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Group Analysis of a Generalized KdV Equation

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Abstract: In this work the Korteweg-de Vries equation which contains an arbitrary function in the nonlinear term is considered and it is referred to as a generalized KdV. This equation has applications in nonlinear solitary wave phenomena in some areas of fluid mechanics, plasma physics and quantum mechanics. The Lie group analysis approach is employed to obtain the possible forms of the arbitrary parameter.

Keywords: Generalized KdV equation, Lie group analysis, Symmetry, Lie algebra

1 Introduction

Nonlinear evolution equations are widely used as mathematical models to describe nonlinear phenomena in various fields of science and engineering. It is desirable to determine the analytical solutions to these equations in order to understand better the complexity involved. The Korteweg-de Vries (KdV) equation [4] is one of these nonlinear evolution equations and it models the propagation of solitary waves on the shallow water surfaces. In recent studies [2,3,10,11], different types of the KdV equation have been investigated to model various situations and the analytical and numerical solution procedures have been employed to solve these types of equations.

In many real life applications the model equations contain arbitrary functions of the dependent variable or its derivatives and independent variables. In solving these equations some special forms of the model parameters (arbitrary functions) are assumed which may lead to approximation of solutions. However, the Lie group based approach known as the method of Lie group classification is a systematic procedure that enables the specification of the possible forms of the arbitrary functions which appear in the equation of interest. Depending upon the equation being considered either the approach based upon the equivalence transformations or the direct analysis approach of the group classification method can be used. The direct analysis is more preferable when the arbitrary functions depend upon only one variable, that is, either a dependent or an independent variable.

In this paper, we consider the generalized Korteweg-de Vries (gKdV) equation

$$u_t + F(u)u_x + u_{xxx} = 0, \quad (1)$$

where the first term denotes the evolution term, the second term is the nonlinear term and lastly, the third term represents dispersion. Some special cases for the function $F(u)$ have been studied in the literature (see for example [2,3,5,8,9,10]). These forms reduce Eq. (1) to the much studied KdV type equations such as the modified KdV (mKdV) and Gardner equations. The current work deals with a systematic way of specifying this arbitrary function, similar work can be found in [2,3,5].

The plan of this work is as follows. In Section 3 the determining equations for Lie point symmetries and the classifying relation are generated. Moreover, the direct analysis of the classifying relation is performed in order to obtain the possible forms of the arbitrary function. In Section 3 the symmetry reduction is performed and where possible exact solutions are obtained. Section 4 presents the summary of our investigations.

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2 Lie group classification

In Lie group analysis (see [1,6,7] for more details), a vector field

$$X = \xi^1(t,x,u)\partial_t + \xi^2(t,x,u)\partial_x + \eta(t,x,u)\partial_u \tag{2}$$

is a generator of Lie point symmetries of Eq. (1) if and only if

$$X^{[3]}\left(u_t + F(u)u_x + u_{xxx}\right)|_{(1)} = 0, \tag{3}$$

where

$$X^{[3]} = X + \zeta_t \partial_{u_t} + \zeta_x \partial_{u_x} + \zeta_{xxx}\partial_{u_{xxx}} \tag{4}$$

is the third-prolongation of the vector field X. The variables ζ's are given by the formulae:

$$\begin{aligned} \zeta_t &= D_t(\eta) - u_t D_t(\xi^1) - u_x D_t(\xi^2), \\ \zeta_x &= D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2), \\ \zeta_{xxx} &= D_x(\zeta_{xx}) - u_{xxt} D_x(\xi^1) - u_{xxx} D_x(\xi^2), \end{aligned} \tag{5}$$

where

$$D_t = \partial_t + u_t\partial_u + \cdots \quad \text{and} \quad D_x = \partial_x + u_x\partial_u + \cdots$$

are the total derivative operators [6].

The splitting of (3) with respect to the powers of the derivatives of u yields the determining equations, which is a system of linear partial differential equations of homogeneous type in ξ^1, ξ^2 and η. To generate the determining equations manually is easy but it is a lengthy and tiring task. As a result, in recent years many computer software packages have been developed to perform this task. The next step is to solve the resulting system for ξ^1, ξ^2 and η. Some of these software packages have functionalities to determine symmetries automatically, that is, without solving the system interactively. However, in some cases the erroneous results may be obtained.

The coefficient functions of the generator of Lie point symmetries (2), namely, ξ^1, ξ^2 and η satisfy the determining equations

$$\xi^1_u = 0, \quad \xi^2_u = 0, \quad \eta_{uu} = 0, \quad \xi^1_x = 0, \tag{6}$$

$$\eta_{xu} - \xi^2_{xx} = 0, \quad 3\xi^2_x - \xi^1_t = 0, \tag{7}$$

$$\eta_x F(u) + \eta_{xxx} + \eta_t = 0, \tag{8}$$

$$\eta F'(u) + \left(\xi^1_t - \xi^2_x\right)F(u) + 3\eta_{xxu} - \xi^2_{xxx} - \xi^2_t = 0, \tag{9}$$

where the subscripts represent partial differentiation with respect to the indicated variables and a 'prime' denotes total derivative with respect to u.

