m!},\ m=0,1,2,\cdots, \tag{1}$$

where $\zeta$ and $\theta>0$ and $T_x(.)$ is the Touchard polynomial. Eq. (1) reduces to the Bell distribution for $\theta=1$. Due to the advancement in technology, the production process has accelerated over time in a way that has never been seen before. Product quality is the crucial aspect of the production process for both the producer and the consumer. A number of statistical quality control approaches are available to evaluate and enhance the quality of the products. One of the specific quality control approach known as an acceptance sampling plan (ASP) specifies the least sample size and acceptance and non-acceptance criteria of a lot. Plans for acceptance sampling are available in many different formats, including variable acceptance plans, attributes acceptance plans, accelerated plans, gradually progressively plans, and group acceptance plans (see the references [11,12,13,14,15,16]). The main goal of these strategies is to protect both the manufacturer and the consumer while evaluating the lot using a small sample size. It is common practice to develop an ASP that provides standards by which a lot may be approved or disapproved based on sample data. For lot acceptability goals, the development of ASP is a frequently discussed topic in quality control and reliability. This topic involves a technique for optimization under restrictions. A group acceptance sampling plan (GASP) constructed on a truncated life test is the outcome of merging GASP and truncated life testing, which presumes that a product's lifespan fits a particular probability distribution. A GASP was given by Jun *et al.* [11] and is an expanded version of the ordinary acceptance sampling plan (OASP). Several authors proposed GASP using extended statistical models, here we mention a few, but are not too limited. Shafiq *et al.* [17] designed a GASP using the odd Perk exponential distribution and compared the GASP with the OASP and concluded that the GASP outperformed than the OASP; Algarni [18] illustrated GASP using three parameters compounded Weibull model; Fayomi *et al.* [9] proposed GASP based on the exponentiated Bell exponential (EBellE) model; Tripathi *et al.* [19] offered a GASP for the inverse log-logistic distribution; Almarashi *et al.* [20] used the MOKw-E (Marshall-Olkin Kumaraswamy exponential) model and designed a GASP using median lifetime as a quality index; Yigiter *et al.* [21] proposed a GASP based on the compounded Weibull-exponential model; Gadde *et al.* [22] used the sized biased Lomax distribution and established a GASP for the resubmitted lot; Rao [23] presented a GASP using the Marshall-Olkin extended Lomax distribution and Aslam [24] illustrated a GASP using the extended exponential distribution. There are more expanded ASP under the lifetime distribution examples and thorough demonstrations are available in the references [25] and [26].

The manuscript is formatted as follows: Section 2 presents the layout of the generalized complementary exponentiated Bell Touchard (CEBT-G) with motivation. Section 3 contained the illustration of the special model along with the graphical presentation of the probability density function (PDF) and hazard rate function (HRF). Further, a GASP is offered using the median as a quality parameter when a product lifetime follows a complementary exponentiated Bell Touchard exponential (CEBTE) model. Additionally, two of the most popular risk indicators, such as value at risk (VaR) and expected shortfall (ES), are shown together with the actuarial data. The actual execution of the real-world data of the suggested model is implemented in Section 4. Finally, Section 5 presents the findings of the study.

## 2 CEBT-G family with motivations

### *2.1 Construction*

In a variety of real-world contexts, complementary risk models are crucial including industrial reliability, quality assurance, biomedical research, and actuarial sciences. Generally, in a parallel system of hazards, the models used in reliability analysis only determine the maximum component lifetime values. Whereas, component failure is reliant on several risk factors, and it is unknown or unobservant which risk factor is more likely to cause or contribute to component failure (Basu *et al.* [27]). These phenomena are known as latent complementary hazards or risks. The key motivation for the development of the proposed family is a complementary latent risk that can effectively deal with the proposed family of distribution. If a system has M independent working or functioning subsystems that all operate at a given specific time. Let's assume that $W_i(i=1,2,\cdots)$ stands for the failure time of $i^{th}$ subsystem. Moreover, each subsystem's failure time follows a zero truncated Bell-Touchard (ZTBT) distribution with PMF $\mathbb{P}(M=m)$ given by

$$\mathbb{P}[M=m]=\frac{\exp\left[\theta\left(1-e^{\zeta}\right)\right]\zeta^{m}T_{m}(\theta)}{m!\left(1-\exp\left[\theta\left(1-e^{\zeta}\right)\right]\right)},\ m=1,2,\cdots, \tag{2}$$

suppose that $T$ represents the maximum time of system failure that is $T = \max(W_1, W_2, \cdots, W_M)$. Then the conditional cumulative distribution function (CDF) of $T$ given $M$ is as follows (following Tahir *et al.* [3])

$$F(t|M=m) = \mathbb{P}[\max(W_1, \ldots W_M) < t|M=m] = [\mathbb{P}^a(T_{m,m} \leq t)]^M = [H^a(t)]^M. \tag{3}$$

Then unconditional CDF of T corresponding to Eq. (3) is as:

$$F(t) = \sum_{m=1}^{\infty} [F(t|M=m)]\,\mathbb{P}(M=m) = \sum_{m=1}^{\infty} [H^a(t)]^m\, \mathbb{P}(M=m), \tag{4}$$

where $\mathbb{P}(M=m)$ represents the PMF of the ZTBT distribution given in Eq. (2).

**Proposition 1.** *The expression of CDF of the CEBT-G family using Eq. (4) is given by*

$$F(t) = \frac{\exp\left[\theta\left(e^{\zeta H^a(t)} - 1\right)\right] - 1}{\exp\left[\theta\left(e^{\zeta} - 1\right)\right] - 1}, \tag{5}$$

*where $H(x)$ represents the baseline CDF.*

**Proof.** Using Eq. (2) in Eq. (4), we get

$$F(t) = \sum_{m=1}^{\infty} [H^a(t)]^m \left(\frac{\exp\left[\theta\left(1-e^{\zeta}\right)\right]\zeta^m T_m(\theta)}{m!\left(1-\exp\left[\theta\left(1-e^{\zeta}\right)\right]\right)}\right), \tag{6}$$

the above expression can also be expressed as follows:

$$F(t) = \left(\frac{\exp\left[\theta\left(1-e^{\zeta}\right)\right]}{\left(1-\exp\left[\theta\left(1-e^{\zeta}\right)\right]\right)}\right) \sum_{m=1}^{\infty} \frac{(\zeta H^a(t))^m}{m!} T_m(\theta), \tag{7}$$

the above expression becomes

$$F(t) = \left(\frac{\exp\left[\theta\left(1-e^{\zeta}\right)\right]}{\left(1-\exp\left[\theta\left(1-e^{\zeta}\right)\right]\right)}\right) \left[\sum_{m=0}^{\infty} \frac{(\zeta H^a(t))^m}{m!} T_m(\theta) - 1\right], \tag{8}$$

following the Castellares *et al.* [10], the functional connection of Touchard polynomials is expressed in the following series

$$\exp\left[\theta\left(e^x - 1\right)\right] = \sum_{m=0}^{\infty} \frac{x^m}{m!} T_m(\theta), \qquad x, \theta \in \mathbb{R}. \tag{9}$$

Comparing Eq. (8) and Eq. (9), we achieved the desired outcomes of the Proposition 1. This completes the proof. □
Listed below are some reasons for adopting the CEBT-G family that make it interesting

- It has a distinctive productive structure based on the several features of Bell numbers and Touchared polynomials, which are never considered before.
- The proposed family is based on the genesis of the Bell-Touchard distribution which is appropriate to underline the over-dispersion (variance>mean) phenomena. The construction of the proposed CEBT-G family is motivated by the difficulty of complementary risk in the presence of latent risks.
- The special model named CEBTE yields flexible PDF and HRF shapes than the classical exponential model that can be used in numerous distinct applied fields such as quality control, actuarial science, medicine, economics, finance and reliability analysis.
- Further, the CEBTE model PDF can be expressed as a linear combination of exponential densities and this trait makes it possible to quickly derive several properties straightly from the exponential distribution.
- Moreover, the proposed CEBTE model has an elegant closed-form quantile and survival function that can be used in quality control, economics and medical data sets for better theoretical and empirical real-world solutions.
- The highly skewed data can be better addressed by the proposed special model.
- Several special cases can be extracted from the proposed family of distributions that make it more attractive.

