

# Upper and Lower Continuity of Soft Multifunctions

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**Abstract:** In this paper, we introduced the notion of soft multifunctions and define the upper and lower inverse of a soft multifunction and prove basic properties. Then using these ideas we introduced the concepts of upper and lower continuous soft multifunctions. Also we obtain some characterizations and several properties concerning upper and lower soft continuous multifunctions.

**Keywords:** soft sets, soft multifunction, soft continuity

## 1 Introduction

There are various types of functions which play an important role in the classical theory of set topology. A great deal of works on such functions has been extended to the setting of multifunctions. A multifunction is a set-valued function. The theory of multifunctions was first codified by Berge [1]. In the last three decades, the theory of multifunctions has advanced in a variety of ways and applications of this theory, can be found for example, in economic theory, noncooperative games, artificial intelligence, medicine, information sciences and decision theory (See [2] and references therein). Papageorgiou [3], Allbrycht and Maltoba [4], Beg [5], Heilpein [6] and Butnairu [7] have started the study of fuzzy multifunctions and obtained several fixed point theorems for fuzzy mappings. On the other hand, a Russian researcher Molodtsov [8] introduced the concept of soft sets as a general mathematical tool for dealing with uncertainty and he successfully applied the soft set theory into several directions, such as smoothness of functions, theory of measurement, game theory, Riemann integration and so on. Then, Maji et al. [9] defined some operations on soft sets and some basic properties of these operations are revealed in [10]. Also, Aktaş and Çağman [12] compared soft sets with fuzzy sets and rough sets. Applications of Soft Set Theory in other disciplines and real life problems are now catching momentum.

We can say that soft sets are a class of special information systems, and both researches of soft sets and information systems are the same formal structures. Information systems have been studied by many

researchers from several domains such as knowledge engineering [21,22], rough set theory [23] and granular computing [24,25]. Pei and Miao [18] showed the operations of information systems are parallel to those of soft sets. They also investigated that there exist some compact connections between soft sets and information systems.

The topological structure of set theories dealing with uncertainties were first studied by Chang [13]. Lashin et al. [14] generalized rough set theory in the framework of topological spaces. Recently, Shabir and Naz [15] introduced the notion of soft topological spaces which are defined over an initial universe with a fixed set of parameters. Also they studied soft separation axioms for soft topological spaces. Zorlutuna et al. [16] introduced the concept of soft continuity of functions and studied some of its properties. Then Aygunoglu and Aygun [17] studied continuous soft functions. Recently, Kharal and Ahmad [18] defined the notion of a mapping on soft classes and studied several properties of images and inverse images of soft sets. Also they applied these concepts to the problem of medical diagnosis in medical expert systems.

In this paper our purpose is two fold. First, we define upper and lower inverse of a soft multifunction and study their various properties. Next, we use these ideas to introduce upper soft continuous multifunctions and lower soft continuous multifunctions. Moreover, we obtain some characterizations and several properties concerning such multifunctions. In addition, we investigate the relationships between soft multifunction and information systems.

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## 2 Preliminaries

Molodtsov [8] defined soft sets in the following manner. Let  $U$  be an initial universe set and  $E$  be a set of parameters. Let  $P(U)$  denote the power set of  $U$ , and let  $B \subseteq E$ .

**Definition 1.**[8] A pair  $(G, B)$  is called a soft set over  $U$ , where  $G$  is a mapping given by

$$G : B \rightarrow P(U)$$

**Definition 2.**[11] For two soft sets  $(G, B)$  and  $(H, C)$  over a common universe  $U$ ,  $(G, B)$  is a soft subset of  $(H, C)$ , denoted by  $(G, B) \widetilde{\subseteq} (H, C)$ , if  $B \subseteq C$  and  $\forall x \in B$ ,  $G(x) \subseteq H(x)$ .

**Definition 3.**[9] Two soft sets  $(G, B)$  and  $(H, C)$  over a common universe  $U$  are said to be soft equal if  $(G, B)$  is a soft subset of  $(H, C)$  and  $(H, C)$  is a soft subset of  $(G, B)$ .

**Definition 4.**[19] The complement of a soft set  $(G, B)$ , denoted by  $(G, B)^c$ , is defined by  $(G, B)^c = (G^c, B)$ .  $G^c : B \rightarrow P(U)$  is a mapping given by  $G^c(x) = U - G(x)$ ,  $\forall x \in B$ .  $G^c$  is called the soft complement function of  $G$ . Clearly,  $(G^c)^c$  is the same as  $G$  and  $((G, B)^c)^c = (G, B)$ .

**Definition 5.**[9] A soft set  $(G, E)$  over  $U$  is said to be a null soft set, denoted by  $\widetilde{\Phi}$ , if  $\forall x \in E$ ,  $G(x) = \emptyset$ .

**Definition 6.**[9] A soft set  $(G, E)$  over  $U$  is said to be an absolute soft set, denoted by  $\widetilde{U}$ , if  $\forall x \in E$ ,  $G(x) = U$ .

$$\text{Clearly, } \widetilde{U}^c = \widetilde{\Phi} \text{ and } \widetilde{\Phi}^c = \widetilde{U}.$$

**Definition 7.**[9] The union of two soft sets  $(G, B)$  and  $(H, C)$  over the common universe  $U$  is the soft set  $(T, D)$ , where  $D = B \cup C$  and for all  $x \in D$ ,

$$T(x) = \begin{cases} G(x) & \text{if } x \in B \setminus C \\ H(x) & \text{if } x \in C \setminus B \\ G(x) \cup H(x) & \text{if } x \in C \cap B \end{cases}$$

This relationship is written as  $(H, C) \widetilde{\cup} (G, B) = (T, D)$ .

