Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/080625

Some Remarks on the Fractional Sumudu Transform and Applications

Adem Kılıçman and Omer Altun*

Department of Mathematics and Institute for Mathematical Research, University Putra Malaysia, 43400 UPM, Serdang, Selangor, Malaysia

Received: 26 Oct. 2013, Revised: 24 Jan. 2014, Accepted: 25 Jan. 2014
Published online: 1 Nov. 2014

Abstract: In this work we study fractional order Sumudu transform. In the development of the definition we use fractional analysis based on the modified Riemann - Liouville derivative, then we name the fractional Sumudu transform. We also establish a relationship between fractional Laplace and Sumudu via duality with complex inversion formula for fractional Sumudu transform and apply new definition to solve fractional differential equations.

Keywords: Sumudu transform, fractional differential operator, Mittag-Leffler function

1 Introduction

In the literature there are numerous integral transforms and widely used in physics, astronomy as well as in engineering. In order to solve the differential equations, the integral transform were extensively used and thus there are several works on the theory and application of integral transform such as the Laplace, Fourier, Mellin and Hankel, to name but a few.

Recently, the Fractional integral transform as a generalization of the classical integral transform, were introduced many years ago in mathematics literature. The original purpose of fractional transform is to solve the some differential equations in engineering as well as in the quantum mechanics. For example, the optics problems can also be interpreted by fractional Fourier transform. In fact, most of the applications of fractional Fourier transform now are applications on optics. The fractional integral transforms have received considerable attention in the literature. Several applications of the fractional integral transform have been suggested, see [11,12,23]. First of all we have the following definition.

Definition 1. The transform

$$g(\alpha) = \int_a^b f(x)K(\alpha,x)\,dx$$

is called the Integral transform and $K(\alpha,x)$ is called the Kernel of the transform.

Now by changing the kernel we will have several types of the integral transforms such as:

– If $K(\alpha,x) = e^{-\alpha x}$ it is known as Laplace transform,

$$L(\alpha) = \int_0^\infty f(x)e^{-\alpha x}\,dx$$

– if we consider the kernel $K(\alpha,x) = xJ_\nu(\alpha x)$ then we obtain the Hankel transform

$$H_\nu(\alpha) = \int_0^\infty f(x)\,x J_\nu(\alpha x)\,dx$$

and

– if $K(x,\alpha) = \dfrac{1}{x-\alpha}$ then we obtain the Hilbert transform

$$H(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(x)}{x-\alpha}\,dx$$

provided that integrals exist.

In the sequence of these transform, in early 90's Watugala [30] introduced a new integral transform, named the Sumudu transform and further applied it to the solution of ordinary differential equation in control engineering problems. For further details and related properties about

* Corresponding author e-mail: akilic@upm.edu.my

Sumudu transform (see [2,3,4,5,8,11]) and many others.

The Sumudu transform is defined over the set of the functions.

$$A = \left\{ f(t) : \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{t/\tau_j}, \right.$$
$$\left. \text{if } t \in (-1)^j \times [0,\infty) \right\}$$

by the following formula

$$G(u) = S[f(t);u] =: \int_0^\infty f(ut)e^{-t}dt,\ u \in (-\tau_1, \tau_2).$$

The existence and the uniqueness was discussed in [19], for further details and properties of the Sumudu transform and its derivatives we refer to [2]. In [3], some fundamental properties of the Sumudu transform were established.

In [17], this new transform was applied to the one-dimensional neutron transport equation. In fact one can easily show that there is a strong relationship between double Sumudu and double Laplace transforms see, [19].

Further in [8], the Sumudu transform was extended to the distributions and some of their properties were also studied in [18]. Recently Kılıçman et al. applied this transform to solve the system of differential equations, see [21].

A very interesting fact about Sumudu transform is that the original function and its Sumudu transform have the same Taylor coefficients except a factor $n!$. Thus if

$$f(t) = \sum_{n=0}^\infty a_n t^n \text{ then } F(u) = \sum_{n=0}^\infty n!\, a_n t^n,$$

see [20].

Similarly, the Sumudu transform sends combinations, $C(m,n)$, into permutations, $P(m,n)$ and hence it will be useful in the discrete systems. Further

$$S(H(t) = \pounds(\delta(t)) = 1 \text{ and } \pounds(H(t)) = S(\delta(t)) = \frac{1}{u}.$$

Thus we further note that since many practical engineering problems involve mechanical or electrical systems acted upon by discontinuous or impulsive forcing terms then the Sumudu transform can be effectively used to solve ordinary differential equations as well as partial differential equations in and engineering problems.