The PDF corresponding to Eq. (5) is given by

$$f(t) = \frac{a\,\zeta\,\theta\,h(t)\,H^{a-1}(t)\,e^{\zeta H^{a}(t)}\exp\left\{\theta\left[e^{\zeta H^{a}(t)}-1\right]\right\}}{\exp\left\{\theta\left[e^{\zeta}-1\right]\right\}-1}, \tag{10}$$

where $H(t)$ and $h(t)$ represent the CDF and PDF of baseline distribution respectively. When $a = 1$ in Eq. (5) , the CEBT-G reduces to the CBT-G. When $\theta = 1$ in Eq. (5) , the CEBT-G reduces to the CEB-G. When $\theta = 1$ and $a = 1$ in Eq. (5) , the CEBT-G reduces to the CB-G. The $p$th quantile function (QF) of the CEBT-G family of distribution is given by

$$Q(p) = H^{-1}\left\{\zeta^{-1}\ln\left\{1+\theta^{-1}\ln\left[1+p\left(\exp\left\{\theta\left[e^{\zeta}-1\right]\right\}-1\right)\right]\right\}\right\}^{\frac{1}{a}}, \tag{11}$$

where $\mathrm{H}^{-1}(x)$ is the inverse function of $H(x)$ and $p$ follows the uniform distribution on the interval $[0,1]$. The well-known actuarial metrics namely VaR and ES can be derived and computed for the CEBTE model by using the aforementioned expression of QF as:

$$VaR_q = H^{-1}\left\{\zeta^{-1}\ln\left\{1+\theta^{-1}\ln\left[1+q\left(\exp\left\{\theta\left[e^{\zeta}-1\right]\right\}-1\right)\right]\right\}\right\}^{\frac{1}{a}}$$

and

$$ES_q(x) = q^{-1}\int_0^q VaR_x dx,$$

where for $0 < q < 1$.

## 3 Development of special Model and Actuarial Metrics

When hazard rates are non-constant and present monotone shapes (Louzada *et al.* [28]), the classical exponential distribution is unable to produce improved fits. Nevertheless, numerous research revealed that the extended exponential distribution or its use as a baseline model yields a better fit (Gupta *et al.* [29]). Here, we define a new special model named the CEBTE, using the exponential model as a baseline model. The novel proposed exponential model yields flexible PDF and HRF shapes. Further, the proposed model is based on several features of Bell numbers and Touchard polynomials and can address the complementary risk in the presence of latent risks with the motivation of series structure. The following are the CDF and PDF of the exponential model, respectively, with parameters $\varpi > 0$ and $t > 0$

$$H(t) = 1-e^{-\varpi t} \text{ and } h(t) = \varpi\,e^{-\varpi t}.$$

Let $T$ be a random variable which follows the CEBTE$(\theta,\zeta,a,\varpi)$ distribution for $t > 0$, and $\theta,\zeta,a,\varpi > 0$, then its associated CDF and PDF, respectively, are given by

$$F(t) = \frac{\exp\left(\theta\left[e^{\zeta\left(1-e^{-\varpi t}\right)^{a}}-1\right]\right)-1}{\exp\left(\theta\left[e^{\zeta}-1\right]\right)-1} \tag{12}$$

and

$$f(t) = \frac{a\,\zeta\,\theta\,\varpi\,e^{-\varpi t}\,\left(1-e^{-\varpi t}\right)^{a-1}\,e^{\zeta\left(1-e^{-\varpi t}\right)^{a}}\exp\left(\theta\left[e^{\zeta\left(1-e^{-\varpi t}\right)^{a}}-1\right]\right)}{\exp\left(\theta\left[e^{\zeta}-1\right]\right)-1}. \tag{13}$$