The manipulation of Eqs. (6)–(9) yields the coefficient functions of Lie point symmetry generator of the form

$$\xi^1 = c_1 t + c_2, \quad \xi^2 = \frac{1}{3}c_1 x + c_5 t + c_6, \quad \eta = c_3 u + c_4, \tag{10}$$

where $c_1,\ldots,c_6$ are arbitrary constants. Eventually we obtain the classifying relation

$$3(c_3 u + c_4)F'(u) + 2c_1 F(u) - 3c_5 = 0. \tag{11}$$

Suppose that $F(u)$ is an arbitrary smooth function of its argument. Then according to the classifying relation (11) the coefficient functions (10) become

$$\xi^1 = c_2, \quad \xi^2 = c_6, \quad \eta = 0. \tag{12}$$

Therefore, we obtain a two-dimensional principal Lie algebra which is spanned by the symmetry generators

$$X_1 = \partial_t, \quad X_2 = \partial_x. \tag{13}$$

Our aim is to obtain the possible forms of the arbitrary parameter, $F(u)$, such that the principal Lie algebra is extended. Ultimately, the direct analysis of the classifying equation (11) leads to the possible forms of $F(u)$ and their corresponding operators which extend the principal Lie algebra (see Table 1).

Table 1: Classification results

No.	F	Condition on consts.	Extra operator(s)
1.	$F_0(\beta+\alpha u)^{-\frac{1}{3\alpha}} + F_1$	$\alpha, F_0 \neq 0$	$X_3 = 3t\partial_t + (2F_1 t + x)\partial_x + 6(\beta+\alpha u)\partial_u$
2.	$F_0 e^{-\frac{u}{3\beta}} + F_1$	$\beta, F_0 \neq 0$	$X_3 = 3t\partial_t + (2F_1 t + x)\partial_x + 6\beta\partial_u$
3.	$F_0 \ln(\gamma+u)$	$F_0 \neq 0$	$X_3 = F_0 t\partial_x + (\gamma+u)\partial_u$
4.	$F_0 u^2 + \widetilde{F}_0 u + F_1$	$F_0, \widetilde{F}_0 \neq 0$	$X_3 = 6F_0 t\partial_t + \left[2F_0(2F_1 t + x) - \widetilde{F}_0^2 t\right]\partial_x - (\widetilde{F}_0 + 2F_0 u)\partial_u$
5.	$\widetilde{F}_0 u + F_1$	$\widetilde{F}_0 \neq 0$	$X_3 = \widetilde{F}_0 t\partial_x + u\partial_u$, $X_4 = 3\widetilde{F}_0 t\partial_t + \widetilde{F}_0 x\partial_x - 2(F_1 + \widetilde{F}_0 u)\partial_u$
6.	F_1	$F_1 \neq 0$	$X_3 = 3t\partial_t + (2F_1 t + x)\partial_x + u\partial_u$, $X_4 = u\partial_u$, $X_\phi = \phi(t,x)\partial_u$

Note: In Table I: F_0, $\widetilde{F}_0$, F_1, α, β and γ are arbitrary constants. The function $\phi(t,x)$ is a solution of the gKdV equation corresponding to $F = F_1$.

3 Exact solutions and symmetry reductions

In order to illustrate the procedure for determining exact solutions and performing similarity reductions, we consider case 5 in Table 1. We seek solutions of the invariant equation

$$u_t + F_1 u_x + \widetilde{F}_0 u u_x + u_{xxx} = 0, \tag{14}$$

the symmetries of which are given by

$$\begin{aligned} X_1 &= \partial_t, \\ X_2 &= \partial_x, \\ X_3 &= \widetilde{F}_0 t\partial_x + \partial_u, \\ X_4 &= 3t\partial_t + (2F_1 t + x)\partial_x - 2u\partial_u. \end{aligned} \tag{15}$$

The optimal system of one-dimensional subalgebras [6,7] is a systematic procedure from which all the possible invariant solutions are obtained. We follow the approach given in [6] to construct the optimal system of

Table 2: Table of commutators

$[X_i,X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	$\widetilde{F}_0 X_2$	$3\widetilde{F}_0 X_1$
X_2	0	0	0	$\widetilde{F}_0 X_2$
X_3	$-\widetilde{F}_0 X_2$	0	0	$-2\widetilde{F}_0 X_3$
X_4	$-3\widetilde{F}_0 X_1$	$-\widetilde{F}_0 X_2$	$2\widetilde{F}_0 X_3$	0

Note: $[X_i,X_j]=X_i(X_j)-X_j(X_i);\ i,j=1,2,3,4$ is the commutator operation.

