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Differential Geometry on $SU(N)$: Left and Right Invariant Vector Fields and One-Forms

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Abstract: In this paper we provide an analytical procedure for explicit calculation of the left and right invariant vector fields and one-forms on $SU(N)$ manifold. The calculations are based on the coset parametrization of $SU(N)$ group. The results enable us to calculate the invariant measure or Haar measure on the group. As an illustrative example, we calculate invariant vector fields and one-forms on $SU(2)$ group.

Keywords: Differential geometry on $SU(N)$, invariant vector fields, invariant one-forms, invariant measure, coset parametrization

1 Introduction

Because of the various applications of the group of unitary transformation in physics, there is a great deal of attention in investigation of the properties of the unitary group $SU(N)$. In view of such considerable interest a lot of work has been devoted to describe and parameterize $SU(N)$ manifold. A generalized Euler angle parametrization for $SU(N)$ and $U(N)$ groups is given by Tilma and Sudarshan [1,2]. Diţă has provided a parametrization of unitary matrices based on the factorization of $N\times N$ unitary matrices [3,4]. Using this parametrization he has provided an explicit parametrization for general N-dimensional Hermitian operators that may be considered either as Hamiltonian or density matrices [5]. The subgroups and the coset spaces of the $SU(3)$ group are also listed in [6] along with a discussion of the geometry of the group manifold which is relevant to the understanding of the geometric phase.

The differential geometry on unitary groups is also an important task in theoretical physics. To achieve this it is important to having a parametrization to construct differential geometry for any unitary group. Byrd [7] has calculated the left and right invariant vector fields and one-forms on $SU(3)$ group. His calculation is based on the Euler angle parametrization for the $SU(3)$ group. He has used the invariant one-forms on $SU(3)$ group, and has studied the geometric phase over the space of 3-level quantum systems. On the other hand, based on the Euler angle parametrization of $SU(3)$ group and the result of [7], Panahi et al [8] have obtained a two-dimensional Hamiltonian on S^2 via Fourier transformation over the three coordinates of the $SU(3)$ Casimir operator defined on $SU(3)/SU(2)$. Also by using the parametrization of $SU(3)/SU(2)$ given in [6], they have constructed right invariant vector fields and the Casimir operator on the symmetric space $SU(3)/SU(2)$ and have obtained the two-dimensional Hamiltonian of a charged particle on S^2 in the presence of an electric field [9].

In this paper we present an analytical procedure for calculation of the left and right invariant vector fields and one-forms on $SU(N)$ group. This calculation is based on the coset parametrization of $SU(N)$. We also use the possibility of factorizing each coset component in terms of a diagonal phase matrix and an orthogonal matrix [4, 10]. By using this coset parametrization, we have recently given an explicit expression for the Bures metric over the space of three-level [11] and N-level [12] quantum systems. This parametrization is convenient for many calculations, in the sense that by using the coset parametrization the calculation can be done in a unique manner for every N. Furthermore the coset spaces appear in physics in several contexts, and provide an elegant way of deducing a high-dimensional theory to a

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lower-dimensional one. This paper, therefore, can be regarded as a further development in the explicit calculation of the differential geometric structure of $SU(N)$. The results of the paper can have applications in studying geometric phase over the space of N-level quantum systems, and also in the context of constructing Hamiltonian on some coset spaces of $SU(N)$ group.

The paper is organized as follows: In section 2, we review briefly the Lie algebra of $SU(N)$ and introduce the generalized Gell-Mann matrices as generators of the algebra. The coset parametrization for $SU(N)$ is introduced in section 3. Based on the coset parametrization, we provide in section 4, a method to construct the left and right invariant vector fields and one-forms on $SU(N)$ manifold. In this section we also obtain the invariant measure or Haar measure on the group. As an illustrative example we obtain the differential geometry on $SU(2)$ in section 5. The paper is concluded in section 6 with a brief conclusion.