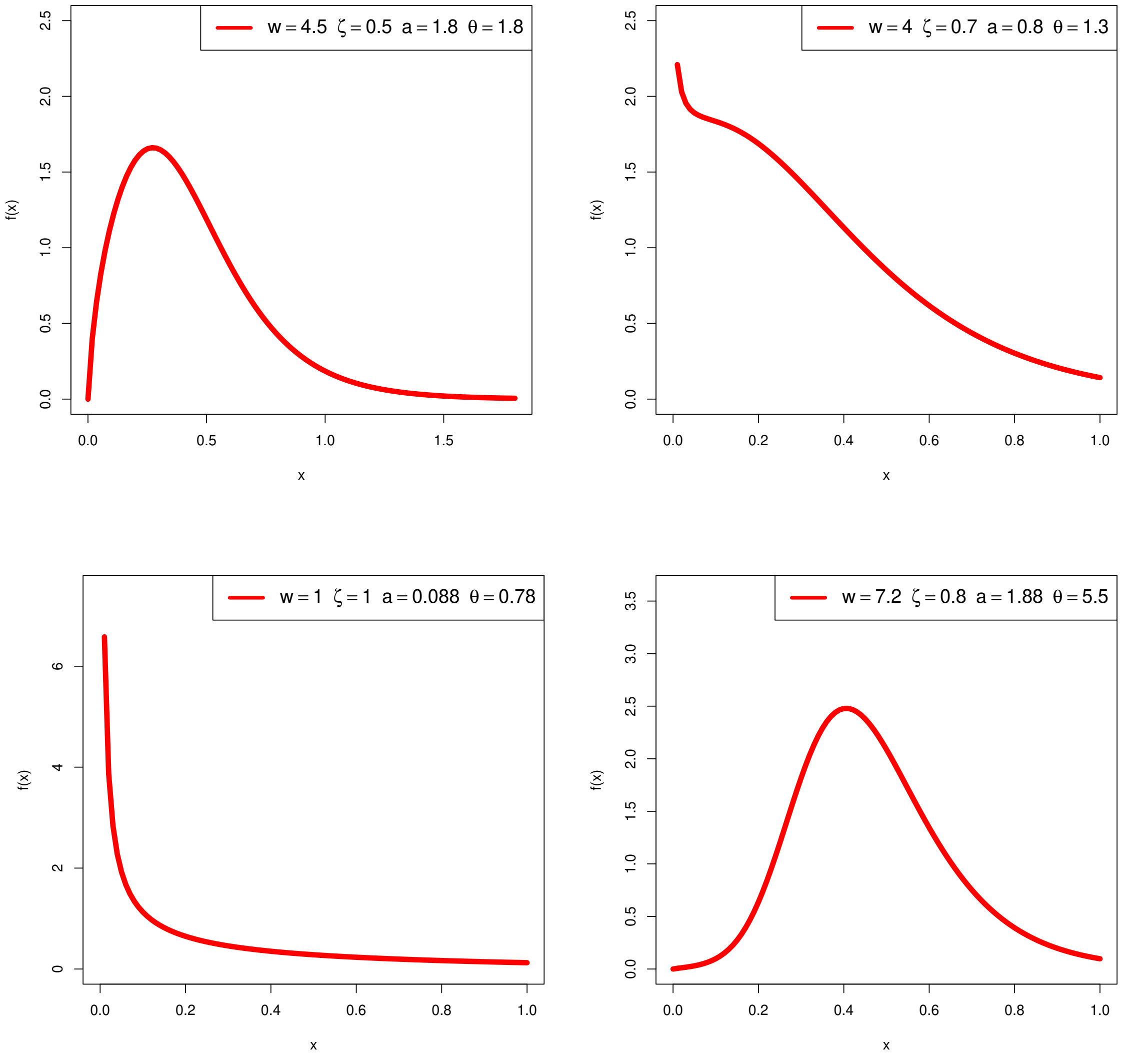


**Fig. 1:** PDF's plots under the CEBTE model at varying parametric values.

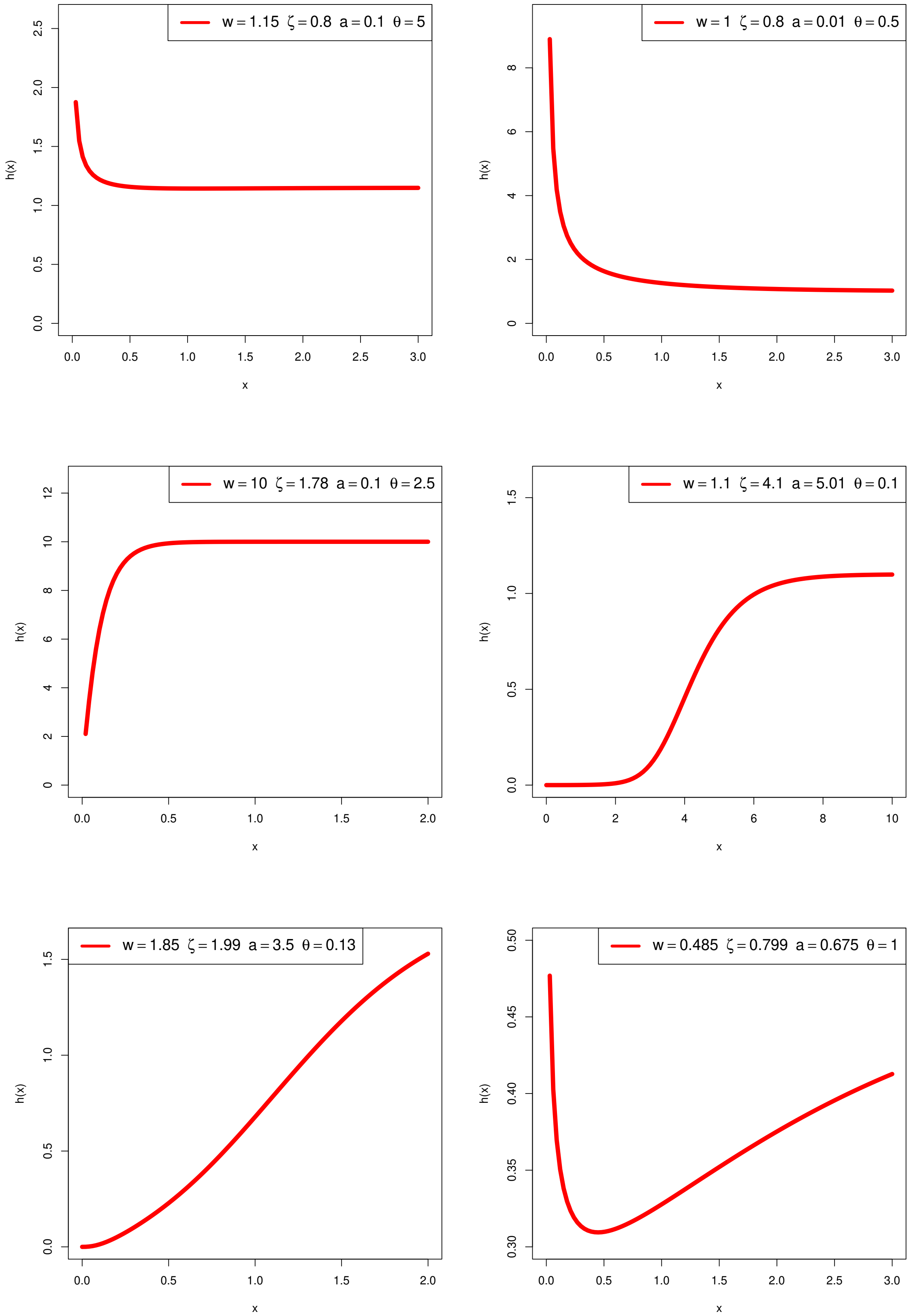


**Fig. 2:** HRF's plots under the CEBTE model at varying parametric values.

The failure rate function of the special model is fascinating for real-world applications in diverse industries since it can adopt a range of shapes (see Figire 2), such as bathtub and reverse bathtub-shaped, increasing, and decreasing trends. When $a=1$ in Eq. (12) , the CEBTE reduces to the CBTE. When $\theta=1$ in Eq. (12) , the CEBTE reduces to the CEBE. When $\theta=1$ and $a=1$ in Eq. (12) , the CEBTE reduces to the CBE. The $p$th quantile of the exponential distribution is given by $t_p=\frac{-1}{\varpi}\ln[1-p]$, by using Eq. (11), the $p$th QF of the CEBTE model is as follows:

$$t_p=\frac{-1}{\varpi}\ln\left[1-\left\{\zeta^{-1}\ln\left\{1+\theta^{-1}\ln\left[1+p\left(\exp\left\{\theta\left[e^{\zeta}-1\right]\right\}-1\right)\right]\right\}\right\}^{\frac{1}{a}}\right]. \tag{14}$$

## *3.1 GASP development using the CEBTE model*

### 3.1.1 Construction

AS ASP decides whether to accept or reject a submitted lot based on the product quality determined by the sample. Unlike OASP, the GASP is economical as it reduced cost and shortens the length of the test or experiment. Based on the fact, that GASP offers a stricter product inspection than the OASP because samples are dispersed among multiple groups and simultaneously tested. Each group is subject to the condition of acceptance to improve product inspection (see the references [30,**?**,**?**]). The mean or median can be used as a quality index to design a GASP. For skewed distribution, the median is typically preferable (Aslam *et al.* [32]). Let $m$ represent the median life of the item or product in a respective lot that is larger than the requisite life, say $m_0$ and we are intrigued in devising a sampling plan. If there is compelling proof that $m>m_0$ at particular points of consumer risk ($\beta$) and producer risk ($\gamma$), we will accept the lot. Otherwise, we must dismiss the entire lot under inspection. The following steps are involved in designing the GASP based on the truncated life test

Step 1In order to ascertain that $n=gr$ given a lot sample size, specify the number of groups ($g$) and then allocate each $g$ a set of $r$ objects.
Step 2Specify the test duration time $t_0$ and set the action limit (or acceptance number $c$) for each group.
Step 3Count the frequency of failures ($F$) for each group after experimenting simultaneously for each of the $g$ groups.
Step 4Approve the lot if $F\le c$.
Step 5If any group experiences $F>c$, the experiment needs to be stopped, and the lot should be rejected.

According to Stephens *et al.* [33], the probability of lot approval is as follows:

$$p_{a(p)}=\left[\sum_{i=0}^{c}\binom{r}{i}p^i\,[1-p]^{r-i}\right]^g, \tag{15}$$

where $p$ is the likelihood that an item in a group would expire just before $t_0$, and it is determined by inserting Eq. (14) in Eq. (12).

$$m=\frac{-1}{\varpi}\ln\left[1-\left\{\zeta^{-1}\ln\left\{1+\theta^{-1}\ln\left[1+0.5\left(\exp\left\{\theta\left[e^{\zeta}-1\right]\right\}-1\right)\right]\right\}\right\}^{\frac{1}{a}}\right]. \tag{16}$$

Consider

$$\pi=\ln\left[1-\left\{\zeta^{-1}\ln\left\{1+\theta^{-1}\ln\left[1+0.5\left(\exp\left\{\theta\left[e^{\zeta}-1\right]\right\}-1\right)\right]\right\}\right\}^{\frac{1}{a}}\right], \tag{17}$$

setting $\varpi=\frac{-\pi}{m}$ and $t=m_0a_1$ in Eq. (12) and the likelihood of failure is given by

$$p=\frac{\exp\left\{\theta\left[e^{\zeta\left[1-e^{\pi a_1(r_2)^{-1}}\right]^a}-1\right]\right\}-1}{\exp\left\{\theta\left[e^{\zeta}-1\right]\right\}-1}. \tag{18}$$

When $r_2$ and $a_1$ are fixed, whereas $r_2=m/m_0$, $p$ can be calculated for chosen $\theta$, $\zeta$, and $a$, using Eq. (18). We will examine the two failure probabilities for the $\beta$ and $\gamma$, denoted as $p_1$ and $p_2$, respectively. Determine the values of the design parameters ($c$ and $g$) that concurrently accomplish the subsequent expressions for the particular values of the constraints $\theta$, $\zeta$, $a$, $r_2$, $a_1$, $\beta$, and $\gamma$

$$p_{a\left(p_1|\frac{m}{m_0}=r_1\right)}=\left[\sum_{i=0}^{c}\binom{r}{i}p_1^i\,[1-p_1]^{r-i}\right]^g\le\beta \tag{19}$$

and

$$p_{a\left(p_2|\frac{m}{m_0}=r_2\right)} = \left[\sum_{i=0}^{c}\binom{r}{i} p_2^i [1-p_2]^{r-i}\right]^g \geq 1-\gamma, \tag{20}$$

where $r_1$ and $r_2$ denote the average ratio of $\gamma$ to that of consumers, and Eq. (19) and Eq. (20) denote the failure probability to be applied to the aforementioned equations

$$p_1 = \frac{\exp\left\{\theta\left[e^{\zeta[1-e^{\pi a_1}]^a}-1\right]\right\}-1}{\exp\left\{\theta\left[e^{\zeta}-1\right]\right\}-1} \tag{21}$$

and

$$p_2 = \frac{\exp\left\{\theta\left[e^{\zeta\left[1-e^{\pi a_1 (r_2)^{-1}}\right]^a}-1\right]\right\}-1}{\exp\left\{\theta\left[e^{\zeta}-1\right]\right\}-1}. \tag{22}$$

