Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/080608

Inter-Temporal Optimal Asset Allocation and Time-Varying Risk Aversion

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Received: 16 Aug. 2013, Revised: 21 Dec. 2013, Accepted: 23 Dec. 2013
Published online: 1 Nov. 2014

Abstract: The inter-temporal optimal decision is related to investors risk preference. In this study, we analyze the optimal asset allocation over investment horizon of invariable risk preference indicated by constant risk aversion. To capture dynamic property of risk aversion, we relax the assumption of constant risk aversion and formulate a time-varying function in response to the impacts of time and wealth. Our general decision model built on time-varying risk aversion allows us to further investigate the inter-temporal optimal asset allocation. The numerical evidences from the model show that the optimal allocation of risky assets in portfolios is significantly related to investors risk aversion and that the time diversification is not existed under the time-varying risk aversion.

Keywords: Optimal asset allocation, heterogeneous investors, time-varying risk aversion, time delay, wealth increment

1 Introduction

The traditional inter-temporal decisions are in general built on the utility functions with constant risk aversion. These studies on inter-temporal decision derive a variety of interesting results, among of which, the concept of time diversification is widely accepted by many portfolio managers. According to time diversification, a young investor with longer investment horizon is in the ability of allocating a larger proportion of portfolio to risky asset to gain a higher return in the future. The validity of this intuitively appealing principle, however, remains debatable.

Samuelson [1] and Merton [2] document that allocation of optimal risky asset in a portfolio is independent of the length of the investment horizon. Regardless of the length of investment horizon, investors would hold the invariable portion of risky assets. Thorley [3] further provides evidence, from both mean-variance and CRRA utilities, which are inconsistent with time diversification. Jan [4] compares safe portfolios with perfectly diversified portfolios and also finds contradictions to time diversification. On the other hand, the concept of time diversification receives direct support from the practice. For example, return on a stock investment exceeds that on a safer bond in long term. Gressis and Philippatos [5] created a multi-period portfolio and verified the possibility of time diversification.

However, there have two major limitations in the existing studies [6]. One is that a great number of early studies mainly focus on the utility of constant relative risk aversion, namely CRRA utility. Another is that those investigations are based on the assumption of constant risk-aversion. The main reason is either to simplify the complex of investors risk preference or to obtain the perfect analytic solution of theoretical model. As the CRRA utility cannot cover the characteristics of numerous heterogeneous investors with different risk preference, focusing only on CRRA utility is obviously inconsistent with investment practices. In addition, investors risk preferences, such as risk tolerance or risk impatience, are varied with time length and/or wealth increment [7,8,9,10,11,12]. The assumption of constant risk-aversion cannot explain the impacts of time and wealth on investors risk preference. Hence, the above two constraint prevent us from capturing the properties of investors risk preference and help us to understand why different opinions exist for the concept of time diversification [13,14,15].

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The objective of our research is to develop an appropriate utility model to capture the dynamic characteristics of heterogeneous investors risk aversion in response to the time delay and wealth change, and further measure the inter-temporal optimal asset allocations of heterogeneous investors.

Motivated by the improvement on the potential flaws described above, we combine several typical utility categories into one uniform utility framework, including the typical mean-variance and risk-neutral utility, to facilitates further analysis of optimal decision for heterogeneous investors compared with time diversification. By selecting a proper form from the exponential and power function, we try to characterize the dynamic properties of risk aversion impacted by the interaction of both time and wealth. Such dynamic treatment of risk aversion is more practical relative to simple constant risk aversion. As a result, we construct a new generalized utility model with time-varying risk aversion, and use numerical illustrations to delineate the optimal risky proportion and clarify the concept of time diversification.

This paper is structured as follows. Section 2 reviews three different utility categories and time diversification. Section 3 discusses a desired function of risk aversion that accounts for the experimental evidence in a risky environment. Section 4 presents our model design and numerical evidence. Section 5 concludes.

2 Optimal asset allocations for traditional utility functions

2.1 Classification of traditional utility functions

In response to the constant, decreased and increased risk aversion as wealth increases, the traditional utility functions are classified into three categories: constant relative risk aversion (CRRA), decreasing relative risk aversion (DRRA), and increasing relative risk aversion (IRRA). The three categories cover heterogeneous investors different risk preferences reflected by a variety of utility forms.

To summarize the categories described above, a synthetic formula of utility function [2,3], is adapted

$$U(W)=\frac{(W-c)^{1-\delta}-1}{1-\delta} \tag{1}$$

Where W denotes the investors wealth, c denotes the parameter discriminating utility category, and δ denotes the risk aversion coefficient.