Table 3: Table of adjoint representations

$\mathrm{Ad}(e^{\varepsilon X_i})\,X_j$	X_1	X_2	X_3	X_4
X_1	X_1	X_2	$X_3-\widetilde{F}_0\varepsilon X_2$	$X_4-3\widetilde{F}_0\varepsilon X_1$
X_2	X_1	X_2	X_3	$X_4-\widetilde{F}_0\varepsilon X_2$
X_3	$X_1+\widetilde{F}_0\varepsilon X_2$	X_2	X_3	$X_4+2\widetilde{F}_0\varepsilon X_3$
X_4	$e^{3\widetilde{F}_0\varepsilon}X_1$	$e^{\widetilde{F}_0\varepsilon}X_2$	$e^{-2\widetilde{F}_0\varepsilon}X_3$	X_4

Note: $\mathrm{Ad}(e^{\varepsilon X_i})\,X_j = X_j-\varepsilon[X_i,X_j]+\frac{1}{2!}\varepsilon^2[X_i,[X_i,X_j]]-\cdots;\ \varepsilon\in \mathbb{R}$ is the adjoint representation.

Eq. (14). Firstly, we construct the table of commutators followed by the table of adjoint representations for the symmetry Lie algebra (15). These are given by Tables 2 and 3 respectively.

Since the symmetry Lie algebra is four-dimensional, we use Table 3 to simplify the operator

$$\Gamma = a_1X_1+a_2X_2+a_3X_3+a_4X_4, \tag{16}$$

where $X_1,\ldots,X_4$ are given by (15) for arbitrary constants a_1, a_2, a_3 and a_4. In simplifying (16) the adjoint operator $\mathrm{Ad}(e^{\varepsilon X_i})$ acts on Γ while different cases of the constants a_1, a_2, a_3 and a_4 are considered. After considering all the cases we obtain the optimal system of one-dimensional subalgebras, which is given by $\{X_1,X_2,\delta X_1+X_3,X_3,X_4\}$ where $\delta=\pm 1$.

Now we construct the invariant solutions with the use of the optimal system. However, the invariance under time and space translations leads to trivial solutions and they are not considered here. The invariant solutions are a foundation on which exact solutions are derived or symmetry reductions can be performed. The remaining cases are presented below.

Case I. The characteristic system for the linear combination δX_1+X_3 is given by

$$\frac{dt}{\delta}=\frac{dx}{\widetilde{F}_0 t}=\frac{du}{1}. \tag{17}$$

Solving the above system yields the invariant solution of the form

$$u(t,x)=\frac{t}{\delta}+f(z), \tag{18}$$

where z is the similarity variable, namely,

$$z=x-\frac{t^2\widetilde{F}_0}{2\delta} \tag{19}$$

and the function $f(z)$ satisfies the first-order equation

$$\left(\frac{df}{dz}\right)^2=C_2-\frac{1}{3\delta}\left[6(z-C_1)+3\delta F_1 f+\delta\widetilde{F}_0 f^2\right]f, \tag{20}$$

where C_1 and C_2 are arbitrary constants.

Case II. The characteristic system for the invariants of X_3 leads to the invariant solution

$$u(t,x)=\frac{x}{t\widetilde{F}_0}+f(t), \tag{21}$$

where $f(t)$ is an arbitrary function of t. Substitution of (21) into Eq. (14) yields a first-order ordinary differential equation (ODE) the solution of which is given by

$$f(t)=\frac{f_0}{t}-\frac{F_1}{\widetilde{F}_0}, \tag{22}$$

for some arbitrary f_0. Thus, we have the exact solution

$$u(t,x)=\frac{1}{t}\left(\frac{x-F_1}{\widetilde{F}_0}+f_0\right). \tag{23}$$

Case III. The solution invariant under X_4 is of the form

$$u(t,x)=\frac{f(z)}{t^{2/3}}-\frac{F_1}{F_0}, \tag{24}$$

where $z=xt^{-1/3}$ is an invariant of the characteristic system for X_4. The function $f(z)$ is an arbitrary smooth function which satisfies the third-order nonlinear ODE

$$3\frac{d^3f}{dz^3}+\left(3\widetilde{F}_0\frac{df}{dz}-2\right)f-z\frac{df}{dz}=0. \tag{25}$$

The last equation (25) may be solved by means of the analytical techniques for solving nonlinear ODEs.

4 Conclusion

This work dealt with the Lie group analysis of a generalized KdV equation. The classifying equation was analyzed using the direct method of group classification for possible forms of the arbitrary function, which include power law, exponential, quadratic, linear and constant forms. The three-, four-and infinite-dimensional symmetry Lie algebra was obtained respectively for these forms of the arbitrary parameter. The optimal system of one-dimensional subalgebras of the invariant equation with the four-dimensional Lie algebra was obtained. As a result, the exact solutions were constructed, or otherwise the symmetry reduction was performed, that is, in the case where exact solutions could not be obtained.

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