### 3.1.2 Discussion

The design parameters under GASP are displayed in Tables 2-3, taking two levels of $\zeta$= (1.25, 1.50) and $r$=(5, 10). The analysis shows that as $\beta$ tends to decrease, the groups increased. Furthermore, as $r_2$ rises, the groups decreased quickly. However, after a certain point, when design parameters remain constant, the likelihood of a lot's acceptance increases. Tables 2-3 showed that effect of $a_1 = (0.5, 1)$ and $r = (5, 10)$, if $\beta = 0.25$, $r_2$=4, $a_1 = 0.5$, $\zeta = 1.250$, and $r$=5 there are 20 groups or 100 $(20\times 5 = 100)$ units are required for life testing. On the other side, a significant decrease in the number of groups may be seen, when $r = 10$, i.e., 4 groups or 40 $(4\times 10 = 40)$ units are needed for life testing. Likewise, 15 items should be required for life testing, when $a_1 = 1$ and $r = 5$, but only 10 items are tested for $a_1 = 1$ and $r = 10$. Here 10 group is therefore preferred. From Table 1, when using median lifetime as a quality parameter, the CEBTE model predicts that as true median life increases, the value of the operating characteristic (OC) function, $p_{a(p)}$ increases and the groups drop for the GASP under consideration. This indicates that the lot being considered will be utilized (accepted) in that case. This is evident from Table 2, under various parameters values, for $\beta$=0.10, $a_1$=1, $\zeta$=1.25, and for $r$=10.

**Table 1:** GASP under CEBTE model, for $\beta$=0.10, $a_1$=1, $\zeta$=1.25, and for $r$=10.

| | | | | |
|---|---|---|---|---|
| $r_2$ | 2 | 4 | 6 | 8 |
| $g$ | 5 | 2 | 1 | 1 |
| $c$ | 5 | 3 | 2 | 2 |
| $P(a)$ | 0.9564 | 0.9810 | 0.9815 | 0.9914 |

Here, we demonstrate the hypothetical example and assume that the producer states that the specified value of $m_0$ is 4000 hours, the lifetime of the units follows the CEBTE distribution with parameter $\zeta = 1.50$, the $\beta = 0.25$, the $\gamma = 0.05$, the $r = 5$, and the actual value of m is 8000h. Since an experimenter wants to run a life test experiment for 1000 hours and we are interested in designing the GASP. We have the following values for the termination ratio($a_1$) = 0.5, $\beta$ = 0.25, and $r_2 = 8000/4000$=2. The design parameters can be calculated as $g$ = 43 and $c$ = 2, by using Table 3. Accordingly, a sample of size 215 $(43\times 5 = 215)$ should be taken, with 5 units being given to each of the 43 groups. The lot is ultimately accepted if there are no more than two units that fail in any group before 1000 hours; if there are more than two, the inspected lot is rejected.

**Table 2:** GASP under the CEBTE model, $\theta = 1.50$, $a = 0.50$ and $\zeta = 1.250$.

| | | $r=5$ | | | | | | $r=10$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $a_1=0.5$ | | | $a_1=1$ | | | $a_1=0.5$ | | | $a_1=1$ | | |
| $\beta$ | $r_2$ | $g$ | $c$ | $p_{a(p)}$ | $g$ | $c$ | $p_{a(p)}$ | $g$ | $c$ | $p_{a(p)}$ | $g$ | $c$ | $p_{a(p)}$ |
| 0.25 | 2 | – | – | – | 44 | 4 | 0.9808 | 32 | 4 | 0.9649 | 3 | 5 | 0.9736 |
| | 4 | 20 | 2 | 0.9831 | 3 | 2 | 0.9799 | 4 | 2 | 0.966 | 1 | 3 | 0.9905 |
| | 6 | 5 | 1 | 0.9546 | 3 | 2 | 0.9685 | 4 | 2 | 0.9875 | 1 | 2 | 0.9815 |
| | 8 | 5 | | 0.9711 | 1 | 1 | 0.9815 | 2 | 1 | 0.9521 | 1 | 2 | 0.9914 |
| 0.1 | 2 | – | – | – | 73 | 4 | 0.9683 | 259 | 5 | 0.9765 | 5 | 5 | 0.9564 |
| | 4 | 33 | 2 | 0.9723 | 4 | 2 | 0.9733 | 15 | 3 | 0.9897 | 2 | 3 | 0.9810 |
| | 6 | 33 | 2 | 0.9903 | 4 | 2 | 0.9923 | 6 | 2 | 0.9813 | 1 | 2 | 0.9815 |
| | 8 | 7 | | 0.9598 | 2 | 1 | 0.9634 | 6 | 2 | 0.9904 | 1 | 2 | 0.9914 |
| 0.05 | 2 | – | – | – | 95 | 4 | 0.9589 | 337 | 5 | 0.9695 | – | – | – |
| | 4 | 43 | 2 | 0.9641 | 5 | 2 | 0.9667 | 20 | 3 | 0.9863 | 2 | 3 | 0.9810 |
| | 6 | 43 | 2 | 0.9874 | 5 | 2 | 0.9904 | 7 | 2 | 0.9782 | 2 | 2 | 0.9633 |
| | 8 | 43 | | 0.9937 | 2 | 1 | 0.9634 | 7 | 2 | 0.9888 | 2 | 2 | 0.9828 |
| 0.01 | 2 | – | – | – | – | – | – | 517 | 5 | 0.9537 | – | – | – |
| | 4 | 535 | 3 | 0.9895 | 7 | 2 | 0.9537 | 30 | 3 | 0.9795 | 2 | 3 | 0.9716 |
| | 6 | 65 | 2 | 0.9810 | 7 | 2 | 0.9866 | 11 | 2 | 0.9659 | 2 | 2 | 0.9633 |
| | 8 | 65 | 2 | 0.9905 | 7 | 2 | 0.9941 | 11 | 2 | 0.9825 | 2 | 2 | 0.9828 |

Remark: Cells for a very large sample size are marked with hyphens (–).

**Table 3:** GASP under the CEBTE model, $\theta = 1.50$, $a = 0.50$ and $\zeta = 1.50$.

| | | $r=5$ | | | | | | $r=10$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $a_1=0.5$ | | | $a_1=1$ | | | $a_1=0.5$ | | | $a_1=1$ | | |
| $\beta$ | $r_2$ | $g$ | $c$ | $p_{a(p)}$ | $g$ | $c$ | $p_{a(p)}$ | $g$ | $c$ | $p_{a(p)}$ | $g$ | $c$ | $p_{a(p)}$ |
| 0.25 | 2 | 43 | 2 | 0.9511 | 7 | 3 | 0.9798 | 22 | 3 | 0.9774 | 2 | 4 | 0.9737 |
| | 4 | 7 | 1 | 0.9747 | 1 | 1 | 0.9773 | 3 | 1 | 0.9544 | 1 | 2 | 0.9884 |
| | 6 | 7 | 1 | 0.9895 | 1 | 1 | 0.9926 | 3 | 1 | 0.9807 | 1 | 1 | 0.9696 |
| | 8 | 7 | 1 | 0.9941 | 1 | 1 | 0.9963 | 3 | 1 | 0.9889 | 1 | 1 | 0.9846 |
| 0.1 | 2 | 792 | 3 | 0.9763 | 12 | 3 | 0.9657 | 36 | 3 | 0.9634 | 3 | 4 | 0.9608 |
| | 4 | 12 | 1 | 0.9570 | 2 | 1 | 0.9552 | 10 | 2 | 0.992 | 1 | 2 | 0.9884 |
| | 6 | 12 | 1 | 0.9821 | 2 | 1 | 0.9852 | 4 | 1 | 0.9743 | 1 | 1 | 0.9696 |
| | 8 | 12 | 1 | 0.9899 | 2 | 1 | 0.9927 | 4 | 1 | 0.9853 | 1 | 1 | 0.9846 |
| 0.05 | 2 | – | – | – | 15 | 3 | 0.9573 | 47 | 3 | 0.9524 | 7 | 5 | 0.986 |
| | 4 | 92 | 2 | 0.9934 | 2 | 1 | 0.9552 | 13 | 2 | 0.9896 | 2 | 2 | 0.977 |
| | 6 | 15 | 1 | 0.9777 | 2 | 1 | 0.9852 | 5 | 1 | 0.968 | 1 | 1 | 0.9696 |
| | 8 | 15 | 1 | 0.9873 | 2 | 1 | 0.9927 | 5 | 1 | 0.9816 | 1 | 1 | 0.9846 |
| 0.01 | 2 | – | – | – | 146 | 4 | 0.9845 | 346 | 4 | 0.978 | 10 | 5 | 0.9801 |
| | 4 | 141 | 2 | 0.9899 | 7 | 2 | 0.9919 | 20 | 2 | 0.9841 | 2 | 2 | 0.977 |
| | 6 | 23 | 1 | 0.9661 | 3 | 1 | 0.9779 | 7 | 1 | 0.9555 | 2 | 2 | 0.9955 |
| | 8 | 23 | 1 | 0.9807 | 3 | 1 | 0.9891 | 7 | 1 | 0.9744 | 2 | 1 | 0.9694 |

Remark: Cells for a very large sample size are marked with hyphens (–).

