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Fractional Hermite-Hadamard Inequalities for some New Classes of Godunova-Levin Functions

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Abstract: In this paper, some new classes of Godunova-Levin functions are introduced and investigated. Several new fractional Hermite-Hadamard inequalities are derived for these new classes of Godunova-Levin functions. The ideas and techniques of this paper may stimulate and inspire further research.

Keywords: convex, s-Godunova-Levin functions, Hermite-Hadamard inequality, Riemann-Liouville fractional integrals.
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1 Introduction

Theory of convex sets and convex functions play an important role in different fields of pure and applied sciences. In recent years, the concept of convexity has been extended and generalized in various directions using novel and innovative techniques, see [1,2,3,4,5,6,9,12, 13,14,16,19]. It is known fact that many important inequalities in the literature are direct consequences of the applications of convex functions, for details see [2,3,4,5, 8,10,11,12,13,14,15,17,18,19]. Dragomir [2,3] has introduced a new class of convex functions, which is called s-Godunova-Levin functions of second kind. Dragomir also showed that for suitable choices of s the class of s-Godunova-Levin functions of second kind reduces to the known classes of convex functions.
Inspired and motivated by the research going on in this field, we also introduce and investigate a new class of convex functions, which is called the s-Godunova-Levin functions of first kind. Some new fractional Hermite-Hadamard inequalities are obtained for these two new extensions of Godunova-Levin functions. The interested readers may explore new and interesting applications of these classes of convex functions in various branches of pure and applied sciences using novel and innovative techniques. This is the main motivation of this paper. Results obtained in this paper may be extended for other classes of convex functions including preinvex functions and coordinated convex functions.

2 Preliminaries

In this section we recall some previously known concepts. First of all let $I=[a,b]\subseteq\mathbb{R}$ be the interval and $\mathbb{R}$ be the set of real numbers.

Definition 1([4]). *A nonnegative function $f: I\to\mathbb{R}$ is said to be P-function, if*

$$f(tx+(1-t)y)\leq f(x)+f(y),\ \forall x,y\in I, t\in[0,1]. \quad (1)$$

Definition 2([6]). *A function $f: I\to\mathbb{R}$ is said to be Godunova-Levin function, if*

$$f(tx+(1-t)y)\ \leq \frac{f(x)}{t}+\frac{f(y)}{1-t},\forall x,y\in I, t\in(0,1). \quad (2)$$

We define a new generalization of Godunova-Levin type of functions, which is called the s-Godunova-Levin functions of first kind.

Definition 3. *A function $f: I\to\mathbb{R}$ is said to be s-Godunova-Levin functions of first kind, if $s\in(0,1]$, we have*

$$f(tx+(1-t)y)\leq\frac{f(x)}{t^s}+\frac{f(y)}{1-t^s},\forall x,y\in I, t\in(0,1). \quad (3)$$

It is obvious that for $s=1$ the definition of s-Godunova-Levin functions of first kind collapses to the definition of Godunova-Levin functions.
Our next definition is due to Dragomir [2,3].

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Definition 4([2,3]). *A function $f : I \to \mathbb{R}$ is said to be s-Godunova-Levin functions of second kind, for $s \in [0,1]$, we have*

$$f(tx+(1-t)y) \le \frac{f(x)}{t^s} + \frac{f(y)}{(1-t)^s}, \forall x,y \in I, t \in (0,1). \quad (4)$$

It is obvious that for $s = 0$, s-Godunova-Levin functions of second kind reduces to the definition of P-functions. If $s = 1$, it then reduces to Godunova-Levin functions.
We now introduce the classes of logarithmic s-Godunova-Levin functions of first and second kinds respectively.

Definition 5. *A function $f : I \to (0,\infty)$ is said to be logarithmic s-Godunova-Levin function of first kind, if $s \in (0,1]$, we have*

$$f(tx+(1-t)y) \le f(x)^{\frac{1}{t^s}} f(y)^{\frac{1}{1-t^s}}, \forall x,y \in I, t \in (0,1).$$

From above inequality, it follows that

$$\ln f(tx+(1-t)y) \le \frac{1}{t^s} \ln f(x) + \frac{1}{1-t^s} \ln f(y).$$

Remark. Note that for $s = 1$ above definition reduces to the definition for logarithmic Godunova-Levin functions, see [14].

