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Some New Comparison Theorems for Double Splittings of Matrices

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Abstract: In this paper, we further investigate the double splitting iterative methods for solving linear systems. Building on the previous work by Song and Song [Convergence for nonnegative double splittings of matrices, Calcolo, (2011) 48: 245-260], some new comparison theorems for the spectral radius of double splittings of matrices under suitable conditions are presented.

Keywords: Linear systems, matrix, double splitting, single splitting, spectral radius, comparison theorem.

1. Introduction

Consider the following linear system

$$Ax = b, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a nonsingular matrix, $b \in \mathbb{R}^{n\times 1}$ is a given vector and $x \in \mathbb{R}^{n\times 1}$ is an unknown vector. In order to solve the linear system (1) by iterative methods, the coefficient matrix A is split into

$$A = M - N,$$

where M is nonsingular, is called a single splitting of A in [1]. Based on the above matrix splitting, the basic iterative method for solving (1) is

$$x^{k+1} = M^{-1}Nx^k + M^{-1}b \equiv Tx^k + M^{-1}b, \tag{2}$$

where $k = 0, 1, \ldots$ and $T = M^{-1}N$ is the iteration matrix in (2). Obviously, the iterative method (2) converges to the unique solution of the linear system (1) if and only if the spectral radius $\rho(M^{-1}N)$ of the iteration matrix is smaller than 1. The spectral radius of the iteration matrix is decisive for the convergence and stability, and the smaller it is, the faster the iterative method converges when the spectral radius is smaller than 1. So far, many comparison theorems of single splitting of matrices have been presented in some papers and books [2–8, 13].

In [1], Woźnicki introduced a double splitting of A, i.e., splitting the matrix A in the form

$$A = P - R - S, \tag{3}$$

where P is a nonsingular matrix. The corresponding iterative scheme is spanned by three successive iterations,

$$x^{k+1} = P^{-1}Rx^k + P^{-1}Sx^{k-1} + P^{-1}b,\ k = 0, 1, 2, \ldots \tag{4}$$

which can be rewritten in the equivalent form

$$\begin{bmatrix} x^{k+1} \\ x^k \end{bmatrix} = \begin{bmatrix} P^{-1}R & P^{-1}S \\ I & 0 \end{bmatrix} \begin{bmatrix} x^k \\ x^{k-1} \end{bmatrix} + \begin{bmatrix} P^{-1}b \\ 0 \end{bmatrix}, \tag{5}$$

where I is the identity matrix. The iterative method given by (5) converges to the unique solution of (1) for all initial vectors x^0, x^1 if and only if the spectral radius of the iteration matrix,

$$W = \begin{bmatrix} P^{-1}R & P^{-1}S \\ I & 0 \end{bmatrix} \tag{6}$$

is less than one, i.e., $\rho(W) < 1$.

Recently, some convergence and comparison results for double splittings of matrices are presented. In [9], Shen and Huang presented some convergence theorems for the double splitting of a monotone matrix or a Hermitian positive definite matrix and obtained two comparison theorems for two double splittings of a monotone matrix. Compared with some results [9], some improved convergence and comparison results for double splitting of a Hermitian positive definite matrix are proposed in [10]. In [12], some convergence results for double splittings of a non-Hermitian positive semidefinite matrix are established. In [11], a comparison theorem for double splittings of different monotone matrices is given. In [13], some convergence and comparison results for

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nonnegative double splittings of matrices are given. In this paper, building on the previous work [13], our basic purpose here is to derive some new comparison theorems for the spectral radius of double splittings of matrices.

2. Preliminaries

For convenience, we give some of the notations, definitions and lemmas which will be used in the sequel.

The matrix A is called nonnegative and denoted $A \geq 0$ if $a_{ij} \geq 0$ for $i, j = 1, 2, \ldots, n$. We write $A \geq B$ $(A > B)$ if $a_{ij} \geq b_{ij}$ $(a_{ij} > b_{ij})$ for $i, j = 1, 2, \ldots, n$. The matrix A is called to be a monotone matrix if $A^{-1} \geq 0$. Matrix A is an L-matrix if $a_{ii} > 0$ $(i = 1, \ldots, n)$ and $a_{ij} < 0$ for all $i, j = 1, \ldots, n; i \neq j$.