**Definition 8.**[11] The intersection of two soft sets  $(G, B)$  and  $(H, C)$  over the common universe  $U$  is the soft set  $(T, D)$ , where  $D = B \cap C$  and for all  $x \in D$ ,  $T(x) = H(x) \cap G(x)$ . This relationship is written as  $(H, C) \widetilde{\cap} (G, B) = (T, D)$ .

For other properties of these operations, we refer to references [9, 11, 19].

**Remark.**[12] Zadeh's fuzzy set may be considered a special case of the soft set.

**Proposition 1.**[15] If  $(G, E)$  and  $(H, E)$  are two soft sets over  $U$ , then

- (1)  $((G, E) \widetilde{\cap} (H, E))^c = (G, E)^c \widetilde{\cap} (H, E)^c$ .
- (2)  $((G, E) \widetilde{\cap} (H, E))^c = (G, E)^c \widetilde{\cap} (H, E)^c$ .

**Definition 9.**[16] Let  $I$  be arbitrary index set and  $\{(G_i, E)\}_{i \in I}$  be soft sets over  $U$ , then

(a) The union of these soft sets is the soft set  $(H, E)$ , where  $H(x) = \bigcup_{i \in I} G_i(x)$  for each  $x \in E$ .

$$\text{We write } \bigcup_{i \in I} (G_i, E) = (H, E).$$

(b) The intersection of these soft sets is the soft set  $(M, E)$ , where  $M(x) = \bigcap_{i \in I} G_i(x)$  for all  $x \in E$ .

$$\text{We write } \bigcap_{i \in I} (G_i, E) = (M, E).$$

**Proposition 2.**[16] Let  $I$  be arbitrary index set and  $\{(G_i, E)\}_{i \in I}$  be soft sets over  $U$ , then

- (1)  $\left[ \bigcup_{i \in I} (G_i, E) \right]^c = \bigcap_{i \in I} (G_i, E)^c$ , and
- (2)  $\left[ \bigcap_{i \in I} (G_i, E) \right]^c = \bigcup_{i \in I} (G_i, E)^c$ .

**Definition 10.**[16] The soft set  $(G, E)$  over  $U$  is called a soft point in  $\widetilde{U}$ , denoted by  $e_G$ , if for the element  $e \in A$ ,  $G(e) \neq \emptyset$  and  $G(e') = \emptyset$  for all  $e' \in E - \{e\}$ .

**Definition 11.**[16] The soft point  $x_G$  is said to be in the soft set  $(H, E)$ , denoted by  $x_G \widetilde{\in} (H, E)$ , if for the element  $x \in E$  and  $G(x) \subseteq H(x)$ .

**Proposition 3.**[?] Let  $x_G \widetilde{\in} \widetilde{U}$  and  $(G, B) \widetilde{\subseteq} \widetilde{U}$ . If  $x_G \widetilde{\in} (G, E)$ , then  $x_G \notin (G, E)^c$ .

**Definition 12.**[15] Let  $\tau$  be the collection of soft sets over a universe  $U$ , then  $\tau$  is called a soft topology on  $U$  if

- T1.  $\widetilde{U}$  and  $\widetilde{\Phi}$  belong to  $\tau$
- T2. the union of any number of soft sets in  $\tau$  belongs to  $\tau$
- T3. the intersection of any two soft sets in  $\tau$  belongs to  $\tau$ .

The triplet  $(U, \tau, E)$  is called soft topological space over  $U$ . The members of  $\tau$  are called soft open sets in  $U$  and complements of their are called soft closed sets in  $U$ .

**Definition 13.**[16] A soft set  $(T, E)$  in a soft topological space  $(U, \tau, E)$  is called a soft neighborhood (briefly: nbd) of the soft point  $x_G \widetilde{\in} \widetilde{U}$  if there exists a soft open set  $(H, E)$  such that  $x_G \widetilde{\in} (H, E) \widetilde{\subseteq} (T, E)$ .

The neighborhood system of a soft point  $x_G$ , denoted by  $N_\tau(x_G)$ , is the family of all its neighborhoods.

**Definition 14.**[16] A soft set  $(T, E)$  in a soft topological space  $(U, \tau, E)$  is called a soft neighborhood (briefly: nbd) of the soft set  $(G, E)$  if there exists a soft open set  $(H, E)$  such that  $(G, E) \widetilde{\subseteq} (H, E) \widetilde{\subseteq} (T, E)$ .

**Theorem 1.**[16] The neighborhood system  $N_\tau(x_G)$  at  $x_G$  in a soft topological space  $(U, \tau, E)$  has the following properties:

- (a) If  $(H, E) \in N_\tau(x_G)$ , then  $x_G \in (H, E)$ ,
- (b) If  $(H, E) \in N_\tau(x_G)$  and  $(H, E) \widetilde{\subseteq} (M, E)$ , then  $(M, E) \in N_\tau(x_G)$ ,

(c) If  $(H, E), (M, E) \in N_\tau(x_G)$ , then  $(G, E) \widetilde{\cap} (M, E) \in N_\tau(x_G)$ ,

(d) If  $(H, E) \in N_\tau(x_G)$ , then there is a  $(M, E) \in N_\tau(x_G)$  such that  $(H, E) \in N_\tau(x'_T) \widetilde{\cap} (M, E)$ .

**Definition 15.** Let  $(U, \tau, E)$  be a soft topological space and let  $(G, E)$  be a soft set over  $U$ .

(a) [15] The soft closure of  $(G, E)$  in  $(U, \tau, E)$  is the soft set  $cl(G, E) = \widetilde{\cap}\{(S, E) : (S, E) \text{ is soft closed and } (G, E) \subseteq (S, E)\}$ .