Definition 6. *A function $f : I \to (0,\infty)$ is said to be logarithmic s-Godunova-Levin function of second kind, if $s \in [0,1]$, we have*

$$f(tx+(1-t)y) \le f(x)^{\frac{1}{t^s}} f(y)^{\frac{1}{(1-t)^s}}, \forall x,y \in I, t \in (0,1).$$

Thus, it follows that

$$\ln f(tx+(1-t)y) \le \frac{1}{t^s} \ln f(x) + \frac{1}{(1-t)^s} \ln f(y).$$

Remark. Note that for $s = 1$ above definition reduces to the definition for logarithmic Godunova-Levin functions, see [14]. Also for $s = 0$ it reduces to logarithmic P-functions, see [14].

Definition 7([7]). *Let $f \in L_1[a,b]$. Then the Riemann-Liouville integrals $J^{\alpha}_{a^+} f(x)$ and $J^{\alpha}_{b^-} f(x)$ of order $\alpha > 0$ with $a \ge 0$ are defined by*

$$J^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a, \quad (5)$$

and

$$J^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b, \quad (6)$$

where

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} x^{\alpha-1} dx,$$

is the well known Gamma function.

Lemma 1([18]). *Let $f : I \to \mathbb{R}$ be twice differentiable function. If $f'' \in L[a,b]$, then*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J^{\alpha}_{a^+} f(b) + J^{\alpha}_{b^-} f(a)\right]$$
$$= \frac{(b-a)^2}{2} \int_0^1 \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} f''(ta+(1-t)b)dt.$$

Lemma 2([10]). *Let $f : [a,b] \to \mathbb{R}$ be twice differentiable function on (a,b) with $a < b$. If $f'' \in L[a,b]$, then, we have*

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J^{\alpha}_{(\frac{a+b}{2})^-} f(a) + J^{\alpha}_{(\frac{a+b}{2})^+} f(b)\right] - f\left(\frac{a+b}{2}\right)$$
$$= \frac{(b-a)^2}{8(\alpha+1)} \int_0^1 (1-t)^{\alpha+1} \left[f''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) + f''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\right] dt.$$

The above two result play key role in proving our main results.

3 Inequalities for s-Godunova-Levin functions of first kind

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be twice differentiable on (a,b) with $a < b$ and $f'' \in L_1[a,b]$. If $|f''|$ is s-Godunova-Levin function of first kind, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J^{\alpha}_{a^+} f(b) + J^{\alpha}_{b^-} f(a)\right]\right|$$
$$\le \frac{(b-a)^2}{2(\alpha+1)} \left[\mathscr{H}_1(\alpha,s)|f''(a)| + \mathscr{H}_2(\alpha,s)|f''(b)|\right],$$

where

$$\mathscr{H}_1(\alpha,s) = \tfrac{1}{1-s} - \tfrac{1}{\alpha-s+2} - \tfrac{\Gamma(\alpha+2)\Gamma(1-s)}{\Gamma(\alpha-s+3)} \quad (7)$$

$$\mathscr{H}_2(\alpha,s) = \tfrac{s+2}{s+1} - \tfrac{1}{\alpha+2} - \tfrac{s\Gamma(s)\Gamma(\alpha+2)}{\Gamma(\alpha+s+3)} - \tfrac{2(\alpha+2)+s}{(\alpha+2)(\alpha+s+2)}. \quad (8)$$

Proof. Using Lemma 1, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J^{\alpha}_{a^+} f(b) + J^{\alpha}_{b^-} f(a)\right]\right|$$
$$\le \frac{(b-a)^2}{2} \int_0^1 \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} |f''(ta+(1-t)b)|dt$$
$$\le \frac{(b-a)^2}{2} \int_0^1 \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} \times \left[\frac{1}{t^s}|f''(a)| + \frac{1}{1-t^s}|f''(b)|\right] dt$$
$$= \frac{(b-a)^2}{2(\alpha+1)} \left\{ \left[\frac{1}{1-s} - \frac{1}{\alpha-s+2} - \frac{\Gamma(\alpha+2)\Gamma(1-s)}{\Gamma(\alpha-s+3)}\right] |f''(a)| \right.$$

$$+\left[\frac{s+2}{s+1}-\frac{1}{\alpha+2}-\frac{s\Gamma(s)\Gamma(\alpha+2)}{\Gamma(\alpha+s+3)}-\frac{2(\alpha+2)+s}{(\alpha+2)(\alpha+s+2)}\right]|f''(b)|\Bigg\}.$$