## 4 Real-life applications

To demonstrate the practical utility of the proposed CEBTE model, real data are applied. The actual data sets are displayed in Table 4 along with some basic descriptive details. Using median life as a quality indicator is preferable to obtain better results because the data sets are skewed to the right. Fifty observations of the carbon fiber's breaking strength under stress are included in the first data set, expressed in Gba units. Almarashi *et al.* [20] used the data set recently and developed, a GASP under the four parameters MOKw-E model. The second data consists of a fleet of 13 Boeing 720 jets' total number of successive air conditioning system failures is included in the data set. Additionally, Kumar *et al.* [34] also used the failure data for lifetime analysis. The third data set shows the survival rates (in days) of 72 guinea pigs exposed to virulent tubercle bacilli. Sivakumar *et al.* [35] used the same data and proposed the GASP using odd generalized exponential log-logistic (OGELL) distribution. The data sets are analyzed using R programming language and the unknown parameters are estimated via the maximum likelihood approach using widely adopted R package AdequacyModel. It provides, a comprehensive output analysis including maximum likelihood estimates (MLEs) with standard errors (SEs), information criterion, and goodness of fit test namely Anderson Darling (AD), Craḿer-Von Mises (CM), and Kolmogorov-Smirnov (KS) tests. The analysis of the first data as follows $\hat{\varpi} = 0.4594961\,(0.08840372)$, $\hat{\zeta} = 2.6202802\,(1.62993317)$, $\hat{a} = 0.0118563\,(0.01164902)$, $\hat{\theta} = 2.0335270\,(3.80770608)$, under KS test, the maximum gap between actual and fitted according to CEBTE model is 0.065441 with $p$-val=0.9830. The fitted second data yields the MLEs (SEs) $\hat{\varpi} = 0.01047087\,(0.001014997)$, $\hat{\zeta} = 2.37034548\,(0.701730513)$, $\hat{a} = 0.01406264\,(0.008131268)$, $\hat{\theta} = 2.68426527\,(2.051831391)$, KS=0.075694 and $p$-val=0.2316. The fitted third data are as follows: $\hat{\varpi} = 1.183425440\,(0.136343)$, $\hat{\zeta} = 9.960142689\,(0.841602)$, $\hat{a} = 0.012643574\,(0.012371)$, $\hat{\theta} = 0.001519992\,(0.00007972559)$. The proposed CEBTE model outperformed (improved KS test 0.065441 and $p$-val=0.9830) than MOKw-E model, recently used by Almarashi *et al.* [20] for data set 1, designing GASP with KS=0.0680 and $p$-val=0.9743. Similarly, the KS test under OGELL distribution fitted to the survival time data [35] remains 0.089 and $p$-val=0.617. The proposed CEBTE model again yields a better fit against OGELL distribution with KS=0.0874 with $p$-value=0.6402. The graphical representation of the data sets shown in Figures 3–7 include the plot of estimated PDF, CDF, hazard rate function (HRF), probability-probability (P-P) plot Kaplan-Meier (K-M) curves, and TTT. Under the CEBTE model, the actual and fitted values for all estimated entities concur well. By using estimated values, Tables 5–8 demonstrate the GASP when a product's life follows a CEBTE model, displaying least design parameters ($g$ and $c$). The analysis finds revealed that the results are consistent with those shown in Table 2 and Table 3.

**Table 4:** Data sets with basic descriptive information's.

| **Data-I: Breaking stress of carbon fibers** | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1.1200 | 0.1700 | 0.6400 | 4.3200 | 1.2200 | 0.3700 | 1.1600 | 1.4200 | 0.0900 | 1.6700 |
| 0.1300 | 0.2500 | 0.0800 | 0.0400 | 2.3500 | 0.2000 | 0.7800 | 0.3400 | 1.0200 | 0.1700 |
| 1.7600 | 2.3900 | 0.5000 | 1.3500 | 3.3600 | 0.4500 | 0.9000 | 2.9200 | 6.5300 | 1.6200 |
| 7.4600 | 3.1900 | 2.4900 | 1.4000 | 7.4900 | 0.5700 | 0.1400 | 0.6300 | 5.2300 | 0.7100 |
| 0.6800 | 0.1200 | 0.0900 | 3.4700 | 5.9300 | 1.8200 | 4.2000 | 7.2900 | 3.1300 | 3.4100 |
| **Descriptive information** | | | | | | | | | |
| $n$ | $x_0$ | $x_m$ | $Q_1$ | $Q_3$ | $\bar{x}$ | $\tilde{x}$ | $\sigma$ | $S_k$ | $K_u$ |
| 50 | 0.0400 | 7.4900 | 0.3900 | 3.0780 | 1.9750 | 1.1900 | 2.1157 | 1.3253 | 0.8032 |

| **Data-II: Successive failures for the air conditioning system** | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 194.00 | 413.00 | 90.00 | 74.00 | 55.00 | 23.00 | 97.00 | 50.00 | 359.00 | 50.00 |
| 130.00 | 487.00 | 57.00 | 102.00 | 15.00 | 14.00 | 10.00 | 57.00 | 320.00 | 261.00 |
| 51.00 | 44.00 | 9.00 | 254.00 | 493.00 | 33.00 | 18.00 | 209.00 | 41.00 | 58.00 |
| 60.00 | 48.00 | 56.00 | 87.00 | 11.00 | 102.00 | 12.00 | 5.00 | 14.00 | 14.00 |
| 29.00 | 37.00 | 186.00 | 29.00 | 104.00 | 7.00 | 4.00 | 72.00 | 270.00 | 283.00 |
| 7.00 | 61.00 | 100.00 | 61.00 | 502.00 | 220.00 | 120.00 | 141.00 | 22.00 | 603.00 |
| 35.00 | 98.00 | 54.00 | 100.00 | 11.00 | 181.00 | 65.00 | 49.00 | 12.00 | 239.00 |
| 14.00 | 18.00 | 39.00 | 3.00 | 12.00 | 5.00 | 32.00 | 9.00 | 438.00 | 43.00 |
| 134.00 | 184.00 | 20.00 | 386.00 | 182.00 | 71.00 | 80.00 | 188.00 | 230.00 | 152.00 |
| 5.00 | 36.00 | 79.00 | 59.00 | 33.00 | 246.00 | 1.00 | 79.00 | 3.00 | 27.00 |
| 201.00 | 84.00 | 27.00 | 156.00 | 21.00 | 16.00 | 88.00 | 130.00 | 14.00 | 118.00 |
| 44.00 | 15.00 | 42.00 | 106.00 | 46.00 | 230.00 | 26.00 | 59.00 | 153.00 | 104.00 |
| 20.00 | 206.00 | 5.00 | 66.00 | 34.00 | 29.00 | 26.00 | 35.00 | 5.00 | 82.00 |
| 31.00 | 118.00 | 326.00 | 12.00 | 54.00 | 36.00 | 34.00 | 18.00 | 25.00 | 120.00 |
| 31.00 | 22.00 | 18.00 | 216.00 | 139.00 | 67.00 | 310.00 | 3.00 | 46.00 | 210.00 |
| 57.00 | 76.00 | 14.00 | 111.00 | 97.00 | 62.00 | 39.00 | 30.00 | 7.00 | 44.00 |
| 11.00 | 63.00 | 23.00 | 22.00 | 23.00 | 14.00 | 18.00 | 13.00 | 34.00 | 16.00 |
| 18.00 | 130.00 | 90.00 | 163.00 | 208.00 | 1.00 | 24.00 | 70.00 | 16.00 | 101.00 |
| 52.00 | 208.00 | 95.00 | 62.00 | 11.00 | 191.00 | 14.00 | 71.00 | | |
| **Descriptive information** | | | | | | | | | |
| $n$ | $x_0$ | $x_m$ | $Q_1$ | $Q_3$ | $\bar{x}$ | $\tilde{x}$ | $\sigma$ | $S_k$ | $K_u$ |
| 188 | 1.00 | 603.00 | 20.750 | 118.00 | 92.0745 | 54.00 | 107.916 | 2.1392 | 5.0231 |