The risk aversion coefficient, δ, takes in general different non-negative constants, representing the various degree of risk aversion. The more the risk averse of investor, the larger the coefficient δ, and vice versa. Based on Eq (1), the relative risk aversion measure, $R(W,c)$, can be expressed as [16]

$$R(W,c)=-\frac{WU''_{WW}(W)}{U'_{W}(W)}=\frac{\delta}{1-(c/W)} \tag{2}$$

Note that $R(W,c)$ is a discriminator to determine the utility category, and dependent on both parameters W and c. When $c=0$, $c>0$ or $c<0$, the values of $R(W,c)$ are respectively constant, decreasing or increasing depending on the condition of wealth increasing, corresponding to the three utility categories mentioned above: CRRA, DRRA and IRRA. Hence, Eq (2) simplifies the three utility categories, and facilitates the analysis of effect on optimal asset allocation from risk aversion under a uniform framework.

2.2 Inter-temporal optimal asset allocations of three utility categories

For the sake of brevity, we assume there are only two assets in the market, risky asset, S, and risk-free asset, T, for investment choice. Then the wealth over t horizons, W_t, can be written as

$$W_t=\lambda_t S_t+(1-\lambda_t)T_t \tag{3}$$

where W_t denotes the investment wealth over t horizons, S_t denotes the income of risky asset over t horizons, T_t denotes the income of risk-free asset over t horizons, and λ_t denotes the initial portion of risky assets at horizon $t-1$.

Given that the risk-free rate on unit horizon, R_F, the risk return on unit horizon, R_S, and the condition of continuous compounded rate, Eq (3) can be rewritten as

$$\begin{aligned} W_t &= \lambda_t W_0 e^{R_S t}+(1-\lambda_t)W_0 e^{R_F t}\\ &= \lambda_t W_0 e^{\mu t+\sigma\varepsilon\sqrt{t}}+(1-\lambda_t)W_0 e^{R_F t} \end{aligned} \tag{4}$$

where W_0 represents investors initial wealth. In general, the return of risky asset, R_S, is assumed as a random variable, following a standard Wiener process: $R_S=\mu t+\sigma\varepsilon\sqrt{t}$, where the expected return, μ, denotes drift ratio, the variance of return, σ, denotes diffuse ratio, and random variable, ε, follows the standard normal distribution. By maximizing expected utility, $E[U(W)]$, an investor makes optimal decision to determine the proportion of wealth allocated to risky asset. As discussed above, a close link between the optimal decision and risk preference is realized by utility function.

Based on Eq (4), Merton [2] drew a conclusion that the optimal proportion allocated to risky asset is independent of investment horizon, which argued against the concept of time diversification. However, this result is only from the CRRA utility category. Whether the other investors also exhibit the similar result needs the evidences of the other two utility categories: DRRA and IRRA. As the analytic solution is hard to obtain, we adopt the numerical illustration to explore the optimal risk proportion of these two utility categories.

Fig. 1 below plots the simulation results of three utility categories. The optimal risk proportion of CRRA, showing as a straight line, stands constant at about 55% level. The visual evidence verifies the derivation in [2]. The curve of DRRA, however, indicates that as the investment horizon increases, the resulting optimal risk percentage shifts toward the upper-right, confirming that investors with DRRA utility prefer more risky asset in the portfolio for a longer investment horizon. The simulation shows the growth of risk proportion from 20% up to 40% as horizon extends from the beginning to the end. In contrast to DRRA, the curve of IRRA moves toward the bottom-right direction. The optimal percentage of risky assets drops to 56% from the original 80%. The differences among three utility categories suggest that the investors with various risk preferences make diverse optimal decisions. Although we find support of time diversification from DRRA illustration, the evidences from the other two categories argue that the occurrence represents only a part of investors risk decisions rather than all heterogeneous investors. In other words, the concept of time diversification is just a particular example of a large number of heterogeneous investors.