2 Preliminary: Lie algebra of $SU(N)$

The group $SU(N)$ of $N\times N$ unitary matrices with unit determinant is generated by the N^2-1 Hermitian, traceless $N\times N$ matrices, that make the basis for the corresponding Lie algebra $su(N)$. By choosing the normalization condition $\mathrm{Tr}(T_iT_j)=\frac{1}{2}\delta_{ij}$ for generators, we can write the $(N-1)$ diagonal generators $\{L_{\alpha_1}^{(1)}\}_{\alpha_1=1}^{N-1}$, i.e. Cartan subalgebra, as ([13], page 187)

$$(L_{\alpha_1}^{(1)})_{k,l}=\frac{1}{\sqrt{2\alpha_1(\alpha_1+1)}}\times\left(\sum_{j=1}^{\alpha_1}\delta_{k,j}\delta_{l,j}-\alpha_1\delta_{k,\alpha_1+1}\delta_{l,\alpha_1+1}\right), \qquad (1)$$

and the remaining $N(N-1)$ non-diagonal generators $\{L_{\alpha_m}^{(m)}\}_{\alpha_m=1}^{2(m-1)}$ (for $m=2,\cdots,N$) as follows

$$(L_{\alpha_m}^{(m)})_{k,l}=\begin{cases}\frac{1}{2}\left(\delta_{\alpha_m,k}\delta_{m,l}+\delta_{\alpha_m,l}\delta_{m,k}\right), \\ \quad \text{for} \quad \alpha_m=1,\cdots,m-1 \\ \frac{-i}{2}\left(\delta_{\alpha_m-m+1,k}\delta_{m,l}-\delta_{\alpha_m-m+1,l}\delta_{m,k}\right) \\ \quad \text{for} \quad \alpha_m=m,\cdots,2(m-1).\end{cases} \qquad (2)$$

In the following sections, we set the range of indexes as $\alpha_1=1,\cdots,N-1$ and $\alpha_m=1,\cdots,2(m-1)$ for $m=2,\cdots,N$. The above $su(N)$ basis is the generalized Gell-Mann matrices, and therefore, for the case of $N=2$ we get the Pauli matrices

$$L_1^{(1)}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, L_1^{(2)}=\frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}, L_2^{(2)}=\frac{1}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}.$$

Also for $N=3$ we get the usual Gell-Mann matrices

$$L_1^{(1)}=\frac{1}{2}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}, \quad L_2^{(1)}=\frac{1}{2\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix},$$

$$L_1^{(2)}=\frac{1}{2}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}, \quad L_2^{(2)}=\frac{1}{2}\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix},$$

$$L_1^{(3)}=\frac{1}{2}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}, \quad L_2^{(3)}=\frac{1}{2}\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},$$

$$L_3^{(3)}=\frac{1}{2}\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix}, \quad L_4^{(3)}=\frac{1}{2}\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}.$$

3 Canonical coset parametrization of $SU(N)$ group

In this section we provide a parametrization for $SU(N)$ group that will be useful in calculating the differential geometry on the group manifold. The parametrization is based on the coset decomposition of unitary matrices.

3.1 Coset factorization of $SU(N)$ group

For every element $U\in SU(N)$, there is a unique decomposition of U into a product of N group elements as [14]

$$U=\Omega^{(N;N)}\Omega^{(N-1;N)}\cdots\Omega^{(2;N)}\Omega^{(1;N)}. \qquad (3)$$

In the above factorization we have

$$\Omega^{(1;N)}\in T^{N-1} \qquad (4)$$

where the $(N-1)$-dimensional torus T^{N-1} is the product of $N-1$ spheres $S^1=T^1$. A typical element for $\Omega^{(1;N)}$ can be represented as

$$\Omega^{(1;N)}=\mathrm{Exp}(i\eta_1L_1^{(1)})\cdots\mathrm{Exp}(i\eta_{N-1}L_{N-1}^{(1)}), \qquad (5)$$

where η_{α_1} for $\alpha_1=1,\cdots,N-1$ are real parameters and $L_{\alpha_1}^{(1)}$ are Cartan generators defined in equation (1). The explicit form for $\Omega^{(1;N)}$ can be expressed as

$$(\Omega^{(1;N)})_{k,l}=\delta_{k,l}\mathrm{Exp}\left(-i\sqrt{\frac{k-1}{2k}}\eta_{k-1}+i\sum_{j=k}^{N-1}\frac{\eta_j}{\sqrt{2j(j+1)}}\right), \qquad (6)$$

with $\eta_0=0$. Also in decomposition (3) we have the cosets

$$\Omega^{(m;N)}\in\frac{U(m)\otimes T^{N-m}}{U(m-1)\otimes T^{N-m+1}}, \quad m=2,\cdots,N. \qquad (7)$$