For the convenience of the reader, firstly we shall give a brief background on the definition of the fractional derivative and basic notations (for more details see [14, 15,16]) and [1].

1.1 Fractional derivative

There are many different starting points for the discussion of classical fractional calculus [26]. One can begin with a generalization of repeated integration. If $f(t)$ is absolutely integrable on $[0,b)$, it can be found that [26,28].

$$\int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_3} dt_2 \int_0^{t2} f(t_1)dt_1 = \frac{\int_0^t (t-t_1)^{n-1} f(t_1)dt_1}{(n+1)!}$$
$$= \frac{1}{(n+1)!} t^{n-1} * f(t)$$

where $n = 1,2,\cdots,$ and $0 \le t \le b$. On writing $\Gamma(n) = (n-1)!$, an immediate generalization in the form of the operation I^α defined for $\alpha > 0$ is

$$(I^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-t_1)^{\alpha-1} f(t_1)dt_1$$
$$= \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t),\ 0 \le t < b, \quad (1)$$

where $\Gamma(\alpha)$ is the Gamma function and $t^{\alpha-1} * f(t) = \int_0^t f(t-t_1)^{\alpha-1}(t_1)dt_1$ is called the convolution product of $t^{\alpha-1}$ and $f(t)$. Eq.(1) is called the Riemann-Liouville fractional integral of order α for the function $f(t)$.

In this study we define the fractional transform as follows:

Definition 2. Let $f : \Re \to \Re, t \to f(t)$ denote a continuous (but not necessarily differentiable) function, and let $h > 0$ denote a constant discretization span. Define the forward operator $FW(h)$ by the equality

$$FW(h) f(t) := f(t+h). \quad (2)$$

Then the fractional difference of order $\alpha, 0 < \alpha < 1$ of $f(t)$ is defined by the expression

$$\Delta^\alpha f(t) := (FW-1)^\alpha$$
$$= \sum_{k=0}^\infty (-1)^k \binom{\alpha}{k} f[t+(\alpha-k)h] \quad (3)$$

and its fractional derivative of order α is defined by the limit.

$$f^{(\alpha)}(t) = \lim_{h \downarrow 0} \frac{\Delta^\alpha f(t)}{h^\alpha}. \quad (4)$$

see the details in [15].

1.2 Modified fractional Riemann-Liouville derivative

G. Jumarie proposed an alternative way to the Riemann-Liouville definition of the fractional derivative, see [15].

Definition 3. Let $f:\Re\to\Re$ be a continuous but not necessarily differentiable function further

(i)Assume that f(t) is a constant K. Then its fractional derivative of order α is

$$D_t^\alpha K = K\Gamma^{-1}(1-\alpha)t^{-\alpha}, \alpha\le 0, \\ = 0,\ \alpha>0 \tag{5}$$

(ii)When f(t) is not a constant, then we will set

$$f(t) = f(0) + (f(t) - f(0)),$$

and its fractional derivative will be defined by the expression

$$f^{(\alpha)}(t) = D_t^\alpha f(0) + D_t^\alpha (f(t) - f(0)),$$

in which, for negative α , one has

$$D_t^\alpha (f(t) - f(0)) := \frac{1}{\Gamma(-\alpha)}\int_o^t (t-\xi)^{-\alpha-1} f(\xi)d\xi, \tag{6}$$

whilst for positive α ,we will set

$$D_t^\alpha (f(t) - f(0)) = D_t^\alpha f(t) = D_t(f^{\alpha-1}(t)). \tag{7}$$

when $n\le\alpha<n+1$, we will set

$$f^{(\alpha)}(t) := (f^{(\alpha-n)}(t))^{(n)}, n\le\alpha<n+1, n\ge 1. \tag{8}$$

We shall refer to this fractional derivative as the modified Riemann-Liouville derivative, and it is in order to point out that this definition is strictly equivalent to the Definition 1, via eq. (3).

1.3 Integration with respect to $(dt)^\alpha$

The integral with respect to $(dx)^\alpha$ is defined as the solution of the fractional differential equation

$$dy = f(x)(dx)^\alpha, x\ge 0, y(0)=0, \tag{9}$$

which is provided by the following results:

Lemma 1. Let $f(x)$ denote a continuous function; then the solution $y(x)$ with $y(0)=0$, of (9) is defined by the equality

$$y = \int_o^x f(\xi)(d\xi)^\alpha \\ = \alpha\int_o^x (x-\xi)^{\alpha-1} f(\xi)d\xi, 0<\alpha<1. \tag{10}$$