This completes the proof. □

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function on (a,b) and $f''\in L[a,b]$. If $|f''|^q$ for $q>1$ is s-Godunova-Levin function of first kind, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b)+J^\alpha_{(b)^-}f(a)\right]\right|$$
$$\leq\frac{(b-a)^2}{2(\alpha+1)}\left(\frac{\alpha}{\alpha+2}\right)^{1-\frac{1}{q}}\times\left\{\mathscr{H}_1(\alpha,s)|f''(a)|^q+\mathscr{H}_2(\alpha,s)|f''(b)|^q\right\}^{\frac{1}{q}},$$

where $\mathscr{H}_1(\alpha,s)$ and $\mathscr{H}_2(\alpha,s)$ are given by (7) and (8) respectively.

Proof. Using Lemma 1, power mean inequality and the fact that $|f''|^q$ is s-Godunova-Levin function of first kind, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b)+J^\alpha_{(b)^-}f(a)\right]\right|$$
$$=\left|\frac{(b-a)^2}{2}\int_0^1\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}f''(ta+(1-t)b)dt\right|$$
$$\leq\frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1 1-(1-t)^{\alpha+1}-t^{\alpha+1}dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1(1-(1-t)^{\alpha+1}-t^{\alpha+1})\left|f''(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq\frac{(b-a)^2}{2(\alpha+1)}\left(\frac{\alpha}{\alpha+2}\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left[\frac{1}{1-s}-\frac{1}{\alpha-s+2}-\frac{\Gamma(\alpha+2)\Gamma(1-s)}{\Gamma(\alpha-s+3)}\right]|f''(a)|^q\right.$$
$$+\left.\left[\frac{s+2}{s+1}-\frac{1}{\alpha+2}-\frac{s\Gamma(s)\Gamma(\alpha+2)}{\Gamma(\alpha+s+3)}-\frac{2(\alpha+2)+s}{(\alpha+2)(\alpha+s+2)}\right]|f''(b)|^q\right\}^{\frac{1}{q}}.$$

This completes the proof. □

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function on (a,b) and $f''\in L[a,b]$. If $|f''|^q$ for $q\geq 1$ and $\frac{1}{p}+\frac{1}{q}=1$ is s-Godunova-Levin function of first kind, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b)+J^\alpha_{(b)^-}f(a)\right]\right|$$
$$\leq\frac{(b-a)^2}{2(\alpha+1)}\left(\frac{p(\alpha+1)-1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\times\left(\frac{1}{1-s}|f''(a)|^q+\frac{s+2}{s+1}|f''(b)|^q\right)^{\frac{1}{q}}.$$

Proof. Using Lemma 1, Holder's inequality and the fact that $|f''|^q$ is s-Godunova-Levin function of first kind, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b)+J^\alpha_{(b)^-}f(a)\right]\right|$$
$$=\left|\frac{(b-a)^2}{2}\int_0^1\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}f''(ta+(1-t)b)dt\right|$$
$$\leq\frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1(1-(1-t)^{\alpha+1}-t^{\alpha+1})^p dt\right)^{\frac{1}{p}}\times\left(\int_0^1\left|f''(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq\frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1(1-(1-t)^{p(\alpha+1)}-t^{p(\alpha+1)})dt\right)^{\frac{1}{p}}\times\left(\int_0^1\left[\frac{1}{t^s}|f''(a)|^q+\frac{1}{1-t^s}|f''(b)|^q\right]dt\right)^{\frac{1}{q}}$$
$$=\frac{(b-a)^2}{2(\alpha+1)}\left(\frac{p(\alpha+1)-1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\times\left(\frac{1}{1-s}|f''(a)|^q+\frac{s+2}{s+1}|f''(b)|^q\right)^{\frac{1}{q}}.$$

This completes the proof. □

4 Inequalities for s-Godunova-Levin functions of second kind

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be differentiable on (a,b) with $a<b$ and $f''\in L_1[a,b]$. If $|f''|$ is s-Godunova-Levin function of second kind, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)\right]\right|$$
$$\leq\frac{(b-a)^2}{2}\mathscr{H}(\alpha,s)[|f''(a)|+|f''(b)|],$$

where

$$\mathscr{H}(\alpha,s)=\frac{1}{1-s}-\frac{1}{\alpha-s+2}-\frac{\Gamma(\alpha+2)\Gamma(1-s)}{\Gamma(\alpha-s+3)}. \qquad (9)$$