A typical coset representative $\Omega^{(m;N)}$ can be written as

$$\Omega^{(m;N)}=\left(\begin{array}{c|c}SU(m)/U(m-1)&O\\ \hline O^T&I_{N-m}\end{array}\right), \qquad (8)$$

where O represents the $m\times(N-m)$ zero matrix, O^T is its transpose and I_{N-m} denotes the unit matrix of order $(N-m)$. The $2(m-1)$-dimensional coset space $SU(m)/U(m-1)$ has the following $m\times m$ matrix representation ([14], page 351)

$$\begin{pmatrix} \cos\sqrt{B^{(m)}[B^{(m)}]^{\dagger}} & B^{(m)}\frac{\sin\sqrt{[B^{(m)}]^{\dagger}B^{(m)}}}{\sqrt{[B^{(m)}]^{\dagger}B^{(m)}}} \\ -\frac{\sin\sqrt{[B^{(m)}]^{\dagger}B^{(m)}}}{\sqrt{[B^{(m)}]^{\dagger}B^{(m)}}}[B^{(m)}]^{\dagger} & \cos\sqrt{[B^{(m)}]^{\dagger}B^{(m)}} \end{pmatrix} \tag{9}$$

where $B^{(m)}$ represents an $(m-1)\times 1$ complex matrix and $[B^{(m)}]^{\dagger}$ is its adjoint.

3.2 Factorization of coset $\Omega^{(m;N)}$

Now by parameterizing the complex vector $B^{(m)}$ as (for $m=2,\cdots,N$)

$$B^{(m)}=(\gamma_1^{(m)}e^{i\xi_1^{(m)}},\gamma_2^{(m)}e^{i\xi_2^{(m)}},\cdots,\gamma_{m-1}^{(m)}e^{i\xi_{m-1}^{(m)}})^T,$$

where $\gamma_i^{(m)}$ and $\xi_i^{(m)}$ are real numbers, the component $\Omega^{(m;N)}$ can be factorized as (for $m=2,\cdots,N$)

$$\Omega^{(m;N)}=X^{(m;N)}R^{(m;N)}X^{(m;N)\dagger}, \tag{10}$$

where $X^{(m;N)}$ is a diagonal $N\times N$ phase matrix with $X_{k,l}^{(m;N)}=\delta_{k,l}\exp(i\xi_k^{(m)})$ and $\xi_i^{(m)}=0$ for $i\geq m$, and $R^{(m;N)}$ is an $N\times N$ orthogonal matrix with the following nonzero elements

$$\begin{aligned}
R_{i,j}^{(m;N)}&=\delta_{i,j}+\hat{\gamma}_i^{(m)}\hat{\gamma}_j^{(m)}(\cos\gamma^{(m)}-1)\\
&\quad\text{for}\quad 1\leq i,j\leq m-1\\
R_{i,m}^{(m;N)}&=-R_{m,i}^{(m;N)}=\hat{\gamma}_i^{(m)}\sin\gamma^{(m)}\\
&\quad\text{for}\quad 1\leq i\leq m-1\\
R_{m,m}^{(m;N)}&=\cos\gamma^{(m)}\\
R_{i,i}^{(m;N)}&=1\quad\text{for}\quad m+1\leq i\leq N,
\end{aligned} \tag{11}$$

where we have defined $\hat{\gamma}_i^{(m)}=\gamma_i^{(m)}/\gamma^{(m)}$ and $\gamma^{(m)}=\sqrt{[B^{(m)}]^{\dagger}B^{(m)}}=\sqrt{\sum_{i=1}^{m-1}(\gamma_i^{(m)})^2}$. As we will see later the important ingredient of our approach in computing the vector fields and one-forms is the possibility of writing the factorization (10).