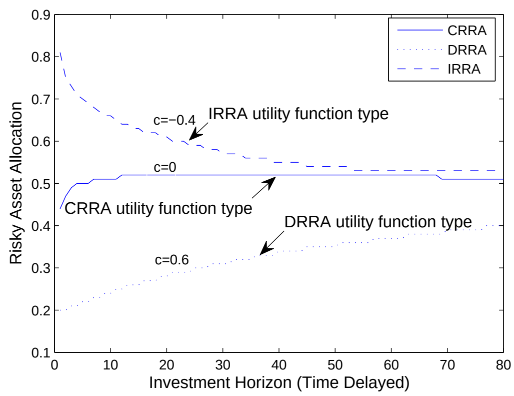

Fig. 1: Optimal asset allocations of three utility categories over investment horizons. Through setting the ranges of $c = 0$, $c > 0$ and $c < 0$, we classify utility functions into the corresponding three categories: CRRA, DRRA and IRRA. The simulation parameters are designed as: (1) the discriminating coefficient $c = 0$ for CRRA, $c = 0.6$ for DRRA and $c = -0.4$ for IRRA; (2) the initial wealth level $W_0 = 1$; (3) the mean $\mu = 0.04$ and standard deviation $\sigma = 0.12$ of Wiener process by means of the empirical data in Chinese A-stock market from Jan., 2006 to Mar., 2011; (3) the risk-free interest rate $R_F = 0.025$ based on yearly fundamental interest rate of central bank of China in 2010; (4) the constant risk-averse coefficient $\delta = 5$.

2.3 Verification of mean-variance and risk-neutral utility function

To test the completeness of the utility classification above, two additional types of utilities: mean-variance utility (MV utility) and risk-neutral utility (RN utility) are included in the discussion. Likewise, the optimal portion of a risky asset is obtained by maximizing the expected utility.

The quadratic utility function, $U(W) = W - \frac{1}{2}\delta W^2$, is used to derive the typical MV utility, where risk aversion, δ, satisfies both conditions: $\delta > 0$ and $\delta W < 1$, to ensure $U^{'}(W) > 0$ and $U^{''}(W) < 0$. By means of the expected form of quadratic utility [16], the MV utility can be obtained

$$\begin{aligned} E[U(W)] &= E[W - \frac{1}{2}\delta W^2] \\ &= [1 - \frac{1}{2}\delta E(W)]E(W) - \frac{1}{2}\delta\sigma^2(W) \\ &= U(E(W), \sigma^2(W)) \end{aligned} \quad (5)$$

The utility of Eq (5) shows a positive relation with wealth mean and a negative relation with wealth variance. Analytically, the MV utility and expected quadratic utility are equivalent and share the same optimal allocation. That $\frac{\partial R(W)}{\partial W} > 0$ indicated by quadratic utility suggests the MV utility actually belongs to IRRA category. The optimal risk proportion decreases with the investment horizon, which is consistent with [3].

Next, for the sake of brevity, we determine a typical RN utility, $U(W) = W$. Due to the features of $U^{'} = 1$ and $U^{''} = 0$, the numerical simulation is a feasible way to explore this optimal allocation. To allow the result more generalized, we still assume that investment wealth follows Eq (4), and then the expected RN utility can be expressed

$$E[U(W)] = \lambda_t W_0 e^{(\mu + \frac{1}{2}\sigma^2)t} + (1 - \lambda_t) W_0 e^{R_F t} \quad (6)$$

By differentiating $E[U(W)]$ with regard to risky share, λ_t, we can obtain the result: $\frac{\partial E[U(W)]}{\partial \lambda_t} = W_0 e^{(\mu + \frac{1}{2}\sigma^2)t} - W_0 e^{R_F t} > 0$. This is consistent with the notion that the expected return on risky asset, μ, is in general greater than the risk-free rate, R_F. $\frac{\partial E[U(W)]}{\partial \lambda_t} > 0$ indicates that if the larger risk proportion is held, the investors would receive the higher expected utility. Therefore, the investors with RN utility would choose a risk percentage as high as possible over the whole horizon. Fig. 2 illustrates such investors always hold 100% risky asset, which is consistent with the above theoretical inference. The findings from RN utility satisfy the feature of CRRA category: the optimal risk fraction remains constant over investment horizons. Hence, RN utility can be viewed as a special case of CRRA when $\delta = 0$. Under this case, the similar RN utility, $U(W) = aW - b$, can be easily derived.

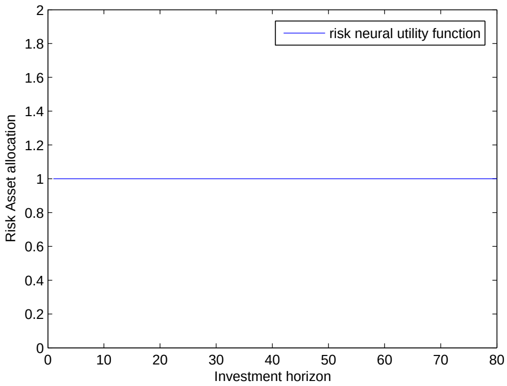

Fig. 2: Optimal asset allocations of RN utility over investment horizons. The parameters here are chosen the same as in Fig. 1.