Lemma 2. *If $m-1<\alpha\le m$, $m\in N$, $f\in C_\mu^m, \mu\ge -1$, then the following two properties hold:*

$$1)\ D^\alpha K^\alpha f(t) = f(t),$$

$$2)(D^\alpha K^\alpha)f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{t^k}{k!}. \tag{11}$$

In fact, Kılıçman and Zhour introduced the Kronecker convolution product and expanded to the Riemann-Liouville fractional integrals of matrices by using the Block Pulse operational matrix as follows:

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-t_1)^{\alpha-1}\phi_m(t_1)dt_1 \simeq F_\alpha \phi_m(t)$$

where

$$F_\alpha = \left(\frac{b}{m}\right)^\alpha \frac{1}{\Gamma(\alpha+2)}\begin{bmatrix} 1 & \xi_2 & \xi_3 & \dots & \xi_m \\ 0 & 1 & \xi_2 & \dots & \xi_{m-1} \\ 0 & 0 & 1 & \dots & \xi_{m-2} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

see [22]. Thus fractional integrals can be represented by using the operational matrices.

2 Sumudu transform of fractional order

The fractional integral transforms have many applications in the engineering. In particular, Fourier transform is one of the most widely used tools in signal processing and optics. Since, the Fractional Fourier transform (FRFT) is a generalization of the conventional Fourier transform and has received much attention in recent years. Several properties of fractional Fourier transform (FRFT) have been studied and many are being investigated at present, see [23]. Thus in the next we recall the fractional sumudu transform. For full account we refer to [11,12].

Definition 4. Let $f(t)$ denote a function which vanishes for negative values of t. Its Sumudu's transform of order α (or its fractional Sumudu's transform) is defined by the following expression, when it is finite:

$$S_\alpha\{f(t)\} =: G_\alpha(u) := \int_0^\infty E_\alpha(-t^\alpha) f(ut)(dt)^\alpha, \\ := \lim_{M\uparrow\infty}\int_0^M E_\alpha(-t^\alpha) f(ut)(dt)^\alpha, \tag{12}$$

where $u\in\mathscr{C}$, and $E_\alpha(x)$ is the Mittag-Leffler function $\sum_{k=0}^{\infty}\frac{x^k}{\alpha k!}$.

Recently Tchuenche and Mbare introduced the double Sumudu transform [29]. Analogously, we define the fractional double Sumudu transform in following way:

Definition 5. Let $f(x,t)$ denote a function which vanish for negative values of x and t. Its double Sumudu transform of fractional order (or its fractional double Sumudu transform) is defined as:

$$S_\alpha^2\{f(t,x)\} :=: G_\alpha^2(u,v) = \int_0^\infty\int_0^\infty E_\alpha[-(t+x)^\alpha] f(ut,vx)(dt)^\alpha(dx)^\alpha \tag{13}$$

where $u,v \in \mathscr{C}$, and $E_\alpha(x)$ is the Mittag-Leffler function.

2.1 The Laplace-Sumudu duality of fractional order

The following definition was given in [15].

Definition 6. Let $f(t)$ denote a function which vanishes for negative values of t. Its Laplace's transform of order α (or its α-th fractional Laplace's transform) is defined by the following expression:

$$\begin{aligned} \mathrm{L}_\alpha\{f(t)\} &=: F_\alpha(u) := \int_0^\infty E_\alpha(-(ut)^\alpha) f(t)(dt)^\alpha, \\ &= \lim_{M\uparrow\infty}\int_0^M E_\alpha(-(ut)^\alpha) f(t)(dt)^\alpha \end{aligned} \tag{14}$$

provided that integral exists.

Theorem 1. If the Laplace transform of fractional order of a function $f(t)$ is

$$\mathrm{L}_\alpha\{f(t)\} = F_\alpha(u)$$

and the Sumudu transform of this function is

$$S_\alpha\{f(t)\} = G_\alpha(u)$$

then

$$G_\alpha(u) = \frac{1}{u^\alpha} F_\alpha\left(\frac{1}{u}\right) \qquad m<\alpha<m+1, \tag{15}$$

where m non negative integer.