4 Differential geometry on $SU(N)$

In this section we will find differential operators corresponding to the generators of $SU(N)$ group. To this aim, we first define $\chi_{\alpha_m}^{(m)}$ $(\alpha_m=1,2,\cdots,2(m-1))$ as the $2(m-1)$ real parameters of the coset component $\Omega^{(m;N)}$ $(m=2,\cdots,N)$ such that

$$\chi_{\alpha_m}^{(m)}=\begin{cases}\gamma_{\alpha_m}^{(m)} & \text{for}\quad \alpha_m=1,\cdots,m-1,\\ \xi_{\alpha_m-m+1}^{(m)} & \text{for}\quad \alpha_m=m,\cdots,2(m-1).\end{cases} \tag{12}$$

4.1 Left invariant vector fields and one-forms

Now in order to calculate left invariant vector fields, we first take derivatives of the group element $U(\eta_{\alpha_1},\chi_{\alpha_m}^{(m)})$ with respect to each set of parameters η_{α_1} and $\chi_{\alpha_m}^{(m)}$, and write the result of differentiation as

$$\frac{\partial}{\partial\eta_{\alpha_1}}U=UA_{\alpha_1}^{(1)},\qquad \frac{\partial}{\partial\chi_{\alpha_m}^{(m)}}U=UA_{\alpha_m}^{(m)}, \tag{13}$$

where the matrices $A_{\alpha_1}^{(1)}$ and $A_{\alpha_m}^{(m)}$ are defined by

$$A_{\alpha_1}^{(1)}=\left[\Omega^{(1,N)\dagger}\frac{\partial\Omega^{(1,N)}}{\partial\eta_{\alpha_1}}\right]=iL_{\alpha_1}^{(1)} \tag{14}$$

$$A_{\alpha_m}^{(m)}=W^{(m,N)\dagger}\left[\frac{\partial\Omega^{(m,N)}}{\partial\chi_{\alpha_m}^{(m)}}\Omega^{(m,N)\dagger}\right]W^{(m,N)} \tag{15}$$

with

$$W^{(m;N)}=\Omega^{(m;N)}\Omega^{(m-1;N)}\cdots\Omega^{(2;N)}\Omega^{(1;N)} \tag{16}$$

and the anti Hermitian matrix $\left[\frac{\partial\Omega^{(m,N)}}{\partial\gamma_{\alpha_m}^{(m)}}\Omega^{(m,N)\dagger}\right]$ has the following nonzero matrix elements

$$\begin{aligned}
&\left[\frac{\partial\Omega^{(m,N)}}{\partial\gamma_{\alpha_m}^{(m)}}\Omega^{(m,N)\dagger}\right]_{r<m,s<m}\\
&\quad=e^{i(\xi_r^{(m)}-\xi_s^{(m)})}\left(\frac{\cos\gamma-1}{\gamma}\right)\left(\hat{\gamma}_r^{(m)}\delta_{s,\alpha_m}-\hat{\gamma}_s^{(m)}\delta_{r,\alpha_m}\right)\\
&\left[\frac{\partial\Omega^{(m,N)}}{\partial\gamma_{\alpha_m}^{(m)}}\Omega^{(m,N)\dagger}\right]_{r<m,m}\\
&\quad=e^{i\xi_r^{(m)}}\left(\hat{\gamma}_r^{(m)}\hat{\gamma}_{\alpha_m}^{(m)}\left(\delta_{r,\alpha_m}-\hat{\gamma}_r^{(m)}\hat{\gamma}_{\alpha_m}^{(m)}\right)\frac{\sin\gamma}{\gamma}\right)\\
&\left[\frac{\partial\Omega^{(m,N)}}{\partial\gamma_{\alpha_m}^{(m)}}\Omega^{(m,N)\dagger}\right]_{m,r<m}=-\left[\frac{\partial\Omega^{(m,N)}}{\partial\gamma_{\alpha_m}^{(m)}}\Omega^{(m,N)\dagger}\right]^*_{r<m,m}\\
&\left[\frac{\partial\Omega^{(m,N)}}{\partial\gamma_{\alpha_m}^{(m)}}\Omega^{(m,N)\dagger}\right]_{m,m}=0,
\end{aligned} \tag{17}$$

and also the matrix $\left[\frac{\partial\Omega^{(m;N)}}{\partial\xi_{\alpha_m}^{(m)}}\Omega^{(m;N)\dagger}\right]$ defined by

$$\begin{aligned}
\left[\frac{\partial\Omega^{(m;N)}}{\partial\xi_{\alpha_m}^{(m)}}\Omega^{(m;N)\dagger}\right]_{r,s}&=i\delta_{r,\alpha_m}\delta_{s,\alpha_m}\\
&-iR_{r,\alpha_m}^{(m;N)}R_{s,\alpha_m}^{(m;N)}e^{i(\xi_r^{(m)}-\xi_s^{(m)})}.
\end{aligned} \tag{18}$$