(b) [16] The soft interior of  $(G, E)$  in  $(U, \tau, E)$  is the soft set  $int(G, E) = \widetilde{\cup}\{(S, E) : (S, E) \text{ is soft open and } (S, E) \subseteq (G, E)\}$ .

$cl(G, E)$  is soft closed and the smallest soft closed set containing  $(G, E)$ , in the sense that it is contained in every soft closed set containing  $(G, E)$ . Similarly, by property T3 for soft open sets,  $int(G, E)$  is soft open and the largest soft open set contained in  $(G, E)$ .

**Corollary 1.** Let  $(U, \tau, E)$  be a soft topological space and let  $(G, E)$  be a soft set over  $U$ . Then

(a) [15]  $(G, E)$  is soft closed iff  $(G, E) = cl(G, E)$ .

(b) [16]  $(G, E)$  is soft open iff  $(G, E) = int(G, E)$ .

**Theorem 2.** [16] A soft set  $(G, E)$  is soft open if and only if for each soft set  $(H, E)$  contained in  $(G, E)$ ,  $(G, E)$  is a soft neighborhood of  $(H, E)$ .

**Proposition 4.** Let  $(U, \tau, E)$  be a soft topological space and let  $(G, E)$  and  $(H, E)$  be soft sets over  $U$ . Then

(a) [15] If  $(G, E) \subseteq (H, E)$ , then  $cl(G, E) \subseteq cl(H, E)$ .

(b) [16] If  $(G, E) \subseteq (H, E)$ , then  $int(G, E) \subseteq int(H, E)$ .

**Theorem 3.** [16] Let  $(U, \tau, E)$  be a soft topological space and let  $(G, E)$  be a soft set over  $U$ . Then

(a)  $(cl(G, E))^c = int((G, E)^c)$ .

(b)  $(int(G, E))^c = cl((G, E)^c)$ .

### 3 Continuity of Soft Multifunctions

Let  $Y$  be an initial universe set and  $E$  be the non-empty set of parameters.

**Definition 16.** A soft multifunction  $F$  from an ordinary topological space  $(X, \tau)$  into a soft topological space  $(Y, \sigma, E)$  assigns to each  $x$  in  $X$  a soft set  $F(x)$  over  $Y$ . A soft multifunction will be denoted by  $F : (X, \tau) \rightarrow (Y, \sigma, E)$ .  $F$  is said to be onto if for each soft set  $(G, E)$  over  $Y$ , there exists a point  $x \in X$  such that  $F(x) = (G, E)$ .

**Definition 17.** For a soft multifunction  $F : (X, \tau) \rightarrow (Y, \sigma, E)$ , the upper inverse  $F^+(G, E)$  and the lower inverse  $F^-(G, E)$  of a soft set  $(G, E)$  over  $Y$  are defined as follows:  $F^+(G, E) = \{x \in X : F(x) \subseteq (G, E)\}$  and  $F^-(G, E) = \{x \in X : F(x) \widetilde{\cap} (G, E) \neq \Phi\}$ . Moreover, for a subset  $M$  of  $X$   $F(M) = \widetilde{\cup}\{F(x) : x \in M\}$ .

**Definition 18.** [3] Let  $(X, \tau)$  be an ordinary topological space and  $(Y, \vartheta)$  be a fuzzy topological space.  $F : (X, \tau) \rightarrow (Y, \vartheta)$  is called a fuzzy multifunction iff for every  $x \in X$ ,  $F(x)$  is a fuzzy set in  $Y$ .

**Remark.** Since every fuzzy set may be considered a soft set, then every fuzzy multifunction may be considered a soft multifunction.

**Example 1.** Let  $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$  be a topology on  $X = \{a, b, c\}$  and let  $\sigma = \{\Phi, \tilde{Y}, (G, E), (H, E)\}$  be a soft topology over  $Y = \{y_1, y_2, y_3\}$  where  $E = \{e_1, e_2, e_3\}$ ,  $G(e_1) = \{y_1\}$ ,  $G(e_2) = \{y_3\}$  and  $G(e_3) = \emptyset$ ,  $H(e_1) = \{y_1, y_2\}$ ,  $H(e_2) = \{y_3\}$  and  $H(e_3) = Y$ . Then the multifunction  $F : (X, \tau) \rightarrow (Y, \sigma, E)$  given by  $F(a) = (G, E)$ ,  $F(b) = (H, E)$  and  $F(c) = \tilde{Y}$  is a soft multifunction but not fuzzy multifunction. Because the soft set  $(H, E)$  is not a fuzzy set.

**Proposition 5.** Let  $M$  be a subset of  $X$ . Then the follows are true for a soft multifunction  $F : (X, \tau) \rightarrow (Y, \sigma, E)$  :

(a)  $M \subset F^+(F(M))$ . If  $F$  is onto  $M = F^+(F(M))$ .

(b)  $M \subset F^-(F(M))$ . If  $F$  is onto  $M = F^-(F(M))$ .

**Proof.** (a) Let  $x \in M$ . Then  $F(x) \subseteq F(M) = \widetilde{\cup}\{F(x) : x \in M\}$  and so  $x \in F^+(F(M))$ . Hence,  $M \subset F^+(F(M))$ .

(b) The proof is similar to (a).

**Proposition 6.** Let  $(G, E)$  be a soft set over  $Y$ . Then the followings are true for a soft multifunction  $F : (X, \tau) \rightarrow (Y, \sigma, E)$  :

(a)  $F^+((G, E)^c) = X - F^-(G, E)$

(b)  $F^-((G, E)^c) = X - F^+(G, E)$ .