Similarly, on using the definition of fractional Sumudu transform, the following operational formulae can easily be obtained:

$$\begin{aligned} S_\alpha\{f(at)\} &= G_\alpha(au), \\ S_\alpha\{f(t-b)\} &= E_\alpha(-b^\alpha)G_\alpha(u), \\ S_\alpha\{E_\alpha(-c^\alpha t^\alpha)f(t)\} &= \frac{1}{(1+cu)^\alpha}G_\alpha\left(\frac{u}{1+cu}\right), \\ S_\alpha\left\{\int_0^t f(t)(dt)^\alpha\right\} &= u^\alpha\Gamma(1+\alpha)G_\alpha(u) \\ S_\alpha\{f^\alpha(t)\} &= \frac{G_\alpha(u)-\Gamma(1+\alpha)f(0)}{u^\alpha}, \end{aligned}$$

see [12]. Now we will obtain very similar properties for the fractional double Sumudu transform. Since proof of these properties are straight. Due to this reason, we will give only statements of these properties:

$$\begin{aligned} S_\alpha^2\{f(at)g(bx)\} &= G_\alpha(au)H_\alpha(bv) \\ S_\alpha^2\{f(at,bx)\} &= G_\alpha^2(au,bv) \\ S_\alpha^2\{f(t-a,x-b)\} &= E_\alpha(-(a+b)^\alpha)G_\alpha^2(au,bv) \\ S_\alpha^2\{\partial_t^\alpha f(t,x)\} &= \frac{G_\alpha^2(u,v)-\Gamma(1+\alpha)f(0,x)}{u^\alpha} \end{aligned}$$

where ∂_t^α is the fractional partial derivative of order $\alpha(0<\alpha<1)$, see [12].

The following two propositions were held in [27].

Proposition 1. If we define the convolution of order of the two functions $f(t)$ and $g(t)$ by the expression

$$(f*g)(x)_\alpha := \int_0^x f(x-v)g(v)(dv)^\alpha, \tag{16}$$

then

$$S_\alpha\{(f(t)*g(t))_\alpha\} = u^\alpha G_\alpha(u)H_\alpha(u)$$

where $G_\alpha(u) = S_\alpha\{(f(t)\}$ and $H_\alpha(u) = S_\alpha\{(g(t)\}$.

Proposition 2. Given the Sumudus transform that we recall here for convenience:

$$G_\alpha(u) = \int_0^\infty E_\alpha(-x^\alpha)f(ux)dx, \quad 0<\alpha<1 \tag{17}$$

one has the inversion formula

$$f(x) = \frac{1}{(M_\alpha)^\alpha}\int_{-i\infty}^{i\infty}\frac{E_\alpha((xu)^\alpha)}{u^\alpha}G\left(\left(\frac{1}{u}\right)^\alpha\right)(du)^\alpha \tag{18}$$

where M_α is the period of the Mitag Leffler function.

3 An application of fractional Sumudu transform

Example 1. Solution of the equation

$$y^{(\alpha)}+y = f(x), \qquad y(0)=0,\ 0<\alpha<1 \tag{19}$$

is given by

$$f(x) = \frac{1}{(M_\alpha)^\alpha}\int_{-i\infty}^{i\infty}\frac{E_\alpha((xu)^\alpha)}{u^\alpha}G_\alpha\left(\left(\frac{1}{u}\right)^\alpha\right)(du)^\alpha \tag{20}$$

Proof. Taking Sumudu transform of (19) both side we can easily get

$$y_\alpha(u) = \frac{u^\alpha}{1+u^\alpha}G_\alpha(u)$$

on using $y(0)=0$ then by applying the complex inversion formula of fractional Sumudu transform we get the following result

$$y(x)=\frac{1}{(M_\alpha)^\alpha}\int_{-i\infty}^{i\infty}\frac{E_\alpha((xu)^\alpha)}{u^\alpha(1+u^\alpha)}G_\alpha\left(\left(\frac{1}{u}\right)^\alpha\right)(du)^\alpha.$$

Now we apply the fractional double Sumudu transform to solve fractional partial differential equation.

*Example 2.*Consider the linear fractional partial differential equation (see [14])

$$\partial_t^\alpha z(x,t)=c\partial_x^\beta z(x,t),\quad x,t\in\Re^+ \tag{21}$$

with the boundary condition

$$z(0,t)=f(t),\quad z(x,0)=g(x)$$

where c is a positive coefficient, and $0<\alpha,\beta<1$.

Proof. Taking fractional double Sumudu transform of (21) both side we can easily get

$$\left(\frac{1}{u^\alpha}-\frac{1}{v^\beta}\right)G_\alpha^2(u,v)=\frac{\Gamma(1+\alpha)}{u^\alpha}f(t)-\frac{\Gamma(1+\beta)}{v^\alpha}g(x)$$

which gives

$$G_\alpha^2(u,v)=\Gamma(1+\alpha)\left(\frac{v^\beta}{v^\beta-u^\alpha}\right)f(t)-\Gamma(1+\beta)\left(\frac{u^\alpha}{v^\beta-u^\alpha}\right)g(x).$$

In the next example we can combine the homotopy and Sumudu transform, see [6].