It should be noted here that the previous utility categories have a common property with constant risk aversion, δ. This artificial setting, apparently, is inconsistent with investors risk preference in real investment activities. In contrast, the investors risk aversion relies closely on the influence of time and wealth. By building a time-varying function, we characterize the mixed impacts on investors risk aversion.

3 Construction of time-varying risk aversion function

Risk aversion is a dependent variable related to time and wealth. However, how to select a proper function to express the relationship becomes a challenging part of this study. To capture the mixed impacts of time and wealth on investors risk aversion, two typical functions, exponential function and power function, are introduced. By identifying quantitatively whether the impacts from time and wealth are consistent with the practice, we have to choose the better function as a proxy of risk aversion function.

3.1 Identification of proper risk-averse function form

In most time-involved decisions, the delay in receiving a return indicates a source of uncertainty. The longer delay implies larger uncertainty. An investor would perceive more risky if the delay of receiving his gains takes longer. Correspondingly, the degree of risk averse of an investor becomes higher. Which function can delineate such a nature? Due to the fact that risk aversion increases with time lengthens, therefore, searching for a proper form from both typical increasing functions becomes more appropriate: (1) the exponential function, $\delta(t) = \exp(\alpha t)$, and (2) the power function, $\delta(t) = (1+\alpha t)^{\beta/\alpha}$, where α and β are both constant coefficients, and t is the length of investment horizon.

As shown in Fig. 3, the major difference between the exponential and power functions is their own concavity. The graph of the power function is concave down on t interval $(0,+\infty)$ where an investor possesses a decreasing marginal risk aversion. This nature implies that given the same investment interval, the investor would manifest less impatient (more risk-tolerant) for the remote payoff than for the equally imminent payoff. Contrarily, the graph of the exponential function is concave up on t interval $(0,+\infty)$ where an investor presents an increasing marginal risk aversion. In the latter case, the investor becomes less patient (more risk-averse) for the distant gain than for the equally immediate gain, conditional on the same investment interval. In summary, the power function represents that the remote outcome over the same investment horizon is preferable while the exponential function shows that the immediate outcome has the priority. Specifically, the slope of the power function at $t = 50$ equals 0.0353, which is less than the slope of 0.0645 at $t = 20$, as shown in Fig. 3, consistent with the above theoretical analysis.

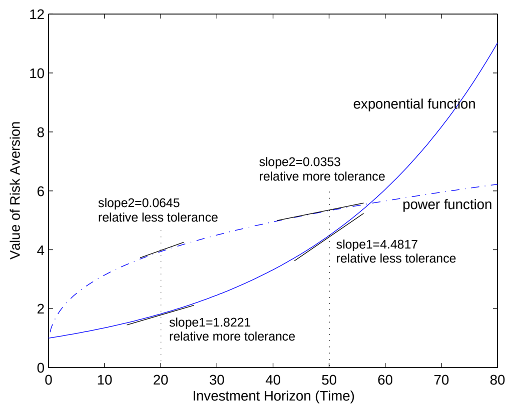

Fig. 3: Risk aversion comparison of exponential function with power function. (1) the exponential functions $\delta(t) = \exp(\alpha t)$ with $\alpha = 0.03$; and (2) the power function $\delta(t) = (1+\alpha t)^{\beta/\alpha}$ with $\alpha = 3$ and $\beta = 1$.

To identify which function is more reasonable, we borrow the survey results in Table 1 of [7] to make a careful comparison.

As shown in the first two columns of Table 1, a large majority of subjects in the imminent future prefer Prospect A over B. However, when all settings remain the same, only the time is delayed by 26 weeks. A large majority of subjects change their preferences from Prospect C to D, i.e., preference reverse. Such evidences from the experiment present a diminishing impatience. If

Table 1: Observations for three different levels of uncertainty

	Probability of monetary reward		
	certainty $p = 1.0$	high certainty $p = 0.9$	middle certainty $p = 0.5$
Imminent future			
A.100 now	82%	54%	39%
B.110 in 4 weeks	18%	46%	61%
Remote future			
C.100 in 26 weeks	37%	25%	33%
D.110 in 30 weeks	63%	75%	67%

Source: Keren and Roelofsma (1995)

the outcome is fixed, subjects are more impatient (less tolerant) over an interval closer to the present, and less impatient (more tolerant) over an equal interval relatively more distant. Comparing to the two functions described above, the power function turns out to be a perfect choice to interpret investors real risk preference.