The matrices $A_{\alpha_1}^{(1)}$ and $A_{\alpha_m}^{(m)}$ can be expanded as a linear combination of the Lie algebra basis (1) and (2) as follows

$$A_{\alpha_1}^{(1)}=\sum_{\beta_1=1}^{N-1}a_{\alpha_1,\beta_1}^{(1,1)}L_{\beta_1}^{(1)}+\sum_{m'=2}^{N}\sum_{\beta_{m'}=1}^{2(m'-1)}a_{\alpha_1,\beta_{m'}}^{(1,m')}L_{\beta_{m'}}^{(m')}, \tag{19}$$

$$A_{\alpha_m}^{(m)}=\sum_{\beta_1=1}^{N-1}a_{\alpha_m,\beta_1}^{(m,1)}L_{\beta_1}^{(1)}+\sum_{m'=2}^{N}\sum_{\beta_{m'}=1}^{2(m'-1)}a_{\alpha_m,\beta_{m'}}^{(m,m')}L_{\beta_{m'}}^{(m')}. \tag{20}$$

The coefficients of the above expansion can be obtained from the orthogonality of the generators of the Lie algebra. We obtain the following results

$$a^{(1,1)}_{\alpha_1,\beta_1} = i\delta_{\alpha_1,\beta_1} \quad (21)$$

$$a^{(1,m)}_{\alpha_1,\beta_m} = 0 \quad (22)$$

$$a^{(m,1)}_{\alpha_m,\beta_1} = \frac{2}{\sqrt{2\beta_1(\beta_1+1)}} \times \left(\sum_{j=1}^{\beta_1}\left(A^{(m)}_{\alpha_m}\right)_{jj} - \beta_1\left(A^{(m)}_{\alpha_m}\right)_{\beta_1+1,\beta_1+1}\right) \quad (23)$$

$$a^{(m,m')}_{\alpha_m,\beta_{m'}} = \left(\left(A^{(m)}_{\alpha_m}\right)_{m',\beta_{m'}} + \left(A^{(m)}_{\alpha_m}\right)_{\beta_{m'},m'}\right) \quad (24)$$

for $\beta_{m'} = 1,\cdots,m-1$,

$$a^{(m,m')}_{\alpha_m,\beta_{m'}} = -i\left(\left(A^{(m)}_{\alpha_m}\right)_{m',\beta_{m'}} - \left(A^{(m)}_{\alpha_m}\right)_{\beta_{m'},m'}\right) \quad (25)$$

for $\beta_{m'} = m',\cdots,2(m-1)$

Therefore by choosing the following order for coordinates

$$\{\eta_1,\cdots,\eta_{N-1}\};\{\gamma^{(2)}_1,\xi^{(2)}_1\};\{\gamma^{(3)}_1,\gamma^{(3)}_2,\xi^{(3)}_1,\xi^{(3)}_2\};\cdots;\{\gamma^{(N)}_1,\cdots,\gamma^{(N)}_{N-1},\xi^{(N)}_1,\cdots,\xi^{(N)}_{N-1}\}. \quad (26)$$

we can define the matrix A by

$$A = \left(\begin{array}{c|c|c|c|c} a^{(1,1)}_{\alpha_1,\beta_1} & a^{(1,2)}_{\alpha_1,\beta_2} & a^{(1,3)}_{\alpha_1,\beta_3} & \cdots & a^{(1,N)}_{\alpha_1,\beta_N} \\ \hline a^{(2,1)}_{\alpha_2,\beta_1} & a^{(2,2)}_{\alpha_2,\beta_2} & a^{(2,3)}_{\alpha_2,\beta_3} & \cdots & a^{(2,N)}_{\alpha_2,\beta_N} \\ \hline a^{(3,1)}_{\alpha_3,\beta_1} & a^{(3,2)}_{\alpha_3,\beta_2} & a^{(3,3)}_{\alpha_3,\beta_3} & \cdots & a^{(3,N)}_{\alpha_3,\beta_N} \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline a^{(N,1)}_{\alpha_N,\beta_1} & a^{(N,2)}_{\alpha_N,\beta_2} & a^{(N,3)}_{\alpha_N,\beta_3} & \cdots & a^{(N,N)}_{\alpha_N,\beta_N} \end{array}\right). \quad (27)$$