*Example 3.*Consider the following one dimensional linear inhomogeneous fractional wave equation:

$$D^\alpha y(t)+y(t)=\frac{x^{(1-\alpha)}}{\Gamma(2-\alpha)}\sin t+x\cos t \qquad 0<\alpha\le 1,\quad x>0, \tag{22}$$

subject to the initial condition

$$y(t,0)=0. \tag{23}$$

The exact for the special case $\alpha=1$ is given by :

$$u(t,x)=x\sin t. \tag{24}$$

Then by using the homotopy Sumudu transform we obtain the following set of partial differential equations

$$\begin{aligned}\frac{\partial y_0}{\partial x}&=\frac{x^{(1-\alpha)}}{\Gamma(2-\alpha)}\sin t+x\cos t,\qquad y_0(t,0)=0\\ \frac{\partial y_1}{\partial x}&=\frac{\partial y_0}{\partial x}-\frac{\partial y_0}{\partial t}-\frac{\partial^\alpha y_0}{\partial x^\alpha},\\ &y_1(t,0)=0\\ \frac{\partial y_2}{\partial x}&=\frac{\partial y_1}{\partial x}-\frac{\partial y_0}{\partial t}-\frac{\partial^\alpha y_1}{\partial x^\alpha},\\ &y_2(t,0)=0,\\ &\vdots\end{aligned} \tag{25}$$

consequently, solving the above equations for y_0,y_1,y_2 the first few components of equation (22) are derived as follows:

$$\begin{aligned}y_0(t,x)&=x\sin t+\frac{x^{(\alpha+1)}}{\Gamma(\alpha+2)}\cos t\\ y_1(t,x)&=-\frac{x^{(\alpha+1)}}{\Gamma(\alpha+2)}\cos t+\frac{x^{(2\alpha+1)}}{\Gamma(2\alpha+2)}\sin t\\ y_2(t,x)&=-\frac{x^{(2\alpha+1)}}{\Gamma(2\alpha+2)}\sin t-\frac{x^{(3\alpha+1)}}{\Gamma(3\alpha+2)}\cos t\\ &\vdots\end{aligned}$$

Hence the series is given by

$$\begin{aligned}y(t,x)&=y_0(t,x)+y_1(t,x)+y_2(t,x)+\cdots\\ &=x\sin t+\frac{x^{(\alpha+1)}}{\Gamma(\alpha+2)}\cos t-\frac{x^{(\alpha+1)}}{\Gamma(\alpha+2)}\cos t+\\ &\quad\frac{x^{(2\alpha+1)}}{\Gamma(2\alpha+2)}\sin t-\frac{x^{(2\alpha+1)}}{\Gamma(2\alpha+2)}\sin t-\\ &\quad\frac{x^{(3\alpha+1)}}{\Gamma(3\alpha+2)}\cos t+\cdots\end{aligned} \tag{26}$$

The idea can also be extended to the partial differential equations as in the following example, see [7].

*Example 4.*Consider the following fractional Black-Scholes option pricing equation as follows

$$\frac{\partial^\alpha v}{\partial t^\alpha}+0.08(2+\sin x)^2x^2\frac{\partial^2 v}{\partial x^2}+0.06\frac{\partial v}{\partial x}-0.06v,\qquad 0<\alpha\le 1 \tag{27}$$

subject to initial condition

$$v(x,0)=\max(x-25e^{-0.06},0). \tag{28}$$

So that the solution $v(x,t)$ of the problem given by

$$\begin{aligned}v(x,t)&=\lim_{p\to 1}\sum_{i=0}^{\infty}p^i u_i(x,t)\\ &=x(1-E_\alpha(-0.06t^\alpha))\\ &\quad+\max(x-25e^{-0.06},0)E_\alpha(-0.06t^\alpha)).\end{aligned}$$

This is the exact solution of the given option pricing equation (27). The solution of equation (27) at the special case $\alpha = 1$ is

$$v(x,t) = x(1 - e^{0.06t} - 1, 0) + \max(x - 25e^{-0.06}, 0)e^{-0.06t}.$$

The Sumudu transform of convolution is given by

$$S\left[\frac{d}{dx}(f*g)(x);\ v\right] = uS\left[\frac{d}{dx}f(x);u\right]S\left[g(x);\ u\right]$$
$$\text{or } uS\left[f(x);\ u\right]S\left[\frac{d}{dx}g(x);\ u\right]. \quad (29)$$

Proposition 3. Let f be a distribution that vanishes below, then if f is Sumudu transformable, then so is f', dom $S[f] \subset \operatorname{dom} S[f']$ and

$$S\left[f'\right] = \frac{1}{u}S\left[f\right](u)$$

for all $u \in \operatorname{dom} S[f]$.