**Table 5:** A GASP under the CEBTE model, $\theta = 2.0330$, $a = 0.0118$ and $\zeta = 2.620$.

| | | $r=5$ | | | | | | $r=10$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $a_1=0.5$ | | | $a_1=1$ | | | $a_1=0.5$ | | | $a_1=1$ | | |
| $\beta$ | $r_2$ | $g$ | $c$ | $p_{a(p)}$ | $g$ | $c$ | $p_{a(p)}$ | $g$ | $c$ | $p_{a(p)}$ | $g$ | $c$ | $p_{a(p)}$ |
| 0.25 | 2 | – | – | – | – | – | – | – | – | – | – | – | – |
| | 4 | 37 | 3 | 0.9716 | 7 | 3 | 0.9609 | 8 | 4 | 0.9762 | 3 | 5 | 0.9849 |
| | 6 | 37 | 3 | 0.9633 | 7 | 3 | 0.9876 | 8 | 4 | 0.9788 | 1 | 3 | 0.9584 |
| | 8 | 7 | 2 | 0.9800 | 3 | 2 | 0.9629 | 3 | 3 | 0.9902 | 1 | 3 | 0.9796 |
| 0.1 | 2 | – | – | – | – | – | – | – | – | – | – | – | – |
| | 4 | 61 | 3 | 0.9536 | 73 | 4 | 0.9814 | 12 | 4 | 0.9646 | 5 | 5 | 0.9749 |
| | 6 | 61 | 3 | 0.9855 | 12 | 3 | 0.9788 | 12 | 4 | 0.9649 | 3 | 4 | 0.9770 |
| | 8 | 12 | 2 | 0.9660 | 4 | 2 | 0.9509 | 5 | 3 | 0.9837 | 2 | 3 | 0.9597 |
| 0.05 | 2 | – | – | – | – | – | – | – | – | – | – | – | – |
| | 4 | 954 | 4 | 0.9815 | 95 | 4 | 0.9759 | 16 | 4 | 0.953 | 7 | 5 | 0.9651 |
| | 6 | 80 | 3 | 0.9810 | 15 | 3 | 0.9735 | 16 | 4 | 0.9512 | 4 | 4 | 0.9695 |
| | 8 | 15 | 2 | 0.9576 | 15 | 3 | 0.9884 | 7 | 3 | 0.9772 | 2 | 3 | 0.9597 |
| 0.01 | 2 | – | – | – | – | – | – | – | – | – | – | – | – |
| | 4 | – | – | – | 146 | 4 | 0.9632 | 75 | 5 | 0.9769 | 10 | 5 | 0.9505 |
| | 6 | 122 | 4 | 0.9712 | 23 | 3 | 0.9597 | 24 | 4 | 0.9818 | 5 | 4 | 0.9620 |
| | 8 | 122 | 3 | 0.9875 | 23 | 3 | 0.9823 | 10 | 3 | 0.9676 | 5 | 4 | 0.9851 |

Remark: Cells for a very large sample size are marked with hyphens (–).

**Table 6:** Proposed GASP under the CEBTE model with least design parameters.

| | Data-I | | | | Data-II | | | | Data-III | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $r_2$ | 2 | 4 | 6 | 8 | 2 | 4 | 6 | 8 | 2 | 4 | 6 | 8 |
| $n$ | – | 100 | 50 | 50 | – | 100 | 50 | 30 | 20 | 10 | 10 | 10 |
| $g$ | – | 10 | 5 | 5 | – | 10 | 5 | 3 | 2 | 1 | 1 | 1 |
| $c$ | – | 5 | 4 | 4 | – | 5 | 4 | 3 | 1 | 1 | 1 | 0 |
| $p_{a(p)}$ | – | 0.9505 | 0.962 | 0.9851 | – | 0.9677 | 0.9765 | 0.9817 | 0.963 | 0.9791 | 0.9978 | 0.9994 |

**Table 7:** A GASP under the CEBTE model, $\theta = 2.6843$, $a = 0.0141$ and $\zeta = 2.3703$.

| | | $r=5$ | | | | | | $r=10$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $a_1=0.5$ | | | $a_1=1$ | | | $a_1=0.5$ | | | $a_1=1$ | | |
| $\beta$ | $r_2$ | $g$ | $c$ | $p_{a(p)}$ | $g$ | $c$ | $p_{a(p)}$ | $g$ | $c$ | $p_{a(p)}$ | $g$ | $c$ | $p_{a(p)}$ |
| 0.25 | 2 | – | – | – | – | – | – | – | – | – | – | – | – |
| | 4 | 42 | 3 | 0.9806 | 7 | 3 | 0.9714 | 9 | 4 | 0.9852 | 2 | 4 | 0.9613 |
| | 6 | 8 | 2 | 0.9735 | 3 | 2 | 0.9507 | 4 | 3 | 0.9842 | 1 | 3 | 0.9712 |
| | 8 | 8 | 2 | 0.9864 | 3 | 2 | 0.9743 | 2 | 2 | 0.9673 | 1 | 3 | 0.9871 |
| 0.1 | 2 | – | – | – | – | – | – | – | – | – | – | – | – |
| | 4 | 70 | 3 | 0.9679 | 12 | 3 | 0.9514 | 14 | 4 | 0.977 | 5 | 5 | 0.9837 |
| | 6 | 13 | 2 | 0.9573 | 12 | 3 | 0.9862 | 6 | 3 | 0.9763 | 3 | 4 | 0.9858 |
| | 8 | 13 | 2 | 0.9780 | 4 | 2 | 0.9659 | 3 | 2 | 0.9513 | 2 | 3 | 0.9743 |
| 0.05 | 2 | – | – | – | – | – | – | – | – | – | – | – | – |
| | 4 | 91 | 3 | 0.9585 | 95 | 4 | 0.9840 | 18 | 4 | 0.9706 | 7 | 5 | 0.9773 |
| | 6 | 91 | 3 | 0.9885 | 15 | 3 | 0.9827 | 7 | 3 | 0.9725 | 4 | 4 | 0.9812 |
| | 8 | 17 | 2 | 0.9714 | 5 | 2 | 0.9576 | 7 | 3 | 0.9882 | 2 | 3 | 0.9743 |
| 0.01 | 2 | – | – | – | – | – | – | – | – | – | – | – | – |
| | 4 | | | | 146 | 4 | 0.9755 | 27 | 4 | 0.9562 | 10 | 5 | 0.9677 |
| | 6 | 140 | 3 | 0.9823 | 23 | 3 | 0.9736 | 11 | 3 | 0.9571 | 5 | 4 | 0.9765 |
| | 8 | 25 | 2 | 0.9582 | 23 | 3 | 0.9893 | 11 | 3 | 0.9816 | 3 | 3 | 0.9817 |

Remark: Cells for a very large sample size are marked with hyphens (–).