Definition 1. [9, 13] *Let A be a nonsingular matrix. Then the double splitting $A = P - R - S$ is*

- *convergent if and only if $\rho(W) < 1$;*
- *a regular double splitting if $P^{-1} \geq 0$, $R \geq 0$ and $S \geq 0$;*
- *a weak regular double splitting if $P^{-1} \geq 0$, $P^{-1}R \geq 0$ and $P^{-1}S \geq 0$;*
- *a nonnegative splitting if $P^{-1}R \geq 0$ and $P^{-1}S \geq 0$.*

Definition 2. *Let A be a nonsingular matrix. The double splitting $A = P - R - S$ is an M-double splitting if P is an M-matrix and $R \geq 0$ and $S \geq 0$.*

Lemma 1. [3] *Let $A \geq 0$. Then*

$$\alpha x \leq Ax, \ x \geq 0, \text{implies } \alpha \leq \rho(A),$$

and

$$Ax \leq \beta x, \ x > 0, \text{implies } \rho(A) \leq \beta.$$

Lemma 2. [9] *Let $A^{-1} \geq 0$ and $A = P - R - S$ be a weak regular double splitting. Then $\rho(W) < 1$.*

Lemma 3. [15] *Let $A \in \mathbb{R}^{n\times n}$ and $A = M_1 - N_1 = M_2 - N_2$ be M-splittings of A (i.e., M_i are M-matrices, $N_i \geq 0, i = 1, 2$) and*

$$N_1 \geq N_2, \ N_1 \neq N_2, \ N_2 \neq 0.$$

Then exactly one of the following statements holds:

(1) $0 \leq \rho(M_2^{-1}N_2) < \rho(M_1^{-1}N_1) < 1$. *In addition, if A is irreducible, the first inequality is also strict.*
(2) $\rho(M_2^{-1}N_2) = \rho(M_1^{-1}N_1) = 1$.
(3) $\rho(M_2^{-1}N_2) > \rho(M_1^{-1}N_1) > 1$.

3. Comparison theorems

Let

$$A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2 \tag{7}$$

be two double splittings of A. Then we define

$$W_1 = \begin{bmatrix} P_1^{-1}R_1 & P_1^{-1}S_1 \\ I & 0 \end{bmatrix} \text{ and } W_2 = \begin{bmatrix} P_2^{-1}R_2 & P_2^{-1}S_2 \\ I & 0 \end{bmatrix}.$$

In [13], some comparison theorems for the spectral radius of double splittings of monotone matrices are given, which are described as follows.

Theorem 1. [13] *Let $A^{-1} \geq 0$, and let the two double splittings (7) be nonnegative and convergent. Suppose*

$$P_1 \leq P_2, \ S_1 \leq S_2,$$

then

$$\rho(W_1) \leq \rho(W_2). \tag{8}$$

Corollary 1. [13] *Let $A^{-1} \geq 0$, and let the two double splittings (7) be nonnegative and convergent. Suppose*

$$R_1 \leq R_2, \ S_1 \leq S_2,$$

then

$$\rho(W_1) \leq \rho(W_2).$$

Theorem 2. [13] *Let $A^{-1} \geq 0$, $A = P_1 - R_1 - S_1$ be regular double splitting, and let $A = P_2 - R_2 - S_2$ be nonnegative and convergent double splitting. Suppose*

$$P_1^{-1} \geq P_2^{-1} \text{ and } P_1^{-1}S_1 \leq P_2^{-1}S_2,$$

then $\rho(W_1) \leq \rho(W_2)$.

Based on Lemma 3, we have the following results.