Now if $p(x)$ is a polynomial function which can be expressed as an infinite series in x as follows:

$$p(x) = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n$$

then the Sumudu Transformation $P(u)$ is defined by

$$P(u) = S(p(x)) = c_0 + c_1u + 2!c_2x^2 + \ldots + n!c_nu^n.$$

Thus we can summarize this as if $P(D)$ is any polynomial in D then Sumudu transform of $P(D)f$ is given by

$$S\left[P(D)f\right] = P(u)S(f)(u).$$

Theorem 2. Let f and g be distribution in W'. Then $f*g$ is a distribution in W' and the Sumudu transform of the convolution is given by

$$S\left[f*g\right] = uF(u)G(u).$$

The following theorem was proved in [21] and discusses the Sumudu transform of convolutions for matrices:

Theorem 3. Let $A(t) = [f_{ij}(t)] \in M_n^I$ and $B(t) = [g_{ij}(t)] \in M_n^I$ be Sumudu Transformable. Then

$$S\left[A(t) \circledast B(t)\right](u) = u\, S\left[A(t)\right]S\left[B(t)\right] \quad (30)$$

where M_n^I the set of $n \times n$ matrices for whose entries are integrable, for more details see [8,21].

The inverse $P^{-1}(u)$ of $P(u)$ will exist provided that u is not a root of the equation $\det\left[P(u)\right] = 0$; hence we let $\widetilde{P}$ denote the adjugate matrix of P by elementary matrix theory we have

$$P(u)^{-1} = \frac{1}{\det\left[P(u)\right]}\widetilde{P}(u). \quad (31)$$

Mathematical models of many physical biological and economic processes involve system of linear constant coefficient ordinary differential

$$\frac{df}{dx} = Af, \qquad f(0) = I \quad (32)$$

Eq(32) was studied by Laplace transform in [24] where f and A are square matrices of the nth order and the elements of A are known constants and also in control theory A is known as the state of companion matrix. The initial condition satisfied by the matrix $f(x)$ is $f(0) = I$ where I is the nth order unit matrix. It is well-known that the equation Eq(32) as the solution with the given initial condition,

$$f(x) = \sum_{k=0}^{\infty}\left(\frac{(Ax)^k}{k!}\right) = e^{Ax} \quad (33)$$

where e^{Ax} is the matrix exponential function. To obtain the solution of Eq(32) by Sumudu transform, we use the following definition

$$Sf(x) = \int_0^{\infty} e^{-\frac{x}{u}} f(x)dx = F(u), \qquad \operatorname{Re} u > 0 \quad (34)$$

and Sumudu transform of derivatives

$$S\left[\frac{df}{dx}\right] = \frac{1}{u}F(u) - \frac{1}{u}f(0) = \frac{1}{u}F(u) - \frac{1}{u}I.$$

The Sumudu transform of Eq(32) is, therefore

$$\left[I - uA\right]F(u) = I$$

hence

$$F(u) = \frac{I}{\left[I - uA\right]}.$$

The matrix $F(u) = [I - uA]$ is the characteristic matrix of A. The matrix $Q(u) = (I - uA)^{-1}$ is called the resolvent of A. If λ is the eigenvalue of A with maximum modulus, then we have the geometric progression expansion,

$$Q(u) = (I - uA)^{-1} = \sum_{k=0}^{\infty}\left((Au)^k\right) = F(u) \quad (35)$$

provided that, $|u| > |\lambda|$.

Next, if we want to solve the fundamental equation

$$y^{(\alpha)} + y = \delta(x), \quad 0 < \alpha < 1 \quad (36)$$

then we need to extend the single Sumudu transform to the delta function as

$$S[\delta(t-a)] = \frac{1}{u}\int_0^{\infty} e^{-t/u}\delta(t-a)dt = \frac{1}{u}e^{-a/u} \quad (37)$$

and the single Sumudu transform of the derivatives of the delta function is given by

$$S[\delta^{(n)}(t-a)] = \frac{1}{u}\int_0^\infty e^{-t/u}\delta^{(n)}(t-a)dt = -\left[\frac{d^n}{dt^n}\left(\frac{1}{u}e^{-t/u}\right)\right]_t = a = \frac{1}{u^{n+1}}e^{-a/u}$$

and

$$S_2[\delta(t-a)\delta(x-b)] = \frac{1}{uv}\int_0^\infty e^{-x/u}\int_0^\infty e^{-t/v}\delta(t-a)\delta(x-b)dtdx = \frac{1}{uv}e^{-a/v-b/u}.$$