Proof. Using Lemma 1, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)\right]\right|$$

$$\leq \frac{(b-a)^2}{2}\int_0^1 \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}|f''(ta+(1-t)b)|dt$$

$$\leq \frac{(b-a)^2}{2}\int_0^1 \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} \times\left[\frac{1}{t^s}|f''(a)|+\frac{1}{(1-t)^s}|f''(b)|\right]dt$$

$$=\frac{(b-a)^2}{2(\alpha+1)} \times\left[\frac{1}{1-s}-\frac{1}{\alpha-s+2}-\frac{\Gamma(\alpha+2)\Gamma(1-s)}{\Gamma(\alpha-s+3)}\right][|f''(a)|+|f''(b)|].$$

This completes the proof. □

Theorem 5. *Let $f: I\to\mathbb{R}$ be a twice differentiable function on I° and $f''\in L[a,b]$. If $|f''|^q$ for $q>1$ is s-Godunova-Levin function of second kind, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b)+J^\alpha_{(b)^-}f(a)\right]\right|$$

$$\leq \frac{(b-a)^2}{2(\alpha+1)}\left(\frac{\alpha}{\alpha+2}\right)^{1-\frac{1}{q}}\left(\mathscr{H}(\alpha,s)\right)^{\frac{1}{q}}\left(|f''(a)|^q+|f''(b)|^q\right)^{\frac{1}{q}},$$

where $\mathscr{H}(\alpha,s)$ is given by (9).

Proof. Using Lemma 1, power mean inequality and the fact that $|f''|^q$ is s-Godunova-Levin function of second kind, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b)+J^\alpha_{(b)^-}f(a)\right]\right|$$

$$=\left|\frac{(b-a)^2}{2}\int_0^1 \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}f''(ta+(1-t)b)dt\right|$$

$$\leq \frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1 1-(1-t)^{\alpha+1}-t^{\alpha+1}dt\right)^{1-\frac{1}{q}} \times\left(\int_0^1(1-(1-t)^{\alpha+1}-t^{\alpha+1})\left|f''(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{(b-a)^2}{2(\alpha+1)}\left(\frac{\alpha}{\alpha+2}\right)^{1-\frac{1}{q}} \times\left(\frac{1}{1-s}-\frac{1}{\alpha-s+2}-\frac{\Gamma(\alpha+2)\Gamma(1-s)}{\Gamma(\alpha-s+3)}\right)^{\frac{1}{q}} \times\left(|f''(a)|^q+|f''(b)|^q\right)^{\frac{1}{q}}$$

$$=\frac{(b-a)^2}{2(\alpha+1)}\left(\frac{\alpha}{\alpha+2}\right)^{1-\frac{1}{q}}\left(\mathscr{H}(\alpha,s)\right)^{\frac{1}{q}}\left(|f''(a)|^q+|f''(b)|^q\right)^{\frac{1}{q}}.$$

This completes the proof. □

Theorem 6. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function on (a,b) and $f''\in L[a,b]$. If $|f''|^q$ for $q\geq 1$ and $\frac{1}{p}+\frac{1}{q}=1$ is s-Godunova-Levin function of second kind, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b)+J^\alpha_{(b)^-}f(a)\right]\right|$$

$$\leq \frac{(b-a)^2}{2(\alpha+1)}\left(\frac{p(\alpha+1)-1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q+|f''(b)|^q}{1-s}\right)^{\frac{1}{q}}.$$

Proof. Using Lemma 1, Holder's inequality and the fact that $|f''|^q$ is s-Godunova-Levin function of second kind, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b)+J^\alpha_{(b)^-}f(a)\right]\right|$$

$$=\left|\frac{(b-a)^2}{2}\int_0^1 \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}f''(ta+(1-t)b)dt\right|$$

$$\leq \frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1(1-(1-t)^{\alpha+1}-t^{\alpha+1})^p dt\right)^{\frac{1}{p}} \times\left(\int_0^1\left|f''(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1(1-(1-t)^{p(\alpha+1)}-t^{p(\alpha+1)})dt\right)^{\frac{1}{p}} \times\left(\int_0^1\left[\frac{1}{t^s}|f''(a)|^q+\frac{1}{(1-t)^s}|f''(b)|^q\right]dt\right)^{\frac{1}{q}}$$

$$=\frac{(b-a)^2}{2(\alpha+1)}\left(\frac{p(\alpha+1)-1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q+|f''(b)|^q}{1-s}\right)^{\frac{1}{q}}.$$