**Proof.** (a) If  $x \in X - F^-(G, E)$  then  $x \notin F^-(G, E)$  which implies  $F(x) \widetilde{\cap} (G, E) = \Phi$  and therefore  $F(x) \subseteq (G, E)^c$ . Thus  $x \in F^+((G, E)^c)$  and  $X - F^-(G, E) \subseteq F^+((G, E)^c)$ . Conversely, If  $x \in F^+((G, E)^c)$  then  $F(x) \subseteq (G, E)^c$  which implies  $F(x) \widetilde{\cap} (G, E) = \Phi$  and therefore  $x \notin F^-(G, E)$ . Thus  $x \in X - F^-(G, E)$  and  $F^+((G, E)^c) \subseteq X - F^-(G, E)$ . (b) If  $x \in X - F^+(G, E)$  then  $x \notin F^+(G, E)$  which implies  $F(x) \not\subseteq (G, E)$  and therefore  $F(x) \widetilde{\cap} ((G, E)^c) \neq \Phi$ . Thus  $x \in F^-((G, E)^c)$  and  $X - F^+(G, E) \subseteq F^-((G, E)^c)$ .

Conversely, If  $x \in F^-((G, E)^c)$  then  $F(x) \widetilde{\cap} ((G, E)^c) \neq \Phi$  which implies  $F(x) \not\subseteq (G, E)$  and therefore  $x \notin F^+(G, E)$ . Thus  $x \in X - F^+(G, E)$  and  $F^-((G, E)^c) \subseteq X - F^+(G, E)$ .

**Proposition 7.** Let  $(G_i, E)$  be soft sets over  $Y$  for each  $i \in I$ . Then the follows are true for a soft multifunction  $F : (X, \tau) \rightarrow (Y, \sigma, E)$  :

(a)  $F^-(\bigcup_{i \in I} (G_i, E)) = \bigcup_{i \in I} (F^-(G_i, E))$ .

(b)  $F^+(\bigcap_{i \in I} (G_i, E)) = \bigcap_{i \in I} (F^+(G_i, E))$ .

*Proof.*(a) For every  $x \in F^{-}(\bigcup_{i \in I}(G_i, E))$ ,  $F(x) \cap (\bigcup_{i \in I}(G_i, E)) \neq \tilde{\Phi}$ . There exists  $i \in I$  such that  $F(x) \cap (G_i, E) \neq \tilde{\Phi}$ . For the same  $i \in I$ ,  $x \in F^{-}(G_i, E)$ . Therefore  $x \in \bigcup_{i \in I}(F^{-}(G_i, E))$ . Thus  $F^{-}(\bigcup_{i \in I}(G_i, E)) \subseteq \bigcup_{i \in I}(F^{-}(G_i, E))$ .  
Conversely, For every  $x \in \bigcup_{i \in I}(F^{-}(G_i, E))$ , there exists  $i \in I$  such that  $x \in F^{-}(G_i, E)$ . For the same  $i \in I$ ,  $F(x) \cap (G_i, E) \neq \tilde{\Phi}$ . Therefore,  $F(x) \cap (\bigcup_{i \in I}(G_i, E)) \neq \tilde{\Phi}$  and  $x \in F^{-}(\bigcup_{i \in I}(G_i, E))$ . Thus  $\bigcup_{i \in I}(F^{-}(G_i, E)) \subseteq F^{-}(\bigcup_{i \in I}(G_i, E))$ .  
(b) The proof is similar of (a).

**Definition 19.** Let  $(X, \tau)$  be an ordinary topological space and  $(Y, \sigma, E)$  be a soft topological space. Then a soft multifunction  $F : (X, \tau) \rightarrow (Y, \sigma, E)$  is said to be

(a) soft upper continuous (briefly: soft u. c.) at a point  $x \in X$  if for each soft open  $(G, E)$  such that  $F(x) \subseteq (G, E)$ , there exists an open neighborhood  $P(x)$  of  $x$  such that  $F(z) \subseteq (G, E)$  for all  $z \in P(x)$ .

(b) soft lower continuous (briefly: soft l. c.) at a point  $x \in X$  if for each soft open  $(G, E)$  such that  $F(x) \cap (G, E) \neq \tilde{\Phi}$ , there exists an open neighborhood  $P(x)$  of  $x$  such that  $F(z) \cap (G, E) \neq \tilde{\Phi}$  for all  $z \in P(x)$ .

(c) soft upper(lower) continuous if  $F$  has this property at every point of  $X$ .

**Example 2.** Let  $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$  be a topology on  $X = \{a, b, c\}$  and let  $\sigma = \{\tilde{\Phi}, \tilde{Y}, (G, E), (H, E)\}$  be a soft topology over  $Y = \{y_1, y_2, y_3\}$  where  $E = \{e_1, e_2, e_3\}$ ,  $G(e_1) = \{y_1\}$ ,  $G(e_2) = \{y_3\}$  and  $G(e_3) = \emptyset$ ,  $H(e_1) = \{y_1, y_2\}$ ,  $H(e_2) = \{y_3\}$  and  $H(e_3) = Y$ . Then the multifunction  $F : (X, \tau) \rightarrow (Y, \sigma, E)$  given by

$F(a) = (G, E)$ ,  $F(b) = (H, E)$  and  $F(c) = \tilde{Y}$  is upper soft multifunction. Because  $F^+(G, E) = \{a\} \in \tau$  and  $F^+(G, E) = \{a, b\} \in \tau$ .

**Proposition 8.** A soft multifunction  $F : (X, \tau) \rightarrow (Y, \sigma, E)$  is soft upper continuous if and only if for all soft open set  $(G, E)$  over  $U$ ,  $F^+(G, E)$  is open in  $X$ .