The Sumudu transform was used also for the nonlinear PDEs such as the following nonlinear time-fractional Harry Dym equation of the form:

$$D_t^\alpha U(x,t) = U^3(x,t)D_x^3 U(x,t), \;\; 0\le\alpha\le 1$$

with the initial condition

$$U(x,0) = \left(a - \frac{3\sqrt{b}}{2}x\right)^{\frac{2}{3}}$$

for course the exact solution is

$$U(x,t) = \left(a - \frac{3\sqrt{b}}{2}(x+bt)\right)^{\frac{2}{3}}.$$

Then by using the homotopy perturbation Sumudu transform then we can approximate the solution as

$$U(x,t) = \left(a - \frac{3\sqrt{b}}{2}(x)\right)^{\frac{2}{3}} - \frac{b^{\frac{3}{2}}t^\alpha}{\Gamma(1+\alpha)}\left(a - \frac{3\sqrt{b}}{2}(x)\right)^{-\frac{1}{3}} + \frac{b^3 t^{2\alpha}}{\Gamma(1+2\alpha)}\left(a - \frac{3\sqrt{b}}{2}(x)\right)^{-\frac{4}{3}} + \ldots$$

and the series solution converges very rapidly, see [17, 27].

We also note that in the classical sense Γ functions is not defined for the negative integers. However in [9] it was proved that

$$\Gamma(-r) = \frac{(-1)^r}{r!}\phi(r) - \frac{(-1)^r}{r!}\gamma \quad (38)$$

for $r = 1,2,\ldots,$ where

$$\phi(r) = \sum_{i=1}^{r}\frac{1}{i}.$$

thus we can extend the definition to the whole real line and,

$$\Gamma(0) = \Gamma'(1) = -\gamma,$$

where γ denotes Euler's constant. Thus this can lead us that we allow the fractional order as negative values and replace

$$S_\alpha\left\{\int_0^t f(t)(dt)^{-\alpha}\right\} = u^{-\alpha}\Gamma(1-\alpha)G_\alpha(u)$$

$$S_\alpha\{f^{-\alpha}(t)\} = \frac{G_\alpha(u) - \Gamma(1-\alpha)f(0)}{u^{-\alpha}}.$$

Acknowledgement

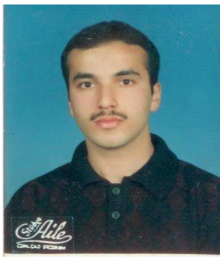
The author gratefully acknowledges that this research was partially supported by the University Putra Malaysia under the ERGS Grant Scheme having project number 5527068. The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References

[1] R. Almeida, A. B. Malinowska and D. F. M. Torres, A fractional calculus of variations for multiple integrals with application to vibrating string, J. Math. Phys., **51**, 033503, 12 pages (2010).
[2] M. A. Asiru, Sumudu transform and the solution of integral equations of convolution type, *Int. J. Math. Edu. Sci. Tech.*, **32**, 906-910 (2001).
[3] M. A. Asiru, Further properties of the Sumudu transform and its applications, *Int. J. Math. Edu. Sci. Tech.*, **33**, 441-449 (2002).
[4] M. A. Asiru, Classroom note: application of the Sumudu transform to discrete dynamic systems, *Int. J. Math. Edu. Sci. Tech.*, **34**, 944-949 (2003).
[5] F. B. M. Belgacem, A. A. Karaballi, and S. L. Kalla, Analytical investigations of the Sumudu transform and applications to integral production equations, *Math. Prob. Eng.*, 103-118 (2003).
[6] A. A. Elbeleze, A. Kılıçman, B. M. Taib. *Application of homotopy perturbation and variational iteration method for Fredholm integro-differential equation of fractional order*, Abstract & Applied Analysis, **2012**, 14 pages (2012).
[7] A. A. Elbeleze, A. Kılıçman, B. M. Taib. *Homotopy Perturbation Method for Fractional Black-Scholes European Option Pricing Equations Using Sumudu Transform*, Mathematical Problems in Engineering, **2013**, 7 pages (2013).
[8] H. Eltayeb, A. Kılıçman, and B. Fisher, A new integral transform and associated distributions, *Int. Trans. Spec. Func.*, **21**, 367-379 (2010).
[9] I Vardi, Determinants of Laplacians and multiple gamma functions, SIAM J. Math. Anal., **19**, 493-507 (1988).
[10] W Goh, J Wimp, Asymptotics for the moments of singular distributions, Journal of Approximation Theory, Elsevier, **74**, 301-334 (1993).