This completes the proof. □

Theorem 7. *Let $f:[a,b]\to\mathbb{R}$ be twice differentiable function on (a,b) with $a<b$. If $f''\in L[a,b]$ and $|f''|$ is s-Godunova-Levin function of second kind, then*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{(\frac{a+b}{2})^-}f(a)+J^\alpha_{(\frac{a+b}{2})^+}f(b)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{(b-a)^2}{2^{3-s}(\alpha+1)}\left\{\mathscr{C}(s,\alpha,t)+\frac{1}{\alpha-s+2}\right\}\left[|f''(a)|+|f''(b)|\right],$$

where $\mathscr{C}(s,\alpha,t)=\int_0^1(1-t)^{\alpha+1}(1+t)^{-s}dt.$

$$=\frac{1}{2+\alpha}\,{}_2F_1(1,s,3+\alpha,-1),$$

where ${}_2F_1(1,s,3+\alpha,-1)$ is hypergeometric function.

Proof. Using Lemma 2 and the fact that $|f''|$ is s-Godunova-Levin function of second kind, we have

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{(\frac{a+b}{2})^-}f(a)+J^\alpha_{(\frac{a+b}{2})^+}f(b)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$
\begin{aligned}
&= \left| \frac{(b-a)^2}{8(\alpha+1)} \int_0^1 (1-t)^{\alpha+1} \left[f''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) \right.\right. \\
&\qquad \left.\left. + f''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) \right] dt \right| \\
&\le \left| \frac{(b-a)^2}{8(\alpha+1)} \int_0^1 (1-t)^{\alpha+1} f''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt \right| \\
&\qquad + \left| \frac{(b-a)^2}{8(\alpha+1)} \int_0^1 (1-t)^{\alpha+1} f''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) dt \right| \\
&\le \frac{(b-a)^2}{8(\alpha+1)} \int_0^1 (1-t)^{\alpha+1} \left| f''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) \right| dt \\
&\qquad + \frac{(b-a)^2}{8(\alpha+1)} \int_0^1 (1-t)^{\alpha+1} \left| f''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) \right| dt \\
&\le \frac{(b-a)^2}{8(\alpha+1)} \left[\int_0^1 (1-t)^{\alpha+1} \right. \\
&\qquad \times \left[\left(\frac{2}{1+t}\right)^s |f''(a)| + \left(\frac{2}{1-t}\right)^s |f''(b)| \right] dt \\
&\qquad \left. + \int_0^1 (1-t)^{\alpha+1} \left[\left(\frac{2}{1-t}\right)^s |f''(a)| + \left(\frac{2}{1+t}\right)^s |f''(b)| \right] dt \right] \\
&= \frac{(b-a)^2}{2^{3-s}(\alpha+1)} \left[|f''(a)| \int_0^1 (1-t)^{\alpha+1}(1+t)^{-s} dt \right. \\
&\qquad + |f''(b)| \int_0^1 (1-t)^{\alpha-s+1} dt + |f''(a)| \int_0^1 (1-t)^{\alpha-s+1} dt \\
&\qquad \left. + |f''(b)| \int_0^1 (1+t)^{\alpha+1}(1-t)^{-s} dt \right] \\
&= \frac{(b-a)^2}{2^{3-s}(\alpha+1)} \left[|f''(a)| \mathscr{C}(s,\alpha,t) + |f''(b)| \frac{1}{\alpha-s+2} \right. \\
&\qquad \left. + |f''(a)| \frac{1}{\alpha-s+2} + |f''(b)| \mathscr{C}(s,\alpha,t) \right] \\
&= \frac{(b-a)^2}{2^{3-s}(\alpha+1)} \left\{ \mathscr{C}(s,\alpha,t) + \frac{1}{\alpha-s+2} \right\} \left[|f''(a)| + |f''(b)| \right].
\end{aligned}
$$