*Proof.* First suppose that  $F$  is soft upper continuous. Let  $(G, E)$  be soft open set over  $Y$  and  $x \in F^+(G, E)$ . Then from Definition 19 we know that there exists an open neighborhood  $P(x)$  of  $x$  such that for all  $z \in P(x)$ ,  $F(z) \subseteq (G, E)$  which means that  $F^+(G, E)$  is open as claimed. The other direction is just the definition of soft upper continuity of  $F$ .

**Proposition 9.**  $F : (X, \tau) \rightarrow (Y, \sigma, E)$  is soft lower continuous multifunction if and only if for every soft set  $(G, E)$  over  $Y$ ,  $F^-(G, E)$  is open set in  $X$ .

*Proof.* First assume that  $F$  is soft lower continuous. Let  $(G, E)$  soft open set over  $Y$  and  $x \in F^-(G, E)$ . Then there is an open neighborhood  $P(x)$  of  $x$  such that  $F(z) \cap (G, E) \neq \tilde{\Phi}$  for all  $z \in P(x)$ . So  $P(x) \subseteq F^-(G, E)$  which implies that  $F^-(G, E)$  is open in  $X$ . Now suppose that  $F^-(G, E)$  is open. Let  $x \in F^-(G, E)$ . Then  $F^-(G, E)$  is an open neighborhood of  $x$  and for all  $z \in F^-(G, E)$  we have  $F(z) \cap (G, E) \neq \tilde{\Phi}$ . So  $F$  is soft lower continuous.

**Theorem 4.** The followings are equivalent for a soft multifunction  $F : (X, \tau) \rightarrow (Y, \sigma, E)$ :

- (a)  $F$  is soft upper continuous
- (b) for each soft closed set  $(G, E)$  over  $Y$ ,  $F^-(G, E)$  is closed in  $X$ .
- (c) for each soft set  $(G, E)$  over  $Y$ , of  $Y$ ,  $cl(F^-(G, E)) \subseteq F^-(cl(G, E))$ .
- (d) for each soft set  $(G, E)$  over  $Y$ ,  $F^+(Int(G, E)) \subseteq Int(F^+(G, E))$ .

*Proof.* (a)  $\Rightarrow$  (b) Let  $(G, E)$  be a closed soft over  $Y$ . Then proposition 1 implies  $(G, E)^c$  is soft open and  $F^+((G, E)^c) = X - F^-(G, E)$ , then since  $F^-(G, E)$  is open and so  $F^-(G, E)$  is closed.

(b)  $\Rightarrow$  (c) Let  $(G, E)$  be any soft set over  $Y$ . Then  $cl(G, E)$  is soft closed set. By (b)  $F^-(cl(G, E))$  is closed in  $X$ . Hence,  $cl(F^-(G, E)) \subseteq F^-(cl(G, E))$  and since  $F^-(G, E) \subseteq F^-(cl(G, E))$ ,  $cl(F^-(G, E)) \subseteq F^-(cl(G, E))$ .

(c)  $\Rightarrow$  (d) Let  $(G, E)$  be any soft set over  $Y$ . By (c),  $cl(F^-(G, E)^c) \subseteq F^-(cl(G, E)^c)$ ,  $X - F^-(cl(G, E)^c) \subseteq int(X - F^-(cl(G, E)^c))$ ,  $X - X - F^+(int(G, E)) \subseteq int F^+(G, E)$ .

(d)  $\Rightarrow$  (a) Let  $(G, E)$  be any soft set over  $Y$ . By (d)  $F^+(Int(G, E)) = F^+(G, E) \subseteq Int(F^+(G, E))$  and so  $F^+(G, E)$  is open in  $X$ . They by proposition (1)  $F$  soft upper continuous.

**Theorem 5.** The following are equivalent for a soft multifunction  $F : (X, \tau) \rightarrow (Y, \sigma, E)$ :

- (a)  $F$  is soft lower continuous.
- (b) for each soft closed set  $(G, E)$  over  $Y$ ,  $F^+(G, E)$  is closed in  $X$ .
- (c) for each soft set  $(G, E)$  over  $Y$ ,  $cl(F^+(G, E)) \subseteq F^+(cl(G, E))$ .
- (d) for each soft set  $(G, E)$  over  $Y$ ,  $F^-(Int(G, E)) \subseteq int(F^-(G, E))$ .

*Proof.* It is similar to that of Theorem 4.

[12] A fuzzy topological space  $(X, \tau)$  can be as well be treated as a soft topological space. Let  $F$  be a fuzzy set with the membership function  $\mu_F$ . Now let us consider the family  $\alpha$ -level sets  $F(\alpha)$  for the function  $\mu_F$ . If we know the family  $F$ , we can find the function  $\mu_F(x)$  by means of  $\mu_F(x) = \sup\{\alpha : x \in F(\alpha)\}$ . Thus every fuzzy set  $F$  may be considered a soft set.  $(F, [0, 1])$ . Then for another soft topological space  $(Y, \sigma)$ , the product soft topological space  $X \times Y$  can be considered in the usual



manner, i.e, the soft sets form  $g_B \times h_C$  with  $g_B \in \tau$  and  $h_C \in \sigma$  form a basis for the product soft topology  $\tau \times \sigma$  on  $X \times Y$ , where any  $(x, y) \in X \times Y$ ,  

$$(g_B \times h_C)(x, y) = \begin{cases} \emptyset : \text{if } x \in B \\ h(y) : \text{if } x \notin B \end{cases} \quad \text{Thus}$$

$$(g_B \times h_C)(x, y) \subseteq h(y).$$
 We now set the following definition.

**Definition 20.** For a soft multifunction  $F : (X, \tau) \rightarrow (Y, \sigma, E)$ , the graph multifunction  $G_F : X \rightarrow X \times Y$  of  $F$  is defined as  $G_F(x) =$  the soft set  $x_1 \times F(x)$ , for every  $x \in X$ . over  $X \times Y$ , where  $x_1$  is the soft set over  $X$ . We shall write  $\{x\} \times F(x)$  for  $x_1 \times F(x)$ .