The third column of Table 1 implies that the preference remains the same regardless of the length delayed. This could be caused by the higher uncertainty. When uncertainty is added up to a certain level, e.g. 50%, it would distort subjects rational decision. At this status, most of subjects could consider the consequences of potential loss more, and incline to choose the larger outcome regardless of the difference of time. This also indicates that using the power function as a proper expression of investors risk aversion is under the condition of high certainty.

3.2 Effects of time and wealth on risk aversion

As the time and wealth, two factors determining the risk aversion, are varied over the whole investment horizon when investors make optimal decision, it is crucial to include them in the select power function. Before designing the detailed function, we first need three reasonable assumptions to clarify the time effect, wealth effect and tradeoff effect of time and wealth, respectively.

Assumption 1 (time delay effect): given a relatively high certainty, p, receiving future outcome and a constant level of original wealth, longer time delay leads to stronger risk aversion, but marginal risk aversion decreases. This feature, called time delay effect, can be explicitly exhibited through a set of inequalities

$$\delta(W,t,p) < \delta(W,t+\Delta t,p), and \frac{\partial^2\delta}{\partial t^2} < 0 \quad (7)$$

where Δt equals the increment of time, and p equals the level of certainty.

Assumption 2 (wealth increment effect): given a fixed investment horizon, say t, and a relatively high certainty, p, investors risk aversion declines as wealth goes up, but increases as wealth decreases. The wealth reduction would cause investors to be more prudent to future risk investment. The marginal value of risk aversion drops. The effect induced by wealth variation, called wealth increment effect, can be expressed as

$$\begin{aligned} \delta(W+|\Delta W|,t,p) &< \delta(W,t,p), \\ \delta(W-|\Delta W|,t,p) &> \delta(W,t,p), \\ and\, \delta(W-|\Delta W|,t,p) &= \theta\delta(W+|\Delta W|,t,p), \\ and \frac{\partial^2\delta}{\partial|\Delta W|^2} &< 0 \end{aligned} \quad (8)$$

where ΔW is the wealth increment, and θ is a constant greater than 1, reflecting the more risk aversion from wealth reduction.

Assumption 3 (tradeoff effect): there exists a tradeoff ratio, k, between time delay and wealth change. The degree of risk aversion from time delay can be offset (amplify) linearly by wealth addition (reduction).

Fig. 4 plots the impact of time and wealth on investors risk aversion. The wealth loss and the time delay commonly cause a higher degree of risk aversion, forming an overlap impact on risk aversion. In contrast, the wealth growth reduces the degree of risk aversion, partially offsetting the risk aversion due to time delay. The change of risk aversion, however, is always above a certain lower level, δ_1, as shown in Fig. 4. This is since investors, even if possessing sufficient wealth, still need to reserve some capital for necessary consumption, such as retirement, education and etc.

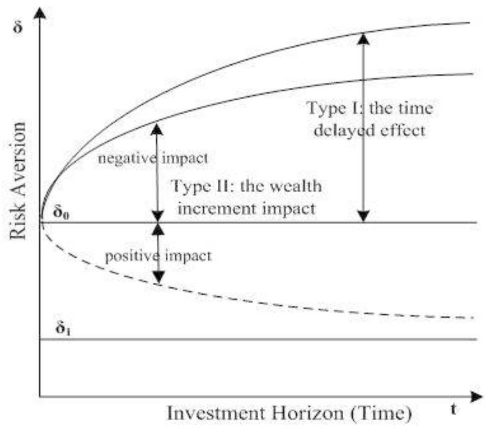

Fig. 4: Impacts of time and wealth on investors risk aversion

In line with the qualitative description above, a time-varying risk aversion model might be more appropriate to capture the dynamic effects of time and wealth on risk aversion than do the constant risk aversion in traditional utility categories.