This completes the proof. □

Theorem 8. *Let $f : [a,b] \to \mathbb{R}$ be twice differentiable function on (a,b) with $a < b$. If $f'' \in L[a,b]$ and $|f''|^q$ is s-Godunova-Levin function of second kind, then*

$$
\begin{aligned}
&\left| \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b) \right] - f\left(\frac{a+b}{2}\right) \right| \\
&\le \frac{2^{\frac{s-3q}{q}}(b-a)^2}{\alpha+1} \left(\frac{1}{p(\alpha+1)+1} \right)^{\frac{1}{p}} \\
&\times \left[\left\{ |f''(a)|^q \left(\frac{-2+2^s}{2^s(s-1)} \right) + |f''(b)|^q \left(\frac{1}{1-s} \right) \right\}^{\frac{1}{q}} \right. \\
&\quad \left. + \left\{ |f''(a)|^q \left(\frac{1}{1-s} \right) + |f''(b)|^q \left(\frac{-2+2^s}{2^s(s-1)} \right) \right\}^{\frac{1}{q}} \right].
\end{aligned}
$$

Proof. Using Lemma 2, Holder's inequality and the fact that $|f''|^q$ is s-Godunova-Levin function of second kind, we have

$$
\begin{aligned}
&\left| \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b) \right] - f\left(\frac{a+b}{2}\right) \right| \\
&= \left| \frac{(b-a)^2}{8(\alpha+1)} \int_0^1 (1-t)^{\alpha+1} \left[f''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) \right.\right. \\
&\qquad \left.\left. + f''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) \right] dt \right| \\
&\le \left| \frac{(b-a)^2}{8(\alpha+1)} \int_0^1 (1-t)^{\alpha+1} f''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt \right| \\
&\qquad + \left| \frac{b-a}{4} \int_0^1 (1-t)^{\alpha+1} f''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) dt \right| \\
&\le \frac{(b-a)^2}{8(\alpha+1)} \left\{ \left(\int_0^1 (1-t)^{p(\alpha+1)} dt \right)^{\frac{1}{p}} \right. \\
&\times \left(\int_0^1 \left| f''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) \right|^q dt \right)^{\frac{1}{q}} + \left(\int_0^1 (1-t)^{p(\alpha+1)} dt \right)^{\frac{1}{p}} \\
&\qquad \left. \times \left(\int_0^1 \left| f''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) \right|^q dt \right)^{\frac{1}{q}} \right\} \\
&\le \frac{(b-a)^2}{8(\alpha+1)} \left(\frac{1}{p(\alpha+1)+1} \right)^{\frac{1}{p}} \\
&\times \left\{ \left(|f''(a)|^q \int_0^1 \left(\frac{2}{1+t}\right)^s dt + |f''(b)|^q \int_0^1 \left(\frac{2}{1-t}\right)^s dt \right)^{\frac{1}{q}} \right. \\
&\qquad \left. + \left(|f''(a)|^q \int_0^1 \left(\frac{2}{1-t}\right)^s dt + |f''(b)|^q \int_0^1 \left(\frac{2}{1+t}\right)^s dt \right)^{\frac{1}{q}} \right\} \\
&= \frac{2^{\frac{s-3q}{q}}(b-a)^2}{\alpha+1} \left(\frac{1}{p(\alpha+1)+1} \right)^{\frac{1}{p}} \\
&\times \left[\left\{ |f''(a)|^q \left(\frac{-2+2^s}{2^s(s-1)} \right) + |f''(b)|^q \left(\frac{1}{1-s} \right) \right\}^{\frac{1}{q}} \right. \\
&\quad \left. + \left\{ |f''(a)|^q \left(\frac{1}{1-s} \right) + |f''(b)|^q \left(\frac{-2+2^s}{2^s(s-1)} \right) \right\}^{\frac{1}{q}} \right].
\end{aligned}
$$

This completes the proof. □

Theorem 9. *Let $f : [a,b] \to \mathbb{R}$ be twice differentiable function on (a,b) with $a < b$. If $f'' \in L[a,b]$ and $|f''|^q$ is* *s-Godunova-Levin function of second kind, then*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{(\frac{a+b}{2})^-}f(a)+J^\alpha_{(\frac{a+b}{2})^+}f(b)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{2^{\frac{s-3q}{q}}(b-a)^2}{\alpha+1}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left(|f''(a)|^q(\mathscr{C}(s,\alpha,t))+|f''(b)|^q\left(\frac{1}{\alpha-s+2}\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(|f''(a)|^q\left(\frac{1}{\alpha-s+2}\right)+|f''(b)|^q(\mathscr{C}(s,\alpha,t))\right)^{\frac{1}{q}}\right\},$$

where $\mathscr{C}(s,\alpha,t)=\int\limits_0^1(1-t)^{\alpha+1}(1+t)^{-s}dt.$

$$=\tfrac{1}{2+\alpha}\ {}_2F_1(1,s,3+\alpha,-1).$$

Proof. Using Lemma 2, power-mean inequality and the fact that $|f''|^q$ is s-Godunova-Levin function of second kind, we have