Theorem 3. *Let $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ be two M-double splittings. If $R_1 \leq R_2, \ S_1 \leq S_2$, then $\rho(W_1) \leq \rho(W_2)$.*

Proof. For $i = 1, 2$, let

$$\mathbb{M}_i = \begin{bmatrix} P_i & S_i \\ 0 & I \end{bmatrix}, \mathbb{N}_i = \begin{bmatrix} R_i + S_i & S_i \\ 0 & I \end{bmatrix}.$$

Then

$$\mathbb{A} = \mathbb{M}_i - \mathbb{N}_i \text{ and } W_i = \mathbb{M}_i^{-1}\mathbb{N}_i.$$

Obviously, $\mathbb{A}$ is nonsingular whenever A is nonsingular. Since $R_1 \leq R_2, \ S_1 \leq S_2$, then we have $R_1 + S_1 \leq R_2 + S_2, \ S_1 \leq S_2$. That is,

$$\mathbb{N}_1 \leq \mathbb{N}_2.$$

From Lemma 3, we obtain that $\rho(W_1) \leq \rho(W_2)$. □

Theorem 4. *Let $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ be two M-double splittings. If $P_1 \leq P_2, \ S_1 \leq S_2$, then $\rho(W_1) \leq \rho(W_2)$.*

Proof. By simple computations, we obtain that

$$R_1 + S_1 = P_1 - A, R_2 + S_2 = P_2 - A.$$

It is not difficulty to find that $R_1 + S_1 \leq R_2 + S_2$. Therefore,

$$\mathbb{N}_1 \leq \mathbb{N}_2.$$

From Lemma 3, we obtain that $\rho(W_1) \leq \rho(W_2)$. □

Compared with Theorem 1 and Corollary 1 [13], the condition $A^{-1} \geq 0$ in Theorems 3 and 4 is not necessary.

Theorem 5. *Let $A \geq 0$, and let $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ be nonnegative splitting. If $P_1^{-1} \geq P_2^{-1}$ and $P_1^{-1}R_1 \geq P_2^{-1}R_2$, then $\rho(W_1) \leq \rho(W_2) < 1$ for $0 < \rho(W_2) < 1$.*

Proof. Obviously, $W_1 \geq 0$ and $W_2 \geq 0$. By the Perron-Frobenius theorem [3], there exists a vector,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 0, \; x \neq 0,$$

such that $W_2 x = \rho(W_2)x$, i.e.,

$$P_2^{-1}R_2 x_1 + P_2^{-1}S_2 x_2 = \rho(W_2)x_1,$$
$$x_1 = \rho(W_2)x_2.$$

Then we have

$$\begin{aligned} & W_1 x - \rho(W_2)x \\ = & \begin{bmatrix} P_1^{-1}R_1x_1 + P_1^{-1}S_1x_2 - \rho(W_2)x_1 \\ x_1 - \rho(W_2)x_2 \end{bmatrix} \\ = & \begin{bmatrix} (P_1^{-1}R_1 - P_2^{-1}R_2)x_1 + \frac{1}{\rho(W_2)}(P_1^{-1}S_1 - P_2^{-1}S_2)x_1 \\ x_1 - \rho(W_2)x_2 \end{bmatrix} \end{aligned}$$

Since $P_1^{-1}R_1 \geq P_2^{-1}R_2$ and $0 < \rho(W_2) < 1$, then

$$\begin{aligned} & W_1 x - \rho(W_2)x \\ \leq & \frac{1}{\rho(W_2)} \begin{bmatrix} (P_1^{-1}R_1 - P_2^{-1}R_2)x_1 + (P_1^{-1}S_1 - P_2^{-1}S_2)x_1 \\ 0 \end{bmatrix} \\ = & \frac{1}{\rho(W_2)} \begin{bmatrix} (P_1^{-1}R_1 + P_1^{-1}S_1)x_1 - (P_2^{-1}R_2 + P_2^{-1}S_2)x_1 \\ 0 \end{bmatrix} \\ = & \frac{1}{\rho(W_2)} \begin{bmatrix} P_1^{-1}(R_1 + S_1)x_1 - P_2^{-1}(R_2 + S_2)x_1 \\ 0 \end{bmatrix} \\ = & \frac{1}{\rho(W_2)} \begin{bmatrix} P_1^{-1}(P_1 - A)x_1 - P_2^{-1}(P_2 - A)x_1 \\ 0 \end{bmatrix} \\ = & \frac{1}{\rho(W_2)} \begin{bmatrix} (P_2^{-1} - P_1^{-1})Ax_1 \\ 0 \end{bmatrix} \leq 0. \end{aligned}$$

From Lemma 1, we obtain that $\rho(W_1) \leq \rho(W_2) < 1$ for $0 < \rho(W_2) < 1$. □

Based on Theorem 5, we have the following result.