With the help of this matrix the differential operators are related to the Lie algebra generators as follows

$$\begin{pmatrix} \partial/\partial\eta_{\alpha_1} \\ \partial/\partial\chi^{(m)}_{\alpha_m} \end{pmatrix} = A \begin{pmatrix} L^{(1)}_{\beta_1} \\ L^{(m')}_{\beta_{m'}} \end{pmatrix}. \quad (28)$$

Now by tacking inverse of the matrix A, the left invariant vector fields on $SU(N)$ group can be obtained by the following equation

$$\begin{pmatrix} \Lambda^{(1)}_{\beta_1} \\ \Lambda^{(m')}_{\beta_{m'}} \end{pmatrix} = A^{-1} \begin{pmatrix} \partial/\partial\eta_{\alpha_1} \\ \partial/\partial\chi^{(m)}_{\alpha_m} \end{pmatrix}. \quad (29)$$

Now in order to calculate the left invariant one-forms, we first expand them in terms of the basis $\mathrm{d}\eta_{\beta_1}$ and $\mathrm{d}\chi^{(m)}_{\beta_m}$

$$\omega^{(1)}_{\alpha_1} = \sum_{\beta_1=1}^{N-1} c^{(1,1)}_{\alpha_1,\beta_1}\mathrm{d}\eta_{\beta_1} + \sum_{m'=2}^{N}\sum_{\beta_{m'}=1}^{2(m'-1)} c^{(1,m')}_{\alpha_1,\beta_{m'}}\mathrm{d}\chi^{(m')}_{\beta_{m'}}, \quad (30)$$

$$\omega^{(m)}_{\alpha_m} = \sum_{\beta_1=1}^{N-1} c^{(m,1)}_{\alpha_m,\beta_1}\mathrm{d}\eta_{\beta_1} + \sum_{m'=2}^{N}\sum_{\beta_{m'}=1}^{2(m'-1)} c^{(m,m')}_{\alpha_m,\beta_{m'}}\mathrm{d}\chi^{(m')}_{\beta_{m'}}. \quad (31)$$

By using the fact that left invariant one-forms are dual to the left invariant vector fields, i.e.

$$(\omega^{(1)}_{\alpha_1},\Lambda^{(1)}_{\alpha'_1}) = \delta_{\alpha_1,\alpha'_1},\ (\omega^{(1)}_{\alpha_1},\Lambda^{(m)}_{\alpha_m}) = 0,$$
$$(\omega^{(m)}_{\alpha_m},\Lambda^{(1)}_{\alpha'_1}) = 0, \qquad (\omega^{(m)}_{\alpha_m},\Lambda^{(m')}_{\alpha'_{m'}}) = \delta_{\alpha_m,\alpha'_{m'}}\delta_{m,m'},$$

we find that $C^T(A^{-1}) = \mathbb{I}$ and therefore, we have $C = A^T$. Consequently, the left invariant one-forms are obtained by tacking the transpose of the matrix A, i.e.

$$\begin{pmatrix} \omega^{(1)}_{\beta_1} \\ \omega^{(m')}_{\beta_{m'}} \end{pmatrix} = A^T \begin{pmatrix} \mathrm{d}\eta_{\alpha_1} \\ \mathrm{d}\chi^{(m)}_{\alpha_m} \end{pmatrix}. \quad (32)$$