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{(\frac{a+b}{2})^-}f(a)+J^\alpha_{(\frac{a+b}{2})^+}f(b)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$=\left|\frac{(b-a)^2}{8(\alpha+1)}\int\limits_0^1(1-t)^{\alpha+1}\left[f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right.\right.$$
$$\left.\left.+f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right]dt\right|$$
$$\leq\left|\frac{(b-a)^2}{8(\alpha+1)}\int\limits_0^1(1-t)^{\alpha+1}f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right|$$
$$+\left|\frac{(b-a)^2}{8(\alpha+1)}\int\limits_0^1(1-t)^{\alpha+1}f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\right|$$
$$\leq\frac{(b-a)^2}{8(\alpha+1)}\left\{\left(\int\limits_0^1(1-t)^{(\alpha+1)}dt\right)^{1-\frac{1}{q}}\right.$$
$$\times\left(\int\limits_0^1(1-t)^{(\alpha+1)}\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int\limits_0^1(1-t)^{(\alpha+1)}dt\right)^{1-\frac{1}{q}}$$
$$\left.\times\left(\int\limits_0^1(1-t)^{(\alpha+1)}\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}$$
$$\leq \frac{2^{\frac{s-3q}{q}}(b-a)^2}{\alpha+1}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left(|f''(a)|^q(\mathscr{C}(s,\alpha,t))+|f''(b)|^q\left(\frac{1}{\alpha-s+2}\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(|f''(a)|^q\left(\frac{1}{\alpha-s+2}\right)+|f''(b)|^q(\mathscr{C}(s,\alpha,t))\right)^{\frac{1}{q}}\right\}.$$

This completes the proof. □

5 Inequalities for logarithmic s-Godunova-Levin functions of second kind

Theorem 10. *If $f:I\to(0,\infty)$ is logarithmically s-Godunova-Levin function of second kind, then for $s\in(0,1)$, we have*

$$f\left(\frac{a+b}{2}\right)^{\frac{1}{2^{s+1}}}$$
$$\leq\exp\left[\frac{1}{b-a}\int_a^b\log f(x)dx\right]\leq[f(a)f(b)]^{\frac{1}{1-s}}.$$

Proof. When f is a logarithmically s-Godunova-Levin function of second kind, we have

$$f\left(\frac{a+b}{2}\right)$$
$$=f\left(\frac{ta+(1-t)b+(1-t)a+tb}{2}\right)$$
$$\leq[f(ta+(1-t)b)]^{2^s}[f((1-t)a+tb)]^{2^s}$$
$$=\{[f(ta+(1-t)b)][f((1-t)a+tb)]\}^{2^s}.$$

Taking the logarithms on both sides of the above inequality yields

$$\ln f\left(\frac{a+b}{2}\right)$$
$$\leq\ln[f(ta+(1-t)b)f((1-t)a+tb)]^{2^s}$$
$$=2^s\ln[f(ta+(1-t)b)f((1-t)a+tb)],$$

which implies that

$$\frac{1}{2^s}\ln f\left(\frac{a+b}{2}\right)$$
$$\leq\ln[f(ta+(1-t)b)f((1-t)a+tb)]$$
$$=\ln f(ta+(1-t)b)+\ln f((1-t)a+tb).$$

Integrating the above inequality with respect to $t\in[0,1]$ procures

$$\frac{1}{2^s}\ln f\left(\frac{a+b}{2}\right)=\frac{2}{b-a}\int_a^b\ln f(x)dx,$$

which is equivalent to

$$\frac{1}{2^{s+1}}\ln f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b\ln f(x)dx. \tag{10}$$