3.3 Constructing time-varying risk aversion model

The dynamic property of risk aversion is addressed by two types of opposite common impacts: one is the overlap impact of time delay and wealth reduction, and the other is the offsetting impact of time delay and wealth addition. Based on the selected power function, we propose a time-varying risk aversion function, which includes the common effects of both wealth change and time delay.

$$\delta(\Delta W,t)=\begin{cases}\delta_0[1+\alpha(t-\frac{\Delta W}{kW_0})]^{\beta/\alpha}+\delta_1,\Delta W\geq 0\\ \delta_0[1+\alpha(t-\theta\frac{\Delta W}{kW_0})]^{\beta/\alpha}+\delta_1,\Delta W<0\end{cases} \quad (9)$$

where δ_0 is the initial risk aversion at horizon $t=0$, δ_1 is the critical degree of risk aversion, k is the tradeoff ratio between wealth change and time delay, and θ is the asymmetric coefficient. θ takes in general value greater than one to reflect the stronger influence from wealth reduction on risk aversion. Other notations in Eq (9) have the same denotation as described earlier in the paper. Considering the continuous growth of wealth, Eq (9) can be rewritten

$$\delta(\Delta W,t)=\begin{cases}\delta_0[1+\alpha(t-\frac{e^{gt}-1}{k})]^{\beta/\alpha}+\delta_1,\Delta W\geq 0\\ \delta_0[1+\alpha(t-\theta\frac{e^{gt}-1}{k})]^{\beta/\alpha}+\delta_1,\Delta W<0\end{cases} \quad (10)$$

where g denotes investors growth rate of wealth. Here we take the parameter α as an odd number to ensure the existence of Eq (10). We notice that the risk aversion given by Eq (10) is independent of investors initial wealth, W_0, satisfying the hypothesis of traditional utility function.

For a given length of time delay, the potential influences of wealth change can be clarified through discussing the following three extreme cases: $\Delta W\to+\infty$, $\Delta W=0$ and $\Delta W\to-\infty$. When $\Delta W\to+\infty$, it means that the investor has enough capital to cover the loss, and then decreases gradually his risk aversion. In contrast, the investor with decreased wealth, i.e.: $\Delta W\to-\infty$, would have less likelihood to recover loss in an investment, and be inclined to being more cautious with future investment decisions. Hence, the degree of risk aversion would be significantly enlarged. Last, for the investors with constant wealth, i.e.: $\Delta W=0$, the time delay becomes the only one factor to affect risk aversion. The value of risk aversion increases as the length of time delay. Fig. 5 illustrates the mixed impacts of time and these three cases of wealth.

As shown in Fig. 5, the evolution of risk aversion exhibits two types of distinct tendencies. The two upward curves represent the increasing risk aversion. The one is caused by the increased time delay alone (solid line), and the other by the combined effects of the time delay and wealth reduction (dashed line). All the values on the curve of the combined effects (dashed line) are significantly greater than those on the curve caused by the pure time effect. This indicates that the decreasing wealth intensifies the risk aversion. In contrast, the reversal curve (dashed-point line) supports the increasing wealth offsets partial uncertainty resulting from time delay. When the marginal increase of risk aversion caused by time delay is greater than the marginal decrease brought by increased wealth, the risk aversion would grow until it is peaked at a certain time level. The simulated peak value is at about $t=60$. And then the risk aversion would drop as the wealth further increases. The numerical observations confirm the previous discussion that increasing wealth partially offsets the uncertainty of future return.

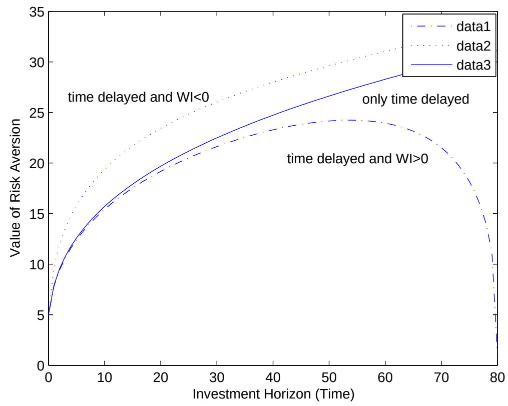

Fig. 5: Numerical illustrations of the time-varying risk aversion. Using the time-varying risk aversion model based on the selected power function characterizes the properties of investors dynamic risk aversion. The parameters are set as follow: (1) the time delay in year $t\in[0,80]$, representing an average investors lifecycle (about 80 years); (2) the initial wealth $W_0=1$; (3) the initial and critical risk aversion $\delta_0=5$ and $\delta_1=0.001$, respectively; (4) the constant coefficients $\alpha=3$ and $\beta=1$; (5) the tradeoff ratio between wealth change and time delay $k=1.5$; (6) the asymmetric coefficient $\theta=30$, depicting the larger impact from wealth reduction; (7) the yearly growth rate of wealth $g=0.06,0,or=-0.06$, corresponding to positive, invariable or negative increment of wealth, respectively.