**Lemma 1.** For a soft multifunction  $F : (X, \tau) \rightarrow (Y, \sigma, E)$ , the followings are hold:

- (a)  $G_F^+(M \times (H, E)) = M \cap F^+(H, E)$   
 (b)  $G_F^-(M \times (H, E)) = M \cap F^-(H, E)$

*Proof.* (a) Let  $M$  be any subset of  $X$  and let  $(H, E)$  be any soft set over  $Y$ . Let  $x \in G_F^+(M \times (H, E))$ . Then  $G_F(x) \subseteq (M \times (H, E))$  that is  $(\{x\} \times F(x)) \subseteq M \times (H, E)$ . Therefore, we have  $x \in M$  and  $F(x) \subseteq (H, E)$ . Hence  $x \in M \cap F^+(H, E)$ .

Conversely, let  $x \in M \cap F^+(H, E)$ . Then  $x \in M$  and  $x \in F^+(H, E)$ . Thus  $x \in M$  and  $F(x) \subseteq (H, E)$  that is  $G_F(x) \subseteq (M \times (H, E))$ . Therefore  $x \in G_F^+(M \times (H, E))$ .

(b) Let  $M$  be any subset of  $X$  and let  $(H, E)$  be any soft set over  $Y$ . Let  $x \in G_F^-(M \times (H, E))$ . Then  $\tilde{\Phi} \neq G_F(x) \cap (M \times (H, E)) = (\{x\} \times F(x)) \cap (M \times (H, E)) = (\{x\} \cap M) \times (F(x) \cap (H, E))$ . Therefore, we have  $x \in M$  and  $F(x) \cap (H, E) \neq \tilde{\Phi}$ . Hence  $x \in M \cap F^-(H, E)$ .

Conversely, let  $x \in M \cap F^-(H, E)$ . Then  $x \in M$  and  $x \in F^-(H, E)$ . Thus  $x \in M$  and  $F(x) \cap (H, E) \neq \tilde{\Phi}$  that is  $G_F(x) \cap (M \times (H, E)) \neq \tilde{\Phi}$ . Therefore  $x \in G_F^-(M \times (H, E))$ .

**Theorem 6.** Let  $F : (X, \tau) \rightarrow (Y, \sigma, E)$  be a soft multifunction. If the soft graph function of  $F$  is soft lower (upper) continuous, then  $F$  is soft lower (upper) continuous.

*Proof.* For a subset  $V$  of  $X$  and  $(G, E)$  a soft set over  $U$ , we take  $(V \times (G, E))(z, y) = \begin{cases} \emptyset, \text{if } z \notin V \\ G(y), \text{if } z \in V \end{cases}$

Let  $x \in X$  and let  $(G, E)$  be soft open set such that  $x \in F^-(G, E)$ . Then we obtain that  $x \in G_F^-(X \times (G, E))$  and  $X \times (G, E)$  is a soft set over  $Y$ . Since the soft graph multifunction  $G_F$  is soft lower continuous, it follows that there exists an open set  $P$  containing  $x$  such that  $P \subseteq G_F^-(X \times (G, E))$ . From here, we obtain that  $P \subseteq F^-(G, E)$ . Thus  $F$  is soft lower continuous.

The proof of the soft upper continuity of  $F$  is similar to the above.

**Theorem 7.** Let  $F : (X, \tau) \rightarrow (Y, \sigma, E)$  be a soft multifunction and  $M$  be an open set of  $X$ . Then the restriction  $F|_M$  is soft upper continuous if  $F$  is soft upper continuous.

*Proof.* Let  $(H, E)$  be any soft open set over  $Y$  such that  $(F|_M)(x) \subseteq (H, E)$ . Since  $F$  is soft upper continuous and  $F(x) = (F|_M)(x) \subseteq (H, E)$ , there exists open set  $U \subseteq X$  containing  $x$  such that  $F(z) \subseteq (H, E)$  for all  $z \in U$ . Put  $U_1 = U \cap M$  then we have  $U_1$  is open set in  $M$  containing  $x$  and  $F(U_1) = (F|_M)(U_1) \subseteq (H, E)$ . This shows that  $F|_M$  is soft upper continuous.

**Theorem 8.** Let  $F : (X, \tau) \rightarrow (Y, \sigma, E)$  be a soft multifunction and  $M$  be an open set of  $X$ . Then  $F$  is soft lower continuous if and only if the restriction  $F|_M$  is soft lower continuous.

*Proof.* Let  $(H, E)$  be any soft open set over  $Y$  such that  $(F|_M)(x) \cap (H, E) \neq \tilde{\Phi}$ . Since  $F(x) = (F|_M)(x)$ , then  $F(x) \cap (H, E) \neq \tilde{\Phi}$ . Also since  $F$  is soft lower continuous there exists open set  $U \subseteq X$  containing  $x$  such that  $F(z) \cap (H, E) \neq \tilde{\Phi}$  for all  $z \in U$ . Put  $U_1 = U \cap M$  then we have  $U_1$  is open set in  $M$  containing  $x$  and  $F(U_1) \cap (H, E) \neq \tilde{\Phi}$ . Therefore  $(F|_M)(U_1) \cap (H, E) \neq \tilde{\Phi}$ . This shows that  $F|_M$  is soft lower continuous.

**Corollary 2.** Let  $F : (X, \tau) \rightarrow (Y, \sigma, E)$  be a soft multifunction and  $\{M_i : i \in I\}$  be an open cover set of  $X$ . The followings are hold:

- (a)  $F$  is soft lower continuous if and only if the restriction  $F|_{M_i}$  is soft lower continuous for every  $i \in I$ .  
 (b)  $F$  is soft upper continuous if and only if the restriction  $F|_{M_i}$  is soft upper continuous for every  $i \in I$ .