4.2 Right invariant vector fields and one-forms

Now in order to calculate right invariant vector fields, we first take derivatives of the group element $U(\eta_{\alpha_1},\chi^{(m)}_{\alpha_m})$ with respect to each set of parameters η_{α_1} and $\chi^{(m)}_{\alpha_m}$, and write the result of differentiation as

$$\frac{\partial}{\partial\eta_{\alpha_1}}U = \widetilde{A}^{(1)}_{\alpha_1}U, \qquad \frac{\partial}{\partial\chi^{(m)}_{\alpha_m}}U = \tilde{A}^{(m)}_{\alpha_m}U, \quad (33)$$

where the matrices $\widetilde{A}^{(1)}_{\alpha_1}$ and $\widetilde{A}^{(m)}_{\alpha_m}$ are defined by

$$\widetilde{A}^{(1)}_{\alpha_1} = \widetilde{W}^{(1,N)}\left[\frac{\partial\Omega^{(1,N)}}{\partial\eta_{\alpha_1}}\Omega^{(1,N)^\dagger}\right]\widetilde{W}^{(1,N)^\dagger} \quad (34)$$

$$\widetilde{A}^{(m)}_{\alpha_m} = \widetilde{W}^{(m,N)}\left[\frac{\partial\Omega^{(m,N)}}{\partial\chi^{(m)}_{\alpha_m}}\Omega^{(m,N)^\dagger}\right]\widetilde{W}^{(m,N)^\dagger}, \quad (35)$$

with

$$\widetilde{W}^{(k;N)} = \Omega^{(N;N)}\Omega^{(N-1;N)}\cdots\Omega^{(k+1;N)} = UW^{(k;N)^\dagger}, \quad (36)$$

and

$$\left[\frac{\partial\Omega^{(1,N)}}{\partial\eta_{\alpha_1}}\Omega^{(1,N)^\dagger}\right] = \left[\Omega^{(1,N)^\dagger}\frac{\partial\Omega^{(1,N)}}{\partial\eta_{\alpha_1}}\right] = iL^{(1)}_{\alpha_1},$$

and the anti Hermitian matrix $\left[\frac{\partial\Omega^{(m,N)}}{\partial\gamma^{(m)}_{\alpha_m}}\Omega^{(m,N)^\dagger}\right]$ has defined in equation (17). The matrices $\widetilde{A}^{(1)}_{\alpha_1}$ and $\widetilde{A}^{(m)}_{\alpha_m}$ can be expanded as a linear combination of the Lie algebra

$$\widetilde{A}^{(1)}_{\alpha_1} = \sum_{\beta_1=1}^{N-1}\tilde{a}^{(1,1)}_{\alpha_1,\beta_1}L^{(1)}_{\beta_1} + \sum_{m'=2}^{N}\sum_{\beta_{m'}=1}^{2(m'-1)}\tilde{a}^{(1,m')}_{\alpha_1,\beta_{m'}}L^{(m')}_{\beta_{m'}}, \quad (37)$$

$$\widetilde{A}^{(m)}_{\alpha_m} = \sum_{\beta_1=1}^{N-1}\tilde{a}^{(m,1)}_{\alpha_m,\beta_1}L^{(1)}_{\beta_1} + \sum_{m'=2}^{N}\sum_{\beta_{m'}=1}^{2(m'-1)}\tilde{a}^{(m,m')}_{\alpha_m,\beta_{m'}}L^{(m')}_{\beta_{m'}}. \quad (38)$$

The coefficients of the above expansion can be obtained from the orthogonality of the generators of the Lie algebra.

We obtain the following results

$$\tilde{a}^{(1,1)}_{\alpha_1,\beta_1} = \frac{2}{\sqrt{2\beta_1(\beta_1+1)}} \times \left(\sum_{j=1}^{\beta_1} \left(\widetilde{A}^{(1)}_{\alpha_1}\right)_{jj} - \beta_1 \left(\widetilde{A}^{(1)}_{\alpha_1}\right)_{\beta_1+1,\beta_1+1} \right) \tag{39}$$

$$\tilde{a}^{(1,m)}_{\alpha_1,\beta_m} = \left(\left(\widetilde{A}^{(1)}_{\alpha_1}\right)_{m,\beta_m} + \left(\widetilde{A}^{(1)}_{\alpha_1}\right)_{\beta_m,m} \right) \quad \text{for } \beta_m = 1,\cdots,m-1 \tag{40}$$

$$\tilde{a}^{(1,m)}_{\alpha_1,\beta_m} = -i\left(\left(\widetilde{A}^{(1)}_{\alpha_1}\right)_{m,\beta_m} - \left(\widetilde{A}^{(1)}_{\alpha_1}\right)_{\beta_m,m} \right) \quad \text{for } \beta_m = m,\cdots,2(m-1) \tag{41}$$