4 Optimal asset allocation of the generalized utility model

4.1 The generalized utility model with time-varying property

By substituting time-varying risk aversion in Eq (10) into the uniform utility framework in Eq (1), we construct the generalized utility model (GUM) with time-varying risk aversion as follows

$$U_G=\frac{(W-c)^{1-\delta(\Delta W,t)}-1}{1-\delta(\Delta W,t)} \quad (11)$$

The generalized utility, U_G, is related to the time delay and wealth change. This is more consistent with investors real risk preference than traditional utility of constant risk aversion. The inter-temporal optimal decision would be impacted by the interaction between these two factors. The simulation illustrations in the next section discover the joint influence on optimal decisions resulting from time and wealth.

4.2 Optimal asset allocation based on GUM

The optimal proportion allocated to risky assets can be obtained by maximizing expected utility. There are three different scenarios of wealth under GUM: constant wealth ($WI = 0$), wealth growth ($WI > 0$) and wealth decrease ($WI < 0$). The numerical simulation still is a feasible alternative to explore the optimal decision as the analytic solution is unavailable.

Fig. 6 shows simulation results when $WI = 0$. In this scenario, only the impact of the time delay on the optimal allocation needs to be accounted for. As mentioned above, when δ is a constant, the optimal risk proportions of CRRA, DRRA and IRRA remains constant, decreasing or increasing over horizon, respectively. However, when considering the time-varying property of risk aversion, the optimal risk proportions of three utility categories share the uniformly decreasing properties. This tendency is due to the increased risk aversion caused by the extension of time delay only, as described in Fig. 5. The greater risk aversion leads to the smaller risk proportion of an optimal portfolio. When the time delay exceeds a certain level, i.e. at about $t = 60$, as shown in Fig. 6, an investor holds nearly zero risky assets in portfolio.

Fig. 6 provides the explicit evidence inconsistent with the concept of time diversification, even including the CRRA category. The conflictive result is simply because time delay affects investors risk attitude. More specifically, time delay intensifies the degree of the investors risk aversion. The investor who is more risk averse would choose less risky asset so that he could achieve the higher expected utility.

Next, we expand the wealth status from $WI = 0$ to $WI > 0$ to explore a more practical situation in which an investor holds increased, instead of constant, wealth. As discussed above, increased wealth enables an investor to better compensate possible loss and thus make them more risk tolerant. When the positive marginal effect of wealth exceeds negative marginal effect from time delay, the curved shape of investors risk aversion with $WI > 0$ reverses downward as demonstrated by an inverted U-Shape in Fig. 5. However, the curves of optimal risk allocation in Fig. 7 show the U-shape, which is caused by the inverted U-shape of risk aversion. This is because when an investor becomes more risk averse, then optimal proportion allocated to risky asset decreases, and vice versa. When investment horizon stands at the level of 60, risky asset allocation reaches the bottom as shown in Fig. 7. The completely opposite shape of curves in Fig. 5 (risk aversion) and in Fig. 7 (optimal risk allocation) illustrates explicitly that investors risk aversion indeed has a significant effect on optimal investment decision.

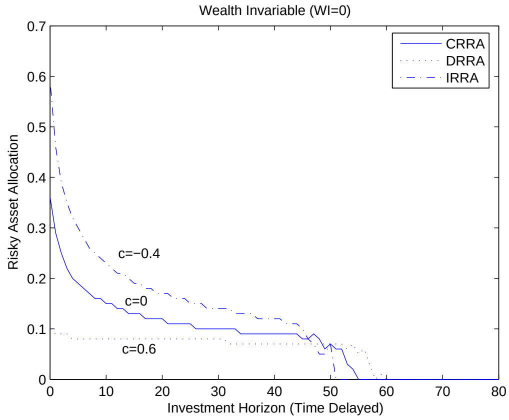

Fig. 6: Optimal asset allocations of GUM with $WI = 0$ over investment horizons. Investors wealth remains constant over the entire investment horizon. Other parameters keep consistent with the settings in Fig. 1 and Fig. 5.