**Table 8:** A GASP under the CEBTE model, $\theta = 0.00152$, $a = 0.01264$ and $\zeta = 9.9601$.

| | | $r=5$ | | | | | | $r=10$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $a_1=0.5$ | | | $a_1=1$ | | | $a_1=0.5$ | | | $a_1=1$ | | |
| $\beta$ | $r_2$ | $g$ | $c$ | $p_{a(p)}$ | $g$ | $c$ | $p_{a(p)}$ | $g$ | $c$ | $p_{a(p)}$ | $g$ | $c$ | $p_{a(p)}$ |
| 0.25 | 2 | 10 | 1 | 0.9510 | 7 | 3 | 0.9892 | 9 | 2 | 0.9886 | 2 | 1 | 0.9630 |
| | 4 | 2 | 0 | 0.9694 | 1 | 1 | 0.9950 | 1 | 0 | 0.9694 | 1 | 1 | 0.9791 |
| | 6 | 2 | 0 | 0.9900 | 1 | 0 | 0.9645 | 1 | 0 | 0.9900 | 1 | 1 | 0.9978 |
| | 8 | 2 | 0 | 0.9960 | 1 | 0 | 0.9846 | 1 | 0 | 0.9960 | 1 | 0 | 0.9994 |
| 0.1 | 2 | 111 | 2 | 0.9872 | 12 | 3 | 0.9815 | 15 | 2 | 0.9810 | 3 | 4 | 0.9803 |
| | 4 | 16 | 1 | 0.9985 | 2 | 1 | 0.9900 | 5 | 1 | 0.9979 | 1 | 1 | 0.9791 |
| | 6 | 4 | 0 | 0.9802 | 1 | 0 | 0.9645 | 2 | 0 | 0.9802 | 1 | 1 | 0.9978 |
| | 8 | 4 | 0 | 0.9920 | 1 | 0 | 0.9846 | 2 | 0 | 0.9920 | 1 | 0 | 0.9694 |
| 0.05 | 2 | 144 | 2 | 0.9834 | 15 | 3 | 0.9769 | 19 | 2 | 0.9790 | 4 | 4 | 0.9738 |
| | 4 | 20 | 1 | 0.9981 | 2 | 1 | 0.9900 | 6 | 1 | 0.9975 | 1 | 1 | 0.9791 |
| | 6 | 5 | 0 | 0.9753 | 1 | 0 | 0.9645 | 3 | 0 | 0.9704 | 1 | 1 | 0.9978 |
| | 8 | 5 | 0 | 0.9900 | 1 | 0 | 0.9846 | 3 | 0 | 0.9881 | 1 | 0 | 0.9694 |
| 0.01 | 2 | 222 | 2 | 0.9746 | 23 | 3 | 0.9648 | 30 | 2 | 0.9624 | 5 | 4 | 0.9674 |
| | 4 | 31 | 1 | 0.9970 | 3 | 1 | 0.9851 | 9 | 1 | 0.9962 | 2 | 1 | 0.9587 |
| | 6 | 7 | 0 | 0.9656 | 3 | 1 | 0.9985 | 4 | 0 | 0.9608 | 2 | 1 | 0.9955 |
| | 8 | 7 | 0 | 0.9861 | 2 | 0 | 0.9694 | 4 | 0 | 0.9841 | 1 | 0 | 0.9694 |

## *4.1 A comparative study*

Recently, Fayomi *et al.* [9] and Almarashi *et al.* [20] proposed a GASP using the extended exponential-based distributions namely the EBellE and the MOKwE distributions respectively. In Table 9, we compare the design parameters of the proposed GASP based on the CEBTE distribution with the MOKwE by Almarashi *et al.* [20] and the EBellE by Fayomi *et al.* [9] distribution. When $\beta = 0.10$, $a_1 = 1$, and $r = 10$, the proposed CEBTE distribution shows the least design parameters $g$ and $c$ as compared to the EBellE and the MOKwE distribution. While the real data comparison yields from Table 8, when $\beta = 0.25$, $a_1 = 1$, and $r = 10$ showing the minimum design parameters and higher OC value against the OGELL distribution by Sivakumar *et al.* [35]. The detailed summary of design parameters, OC values and $n$ based on third data are presented in Table 10. Furthermore, Figure 6 is designed to compare the design parameters of the CEBTE model versus the MOKwE, EBellE and OGELL models for the $r_2 = 4$. It is quite clear from the Tables 9-10 and Figure 6 that the proposed GASP under the CEBTE model has optimized values of the design parameters as compared to the MOKwE, EBellE and ODELL models.

**Table 9:** Comparison between CEBTE, MOKwE and EBellE distributions.

| | CEBTE | | | MOKwE | | | EBellE | | |
|---|---|---|---|---|---|---|---|---|---|
| $r_2$ | $g$ | $c$ | $p_{a(p)}$ | $g$ | $c$ | $p_{a(p)}$ | $g$ | $c$ | $p_{a(p)}$ |
| 4 | 2 | 3 | 0.9810 | 6 | 5 | 0.9809 | 5 | 5 | 0.9843 |
| 6 | 1 | 2 | 0.9815 | 3 | 4 | 0.9878 | 2 | 3 | 0.9503 |
| 8 | 1 | 2 | 0.9914 | 2 | 3 | 0.9794 | 2 | 3 | 0.9800 |

**Table 10:** Comparison of the CEBTE and the OGELL distribution using actual data.

| CEBTE | | | | OGELL | | | |
|---|---|---|---|---|---|---|---|
| $r_2$ | 4 | 6 | 8 | $r_2$ | 4 | 6 | 8 |
| $n$ | 10 | 10 | 10 | $n$ | 30 | 20 | 10 |
| $g$ | 1 | 1 | 1 | $g$ | 3 | 2 | 1 |
| $c$ | 1 | 1 | 0 | $c$ | 2 | 1 | 0 |
| $p_{a(p)}$ | 0.9791 | 0.9978 | 0.9994 | $p_{a(p)}$ | 0.9553 | 0.9868 | 0.962 |