Since f is logarithmically s-Godunova-Levin function of second kind, then for $s\in(0,1)$, we have

$$\ln f(ta+(1-t)b)\leq\frac{1}{t^s}\ln f(a)+\frac{1}{(1-t)^s}\ln f(b).$$

Integrating on both sides of the above inequality with respect to $t \in [0,1]$ generates

$$\frac{1}{b-a}\int_a^b \ln f(x)dx \le \{\ln[f(a)]+\ln[f(b)]\}\int_0^1 \frac{1}{t^s}dt. \quad (11)$$

Combining (10) and (11) gives

$$\ln f\left(\frac{a+b}{2}\right)^{\frac{1}{2^{s+1}}} \le \frac{1}{b-a}\int_a^b \ln f(x)dx \le \ln[f(a)f(b)]\int_0^1 \frac{1}{t^s}dt.$$

Taking the power of e on all sides of the above inequality produces

$$f\left(\frac{a+b}{2}\right)^{\frac{1}{2^{s+1}}} \le \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)dx\right] \le e^{\ln[f(a)f(b)]\int_0^1 \frac{1}{t^s}dt} = [f(a)f(b)]^{\frac{1}{1-s}}.$$

This completes the proof. □

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References

[1] G. Cristescu, L. Lupsa, Non-connected Convexities and Applications, Kluwer Academic Publishers, Dordrecht, Holland, (2002).

[2] S. S. Dragomir, Inequalities of hermite-hadamard type for h-convex functions on linear spaces, preprint.

[3] S. S. Dragomir, n-points inequalities of hermite-hadamard type for h-convex functions on linear spaces, preprint.

[4] S. S. Dragomir, J. Pečarić and L. E. Persson, Some inequalities of Hadamard type, Soochow J. Math, **21**, 335-341 (1995).

[5] S. S. Dragomir, C. E. M. Pearce, Selected topics on Hermite-Hadamard inequalities and applications, Victoria University, Australia, (2000).

[6] E. K. Godunova and V. I. Levin, Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions. (Russian) Numerical mathematics and mathematical physics (Russian), Moskov. Gos. Ped. Inst., Moscow, **166**, 138-142 (1985).

[7] U. N. Katugampola, New approach to a generalized fractional integral, Appl. Math. Comput., **218**, 860-865 (2011).

[8] W. Liu, J. Park, A generalization of the companion of Ostrowski-like inequality and applications, Appl. Math. Inf. Sci., **7**, 273-278 (2013).

[9] D. S. Mitrinovic, J. Pecaric, Note on a class of functions of Godunova and Levin, C. R. Math. Rep. Acad. Sci. Can., **12**, 33-36 (1990).

[10] M. A. Noor, M. U. Awan, Some Integral inequalities for two kinds of convexities via fractional integrals, Trans. J. Math. Mech., **5**, (2013).

[11] M. A. Noor, G. Cristescu, M. U. Awan, Generalized Fractional Hermite-Hadamard Inequalities for twice differentiable s-convex functions, Filomat, (2014).

[12] A Brndsted, RT Rockafellar, On the subdifferentiability of convex functions, Proceedings of the American Mathematical Society, **16**, (1965).

[13] M. A. Noor, K. I. Noor, M. U. Awan, J. Li, On Hermite-Hadamard Inequalities for h-preinvex functions, Filomat, (2014), (inpress).

[14] M. A. Noor, F. Qi, M. U. Awan, Some HermiteHadamard type inequalities for log $-h$-convex functions, Analysis, **33**, 367-375 (2013).

[15] M. E. Ozdemir, C. Yildiz, A. O. Akdemir, On the co-ordinated convex functions, Appl. Math. Infor. Sci., **8**, 1085-1091 (2014).

[16] M. Radulescu. S. Radulescu, P. Alexandrescu, On the Godunova-Levin-Schur class of functions, Math. Inequal. Appl., **12**, 853-862 (2009).

[17] M. Z. Sarikaya, E. Set, H. Yaldiz and N. Basak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, Mathematical and Computer Modelling, **57**, 2403-2407 (2013).

[18] J. Wang, X. Li, M. Feckan, Y. Zhou, Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity. Applicable Analysis, (2012), http://dxdoi.org/10.1080/00036811.2012.727986.

[19] T. -Y. Zhang, A. -P. Ji, F. Qi, On integral inequalities of Hermite-Hadamard type for s-geometrically convex functions, Abstr. Appl. Anal. 2012, Art. ID 560586, 14 pp.

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