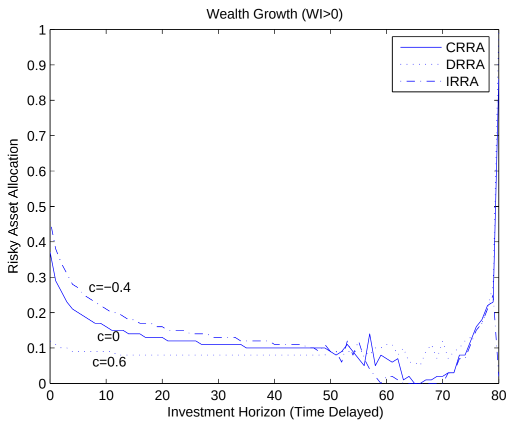

Fig. 7: Optimal asset allocations of GUM with $WI > 0$ over investment horizons. Investors wealth continues to increase over the whole horizon. The growth rate of wealth is set as $g = 0.06$, slightly over the actual CPI of 0.051 in April, 2011 in China. Other parameters remain the same as the settings in Fig. 1 and Fig. 5.

Last, when $WI < 0$, the investor suffers from double pressures from time delay and wealth reduction. The possibility to compensate the potential loss falls dramatically as the investment horizon lengthens. Hence, the degree of risk aversion would obviously be magnified.

The overlap effect leads the investor to hold smaller risk proportion over an equal horizon. The critical horizon that the investor exchanges all risky asset for the other safe investment vehicles approximately stands at a horizon of $t = 55$.

There is a surprising similarity between Fig. 6 and Fig. 8, such as decreasing curves and critical horizon. The only distinction between $WI < 0$ and $WI = 0$ lies in the decreasing slope. Due to higher risk aversion when $WI < 0$, the optimal percentage of holding a risky asset, as shown in Fig. 8, declines faster than that of $WI = 0$, as shown in Fig. 6. The notion that younger investor holds a larger proportion of risky assets focuses only on the possible compensation of wealth growth, but neglects wealth reduction and time uncertainty. Therefore, the opinion of time diversification is flawed when we fully consider the combined effects of both time and wealth on risk aversion.

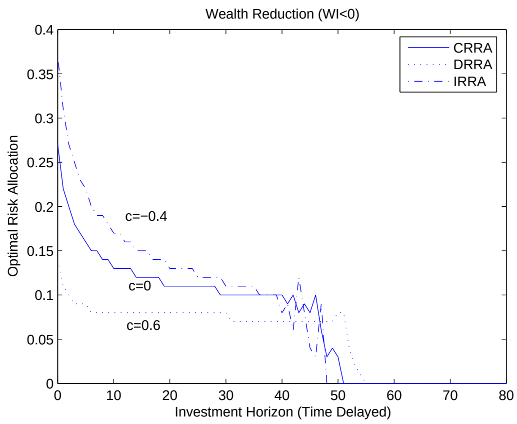

Fig. 8: Optimal risk allocations of GUM with $WI < 0$ over investment horizons. Investors wealth continues decreasing over the whole horizon. The decreasing rate of wealth is set as $g = -0.06$. Further parameters are chosen as in Fig. 1 and Fig. 5.

5 Conclusions

By introducing time delay and wealth change into the power function, we develop a model of the time-varying risk aversion to capture investors more practical risk preference. Based on the time-varying risk aversion, a newly generalized utility model (GUM) under the uniform utility framework has been built to further investigate the inter-temporal optimal decisions of heterogeneous investors.

The evidences from the numerical simulation support the arguments that the optimal proportion allocated to risky asset depends significantly on the investors risk aversion. And the concept of time diversification is proved to be only a coincidence with traditional DRRA category of constant risk aversion. Under GUM, the optimal risk proportion of three utility categories, such as DRRA, CRRA and IRRA, show obviously different tendencies from the concept of time diversification. Almost all short-term investors in three wealth statuses choose relatively riskier portfolios than long-term investors. The evolution of the optimal risk proportion is opposite to the change of the risk aversion curve. Our study shed light on further research that provides better approximation of the tradeoff ratio between wealth increment and time delay.

Acknowledgement

Aids from the National Natural Science Foundation of China under Grant No. 71073049, 71031004, 71221001, 71210107022, 71371008 and 71001093, the Program for New Century Excellent Talents in University under Grant No. NCET-13-0193, and the Research Fund of the Doctoral Program of High Education under Grant No. 20090161120034, are gratefully acknowledged.

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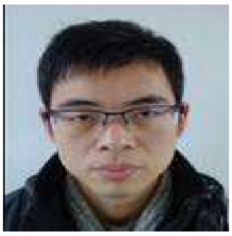

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