*Proof.* (a): Let  $x \in X$  and  $x \in M_i$  for an  $i \in I$ . Let  $(G, E)$  be a soft closed set over  $Y$  with  $F|_{M_i}(x) \cap (G, E) \neq \tilde{\Phi}$ . Since  $F$  is soft lower continuous and  $F(x) = F|_{M_i}(x)$ , there exists an open set  $P$  containing  $x$  such that  $P \subseteq F^-(G, E)$ . Take  $P_1 = P \cap M_i$ . Then  $P_1$  is an open set in  $M_i$  containing  $x$ . We have  $P_1 \subseteq F^-(G, E)$ . Thus  $F|_{M_i}$  is soft lower continuous.

Conversely, let  $x \in X$  and  $(G, E)$  be a soft closed set over  $Y$  with  $F(x) \subseteq (G, E)$ . Since  $\{M_i : i \in I\}$  is an open cover set of  $X$ , then  $x \in M_{i_0}$  for an  $i_0 \in I$ . We have  $F|_{M_{i_0}}(x) = F(x)$  and hence  $x \in F|_{M_{i_0}}^-(G, E)$ . Since  $F|_{M_{i_0}}$  is soft lower continuous, there exists an open set  $B = G \cap M_{i_0}$  in  $M_{i_0}$  such that  $x \in B$  and  $F^-(G, E) \cap M_{i_0} = F|_{M_{i_0}}^-(G, E) \supset B = G \cap M_{i_0}$ , where  $G$  is an open set in  $X$ . We have  $x \in B = G \cap M_{i_0} \subseteq F|_{M_{i_0}}^-(G, E) = F^-(G, E) \cap M_{i_0} \subseteq F^-(G, E)$ . Hence  $F$  is soft lower continuous.

(b): It is similar to the proof (a).

**Definition 21.** Let  $F : (X, \tau) \rightarrow (Y, \sigma)$  be a multifunction and let  $G : (Y, \sigma) \rightarrow (Z, \vartheta, E)$  be a soft multifunction. Then the soft multifunction  $G \circ F : (X, \tau) \rightarrow (Z, \vartheta, E)$  is defined by  $(G \circ F)(x) = G(F(x))$ .

**Proposition 10.** Let  $F : (X, \tau) \rightarrow (Y, \sigma)$  be a multifunction and let  $G : (Y, \sigma) \rightarrow (Z, \vartheta, E)$  be a soft multifunction. Then we have

- (a)  $(G \circ F)^+(H, E) = F^+(G^+(H, E))$
- (b)  $(G \circ F)^-(H, E) = F^-(G^-(H, E))$

*Proof.* Clear from the Definitions 17 and 20.

**Definition 22.** [1] Let  $(X, \tau)$  and  $(Y, \sigma)$  be two ordinary topological spaces. Then a multifunction  $F : (X, \tau) \rightarrow (Y, \sigma)$  is said to be

(a) upper semi continuous if for each open set  $V$  in  $Y$ ,  $F^+(V)$  is an open set in  $X$ .

(b) lower semi continuous if for each open set  $V$  in  $Y$ ,  $F^-(V)$  is an open set in  $X$ .

**Theorem 9.** Let  $F : (X, \tau) \rightarrow (Y, \sigma)$  be a multifunction and let  $G : (Y, \sigma) \rightarrow (Z, \vartheta, E)$  be a soft multifunction. If  $F$  is upper (lower) semi continuous and  $G$  is soft upper (lower) continuous then  $G \circ F$  is soft upper (lower) continuous.

*Proof.* Let  $(H, E)$  be any soft open set over  $Z$ . Since  $G$  is soft upper continuous then  $G^+(H, E)$  is open in  $Y$ . Since  $F$  is upper semi continuous then  $F^+(G^+(H, E)) = (G \circ F)^+(H, E)$  is open in  $X$ . Therefore  $G \circ F$  is soft upper continuous.

**Definition 23.** [16] A family  $\Psi$  of soft sets is a cover of a soft set  $(G, E)$  if  $(G, E) \subseteq \bigcup \{(G_i, E) : (G_i, E) \in \Psi, i \in I\}$ . It is soft open cover if each member of  $\Psi$  is a soft open set. A subcover of  $\Psi$  is a subfamily of  $\Psi$  which is also cover.

**Definition 24.** [16] A soft topological space  $(Y, \sigma, E)$  is compact if each soft open cover of  $\tilde{Y}$  has a finite subcover.

**Theorem 10.** The image of a soft compact set under soft upper continuous multifunction is soft compact.

*Proof.* Let  $F : (X, \tau) \rightarrow (Y, \sigma, E)$  be an onto soft multifunction and let  $\Psi = \{(G_i, E) : i \in I\}$  be a cover of  $\tilde{Y}$  by soft open sets. Then since  $F$  is soft upper continuous, the family of all open sets of the form  $F^+(G_i, E)$ , for  $(G_i, E) \in \Psi$  is an open cover of  $X$  which has a finite subcover. However since  $F$  is surjective, then it is easily seen that  $F(F^+(G_i, E)) = (G_i, E)$  for any soft set  $(G_i, E)$  over  $Y$ . There is the family of image members of subcover is a finite subfamily of  $\Psi$  which covers  $\tilde{Y}$ . Consequently  $(Y, \sigma, E)$  is soft compact.

## 4 Soft Multifunctions in Information Systems

In this section, we introduce that there exist some connections between soft multifunction and information systems. It is well known that soft sets are a class of special information systems, and both researches of soft sets and information systems are the same formal structures.