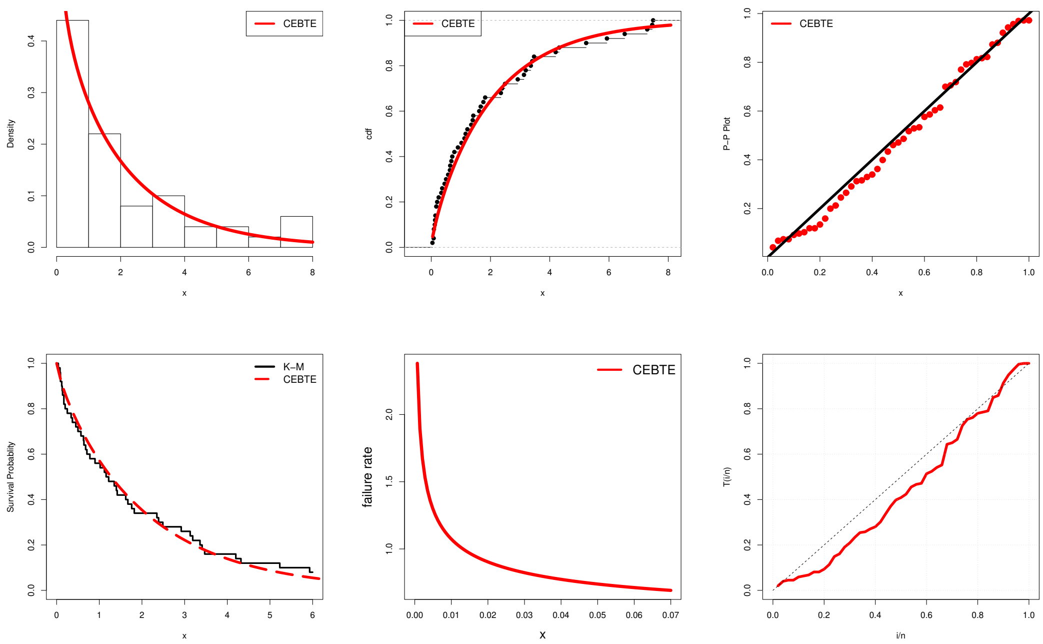


**Fig. 3:** Estimated plots based on the CEBTE model, Data-I.

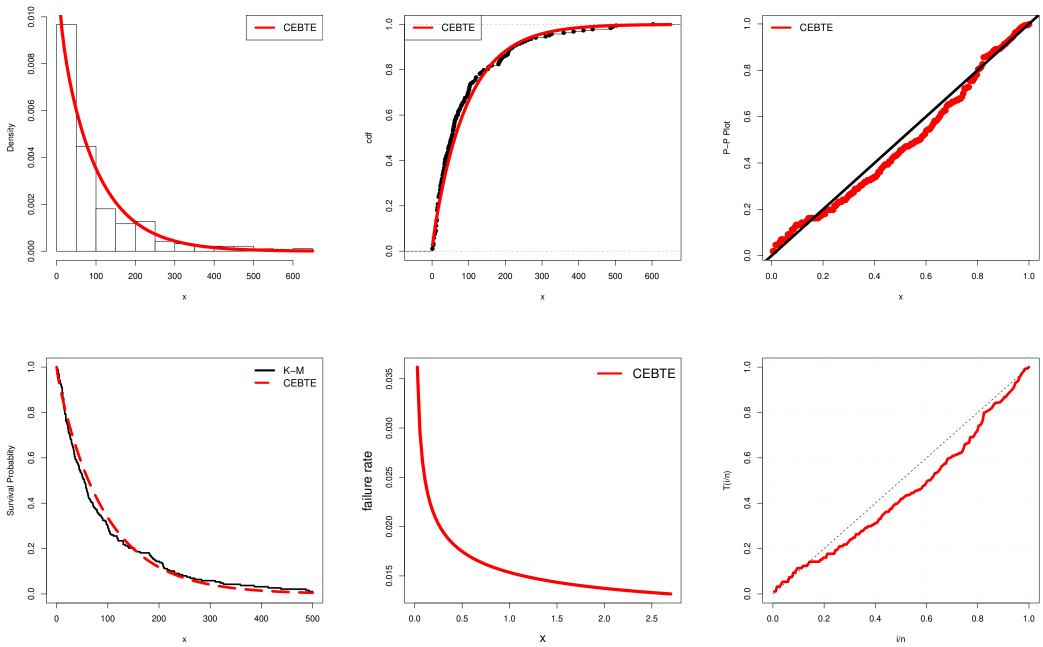


**Fig. 4:** Estimated plots based on the CEBTE model, Data-II.

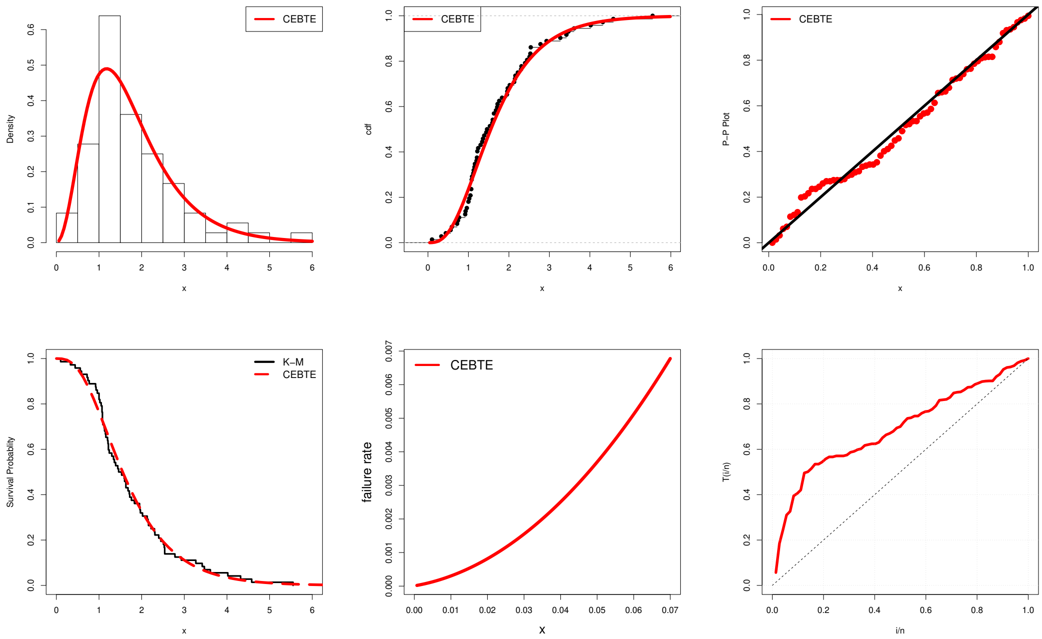


**Fig. 5:** Estimated plots based on the CEBTE model, Data-III.

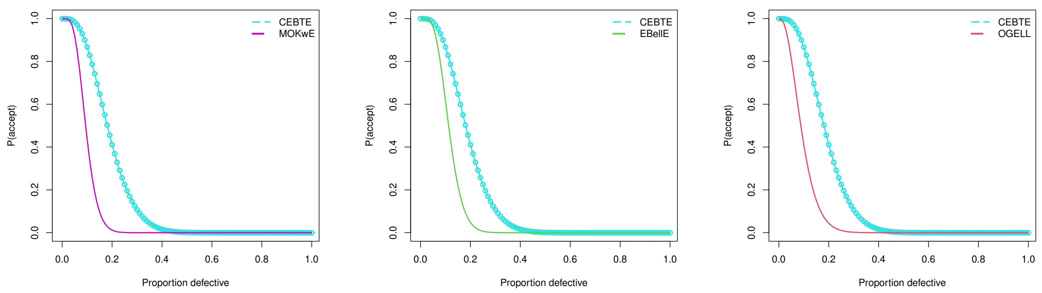


**Fig. 6:** OC curves of the CEBTE model versus MOKwE, EBellE and OGELL models based on the sample sizes based on the Tables 9-10.

## *4.2 Actuarial data and risk measures*

Recently, Afify *et al.* [36] used the unemployment insurance-related claims data and applied the alpha power exponentiated exponential (APEE) distribution. The fitted APEE model showed the goodness-of-fit measures such as AD(0.70096), CM(0.12743), and KS(0.09917) tests with p-value(0.61833). On the other side, the proposed CEBTE model is fitted to the same unemployment insurance-related claims data and with improved goodness-of-fit measures such as AD(0.6961), CM(0.1257), and KS(0.0970) tests with p-value(0.6458) compare to the APEE model. The MLEs of the fitted CEBTE model are as follows: $\hat{\varpi} = 0.03414481\,(0.008431383)$, $\hat{\zeta} = 0.33161910\,(0.261097963)$, $\hat{a} = 12.76246655\,(4.345384632)$, $\hat{\theta} = 6.26096893\,(5.836776096)$. The following Figure 7 shows the good agreement between actual and predicted based on the CEBTE model. Figure 8 shows the graphical illustration of the two commonly used risk measures namely VaR and ES.

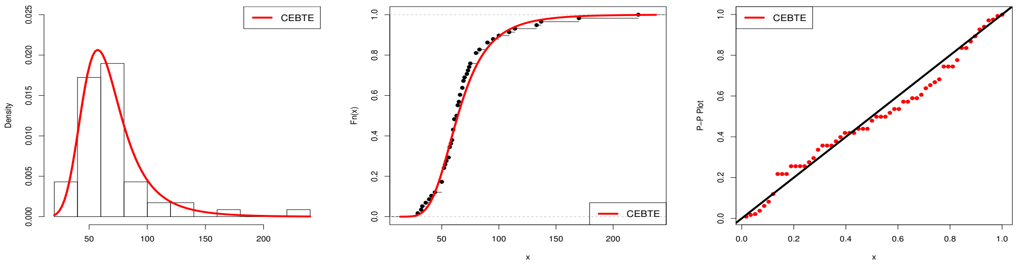


**Fig. 7:** Estimated plots based on CEBTE model, Data-IV.

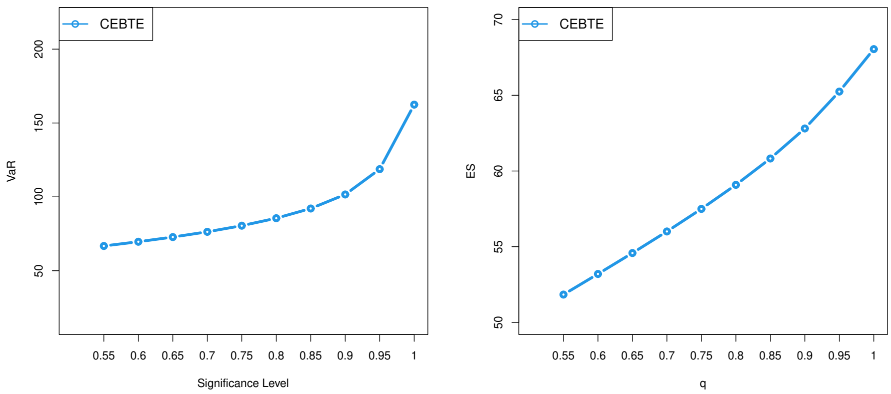


**Fig. 8:** VaR and ES based on MLEs, Data-IV.

## 5 Concluding Remarks

In this article, we presented a new CEBT-G family of distributions by compounding approach. Furthermore, the linear representation of the densities is presented and can be utilized to determine a number of significant aspects of the special model. A special model named CEBTE with a flexible failure rate function is presented with the proposed GASP by taking the median lifetime as a quality parameter. The analysis of the data yields optimized outcomes of the GASP for the CEBTE model so as the number of groups and acceptance numbers tend to decrease rather than the OC values steadily increase as the true median life tends to increase. More specifically, the GASP may be used to recruit embedded items for a trial to be run simultaneously, which will be useful in optimizing the test time and cost as multiple objects can be evaluated simultaneously. The proposed CEBTE distribution produces convincing results based on the comparison of real data sets as well. In addition, to enhance the goodness of fit metrics, the suggested model also minimized design parameters to achieve better lot quality checks within limited financial and time limitations. The proposed family has a closed-form quantile function, which allows it to be further extended to regression analysis and quantile regression. Special models, however, can offer potentially better results with over-dispersion data.

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