Corollary 2. *Let*

$$A_1 = P_1 - R_1 - S_1, A_2 = P_2 - R_2 - S_2$$

be nonnegative splitting. If $P_1^{-1}A_1 \geq P_2^{-1}A_2$ *and* $P_1^{-1}R_1 \geq P_2^{-1}R_2$, *then* $\rho(W_1) \leq \rho(W_2) < 1$ *for* $0 < \rho(W_2) < 1$.

By investigating Corollary 2, it is easy to see that the conditioners of Corollary 2 are weaker than that of Theorem 3.1 [11]. That is, the result of Corollary 2 holds without $A_1^{-1} \geq 0$ and $A_2^{-1} \geq 0$.

Similarly, we have the following result.

Theorem 6. *Let* $A \geq 0$, *and let* $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ *be nonnegative splitting. If* $P_1^{-1} \geq P_2^{-1}$ *and* $P_1^{-1}S_1 \leq P_2^{-1}S_2$, *then* $\rho(W_1) \leq \rho(W_2) < 1$ *for* $0 < \rho(W_2) < 1$.

Compared with Theorem 2 [13], the condition $A^{-1} \geq 0$ in Theorem 6 is not necessary and instead of it is $A \geq 0$. By investigating Theorem 6, it is easy to see that the conditioners of Theorem 6 are weaker than that of Theorem 2 [13].

Corollary 3. *Let*

$$A_1 = P_1 - R_1 - S_1, A_2 = P_2 - R_2 - S_2$$

be nonnegative splitting. If $P_1^{-1}A_1 \geq P_2^{-1}A_2$ *and* $P_1^{-1}S_1 \leq P_2^{-1}S_2$, *then* $\rho(W_1) \leq \rho(W_2) < 1$ *for* $0 < \rho(W_2) < 1$.

In fact, Corollaries 2 and 3 are mainly results in [16], which implies that Theorems 5 and 6 extend the results of Corollaries 2 and 3 in [16].

4. Numerical examples

In this section, we make use of two examples to illustrate Theorem 3, Theorem 4 and Theorem 5.

Example 4.1 Let

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix},$$

$$P_1 = \begin{bmatrix} 2 & 0 \\ -1 & 4 \end{bmatrix}, R_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, S_1 = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}, R_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, S_2 = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}.$$

In this case, one can easily see that $R_1 \leq R_2$, $S_1 \leq S_2$, which satisfies the conditions of Theorem 3. In the meanwhile, one can also easily see that $P_1 \leq P_2$, $S_1 \leq S_2$, which satisfies the conditions of Theorem 4.

By the simple computations, we have $\rho(W_1) = 0.8846$ and $\rho(W_2) = 0.9164$. Clearly, $\rho(W_1) \leq \rho(W_2) < 1$ hold. That is to say, Theorems 3 and 4 holds true.

Example 4.2 Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

$$P_1 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, R_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, S_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}, R_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, S_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then

$$P_1^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}, \; P_2^{-1} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{5} \end{bmatrix}.$$

and

$$P_1^{-1}R_1 = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}, \; P_2^{-1}R_2 = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{5} \end{bmatrix}.$$

That is to say, we have $P_1^{-1} \geq P_2^{-1}$ and $P_1^{-1}R_1 \geq P_2^{-1}R_2$, which satisfies the conditions of Theorem 5.

By the simple computations, we have $\rho(W_1) = 0.7676$ and $\rho(W_2) = 0.7696$. Clearly, $\rho(W_1) \leq \rho(W_2) < 1$ holds. That is to say, Theorem 5 holds true.

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