Let us first introduce the definition of set-valued (multifunction) information systems given by Pei and Miao in [18] as follows:

**Definition 25.** [18] The quadruple  $(U, A, F, V)$  is called an information system, where  $U = \{x_1, x_2, \dots, x_n\}$  is a containing all interested objects,  $A = \{a_1, a_2, \dots, a_m\}$  is a set of attributes,  $V = \bigcup_{i=1}^m V_i$  where  $V_j$  is the value set of the attribute  $a_j$ , and  $F = \{f_1, f_2, \dots, f_m\}$  where  $f_j : U \rightarrow V_j$ .

It is assumed that if  $f_j : U \rightarrow P(V_j)$  is a mapping from  $U$  to the power set of  $V_j$  for all  $j \leq m$ , then the corresponding information systems are called set-valued (multifunction) information systems.

**Definition 26.** In Definition 25, if the functions  $f_j : U \rightarrow P(V_j)$  are taken as a soft multifunction, then the corresponding information systems are called soft set-valued information systems.

To illustrate this definition, let us consider following convenient example:

**Example 3.** Let us consider a soft-valued information system  $I = (U, G, A, V)$  is defined by

$$\begin{aligned} U &= \{x_1, x_2\} \\ A &= \{a_1 = \text{house}, a_2 = \text{car}, a_3 = \text{city}\} \\ V &= V_{\text{house}} \cup V_{\text{car}} \cup V_{\text{city}} \end{aligned}$$

where  $V_{\text{house}} = \{h_1, h_2, h_3\}$ ,  $V_{\text{car}} = \{b_1, b_2, b_3, b_4\}$  and  $V_{\text{city}} = \{c_1, c_2\}$ .

$$G = \{G_1, G_2, G_3\}$$

where, if  $E = \{e_1 = \text{cheap}, e_2 = \text{beautiful}, e_3 = \text{comfortable}\}$ , then

$$\begin{aligned} G_1 : U &\rightarrow V_{\text{city}}, G_1(x_1) = (F_1, E) \text{ and } G_1(x_2) = (F_2, E) \\ G_2 : U &\rightarrow V_{\text{car}}, G_2(x_1) = (H_1, E) \text{ and } G_2(x_2) = (H_2, E) \\ G_3 : U &\rightarrow V_{\text{city}}, G_3(x_1) = (K_1, E) \text{ and } G_3(x_2) = (K_2, E) \end{aligned}$$

$$\begin{aligned} F_1 &: E \rightarrow P(V_{\text{house}}), \\ F_1(e_1) &= \{h_1, h_3\}, F_1(e_2) = \{h_2\}, F_1(e_3) = \{h_2, h_3\} \\ F_2 &: E \rightarrow P(V_{\text{house}}), \\ F_2(e_1) &= \{h_1, h_2\}, F_2(e_2) = \{h_3\}, F_2(e_3) = \{h_2, h_3\} \end{aligned}$$

$$\begin{aligned} H_1 &: E \rightarrow P(V_{\text{car}}), \\ H_1(e_1) &= \{b_1, b_3, b_4\}, H_1(e_2) = \emptyset, H_1(e_3) = \{b_2, b_4\} \\ H_2 &: E \rightarrow P(V_{\text{car}}), \\ H_2(e_1) &= \emptyset, H_2(e_2) = \{b_3\}, H_2(e_3) = \{b_1, b_4\} \end{aligned}$$

$$\begin{array}{lcl} K_1 & : & E \rightarrow P(V_{city}), \\ K_1(e_1) = \{c_1, c_2\}, K_1(e_2) = \{c_2\}, K_1(e_3) = \{c_1\} \\ K_2 & : & E \rightarrow P(V_{city}), \\ K_2(e_1) = \{c_1\}, K_2(e_2) = \{c_2\}, K_2(e_3) = \emptyset. \end{array}$$

The function  $\rho : U \times A \rightarrow S(V)$  where  $S(V)$  is all soft sets over  $V$  is defined by the following table:

$U$	$e_1$	$e_2$	$e_3$
$x_1$	$(F_1, E)$	$(H_1, E)$	$(K_1, E)$
$x_2$	$(F_2, E)$	$(H_2, E)$	$(K_2, E)$

For every  $x \in U$  we define the function  $\rho_x : A \rightarrow S(V)$  such that  $\rho_x(e) = \rho(x, e)$ . We shall call this function soft-valued information about  $x$  in  $I$ .

For instance, in this example, let us consider  $x_2$  to form the soft-valued information as follows:

$$\rho_{x_2} = \begin{array}{ccc} \text{house} & \text{car} & \text{city} \\ (F_2, E) & (H_2, E) & (K_2, E) \end{array}$$

$$\text{House} : e_1 = \text{cheap}, e_2 = \text{beautiful}, e_3 = \text{comfortable} \\ \{h_1, h_2\} \quad \{h_3\} \quad \{h_2, h_3\}$$

$$\text{Car} : e_1 = \text{cheap}, e_2 = \text{beautiful}, e_3 = \text{comfortable} \\ \emptyset \quad \{b_3\} \quad \{b_1, b_4\}$$

$$\text{City} : e_1 = \text{cheap}, e_2 = \text{beautiful}, e_3 = \text{comfortable} \\ \{c_1\} \quad \{c_2\} \quad \emptyset$$

## 5 Conclusion

In this paper our purpose is two fold. First, we define upper and lower inverse of a soft multifunction and study their various properties. Next, we use these ideas to introduce upper soft continuous multifunctions and lower soft continuous multifunctions. Moreover, we obtain some characterizations and several properties concerning such multifunctions. We expect that results in this paper will be basis for further applications of soft mappings in soft sets theory and corresponding information systems.

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