[11] V. G. Gupta and B. Sharma, Application of Sumudu Transform in Reaction-Diffusion Systems and Nonlinear Waves, *Appl. Math. Sci.*, **4**, 435-446 (2010).
[12] V. G. Gupta, S. Bhavna and A. Kılıçman, *A note on fractional Sumudu transform*, Journal of Applied Mathematics, **2010**, Article ID 154189, doi: 10.1155/2010/154189.
[13] M. G. M. Hussain and F. B. M. Belgacem, Transient solutions of Maxwell's equations based on Sumudu transform, *Progress In Electromagnetics Research(PIER)*, **74**, 273–289 (2007).
[14] G. Jumarie, Fractional partial differential equations and modified RiemannLiouville derivatives. Method for solution, *J. Appl. Math. Computing* , **24**, 31–48 (2007).
[15] G. Jumarie, Table of some basic fractional calculus formulae derived from a modified RiemannLiouville derivative for non-differentiable functions, *Appl. Math. Lett.*, **22**, 378-385 (2009).
[16] G. Jumarie, Laplaces transform of fractional order via the MittagLeffler function and modified RiemannLiouville derivative, *Appl. Math. Lett.*, **22**, 1659–1664 (2009).
[17] A. Kadem, Solving the one-dimensional neutron transport equation using Chebyshev polynomials and the Sumudu transform, *Analele Universitatii din Oradea. Fascicola Matematica*, **XII**, 153–171 (2005).
[18] A. Kılıçman and H. Eltayeb. On the applications of Laplace and Sumudu transforms, *Journal of the Franklin Institute*, doi:10.1016/j.jfranklin.2010.03.008.
[19] A. Kılıçman & H. Eltayeb, A note On Integral Transforms and Partial Differential Equations, *Applied Mathematical Sciences*, **4**, 109–118 (2010).
[20] A. Kılıçman, H. Eltayeb & K. A. M. Atan, A Note On The Comparison Between Laplace and Sumudu Transforms, *Bulletin of the Iranian Mathematical Society*, **37**, 131–141 (2011).
[21] A. Kılıçman, H. Eltayeb and P. Ravi Agarwal, On Sumudu Transform and System of Differential Equations, *Abstract and Applied Analysis*, Volume 2010, Article ID 598702, 11 pages doi:10.1155/2010/598702.
[22] A. Kılıçman and Z. A. A. Al Zhour, *Kronecker operational matrices for fractional calculus and some applications*. Appl. Math. Comput., **187** , 250–265 (2007).
[23] D. Mendlovic and H. M. Ozaktas, *Fractional Fourier transforms and their optical implementation: I*, Journal of the Optical Society of America A, **10**, 1875-1881 (1993).
[24] C. Moler & C. V. Loan, *Nineteen Dubious Ways to Compute The Exponential of A Matrix,* SIAM Review, **20**, 801–836 (1978).
[25] L. A. Pipes, "Matrix Methods for Engineering," Prentice-Hall, Inc., Englewood Cliffs, N. J., (1963).
[26] B. Ross., Fractional Calculus and its Applications, Springer-Verlag, Berlin, (1975).
[27] J. Singh, D. Kumar, and A. Kılıçman, Homotopy perturbation method for fractional gas dynamics equation using Sumudu transform, Abstract and Applied Analysis, **2013**, 8 pages (2013).
[28] H. Sumita, The Matrix Laguerre Transform, *Appl. Math. and Computation*, **15**, 1–28 (1984).
[29] J. M. Tchuenche and N. S. Mbare, An application of the double Sumudu transform, *Applied Mathematical Sciences*, **1**, 31–39 (2007).
[30] G. K. Watugala, Sumudu transform: a new integral transform to solve differential equations and control engineering problems, *Int. J. Math. Edu. Sci. Tech.*, **24**, 35-43 (1993).

Adem Kılıçman is full Professor in the Department of Mathematics at University Putra Malaysia. He received his Bachelor and Master degrees from Hacettepe University in 1989 and 1991 respectively, Turkey. He obtained his PhD from University of Leicester in 1995, UK. He has been actively involved several academic activities in the Faculty of Science and Institute of Mathematical Research (INSPEM). Adem Kilicman is also member of some Associations; PERSAMA, SIAM, IAENG, AMS. His research areas include Differential Equations, Functional Analysis and Topology. He has published research articles in reputed international journals of mathematical and engineering sciences. He is referee and editor of some mathematical journals.



Omer Altun was born in Tokat, Turkey on 30th November 1984. In 2008, He was awarded Bachelor of Science Mathematics with Education in University of Ataturk, Erzurum Turkey. He obtained his Master in University Putra Malaysia, then currently he is studying towards his PhD and his research area differential transformation method, ordinary differential equation, boundary value problem and convolutions.