$$\tilde{a}^{(m,1)}_{\alpha_m,\beta_1} = \frac{2}{\sqrt{2\beta_1(\beta_1+1)}} \times \left(\sum_{j=1}^{\beta_1} \left(\widetilde{A}^{(m)}_{\alpha_m}\right)_{jj} - \beta_1 \left(\widetilde{A}^{(m)}_{\alpha_m}\right)_{\beta_1+1,\beta_1+1} \right) \tag{42}$$

$$\tilde{a}^{(m,m')}_{\alpha_m,\beta_{m'}} = \left(\left(\widetilde{A}^{(m)}_{\alpha_m}\right)_{m',\beta_{m'}} + \left(\widetilde{A}^{(m)}_{\alpha_m}\right)_{\beta_{m'},m'} \right) \quad \text{for } \beta_{m'} = 1,\cdots,m-1 \tag{43}$$

$$\tilde{a}^{(m,m')}_{\alpha_m,\beta_{m'}} = -i\left(\left(\widetilde{A}^{(m)}_{\alpha_m}\right)_{m',\beta_{m'}} - \left(\widetilde{A}^{(m)}_{\alpha_m}\right)_{\beta_{m'},m'} \right) \quad \text{for } \beta_{m'} = m',\cdots,2(m-1) \tag{44}$$

By choosing the coordinates order given in (26), we get the following representation for matrix $\widetilde{A}$

$$\widetilde{A} = \left(\begin{array}{c|c|c|c|c} \tilde{a}^{(1,1)}_{\alpha_1,\beta_1} & \tilde{a}^{(1,2)}_{\alpha_1,\beta_2} & \tilde{a}^{(1,3)}_{\alpha_1,\beta_3} & \cdots & \tilde{a}^{(1,N)}_{\alpha_1,\beta_N} \\ \hline \tilde{a}^{(2,1)}_{\alpha_2,\beta_1} & \tilde{a}^{(2,2)}_{\alpha_2,\beta_2} & \tilde{a}^{(2,3)}_{\alpha_2,\beta_3} & \cdots & \tilde{a}^{(2,N)}_{\alpha_2,\beta_N} \\ \hline \tilde{a}^{(3,1)}_{\alpha_3,\beta_1} & \tilde{a}^{(3,2)}_{\alpha_3,\beta_2} & \tilde{a}^{(3,3)}_{\alpha_3,\beta_3} & \cdots & \tilde{a}^{(3,N)}_{\alpha_3,\beta_N} \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline \tilde{a}^{(N,1)}_{\alpha_N,\beta_1} & \tilde{a}^{(N,2)}_{\alpha_N,\beta_2} & \tilde{a}^{(N,3)}_{\alpha_N,\beta_3} & \cdots & \tilde{a}^{(N,N)}_{\alpha_N,\beta_N} \end{array} \right), \tag{45}$$

where can be used to write the differential operators in terms of the Lie algebra generators as

$$\begin{pmatrix} \partial/\partial\eta_{\alpha_1} \\ \\ \partial/\partial\chi^{(m)}_{\alpha_m} \end{pmatrix} = \widetilde{A} \begin{pmatrix} L^{(1)}_{\beta_1} \\ \\ L^{(m')}_{\beta_{m'}} \end{pmatrix}. \tag{46}$$

Consequently, the right invariant vector fields on the $SU(N)$ group can be obtained by the following

$$\begin{pmatrix} \widetilde{\Lambda}^{(1)}_{\beta_1} \\ \\ \widetilde{\Lambda}^{(m')}_{\beta_{m'}} \end{pmatrix} = \widetilde{A}^{-1} \begin{pmatrix} \partial/\partial\eta_{\alpha_1} \\ \\ \partial/\partial\chi^{(m)}_{\alpha_m} \end{pmatrix}. \tag{47}$$

Finally, similar to the case of left invariant one-forms by defining $\widetilde{A}^T$ as the transpose of $\widetilde{A}$, the right invariant one-forms can be written as

$$\begin{pmatrix} \widetilde{\omega}^{(1)}_{\beta_1} \\ \\ \widetilde{\omega}^{(m')}_{\beta_{m'}} \end{pmatrix} = \widetilde{A}^T \begin{pmatrix} \mathrm{d}\eta_{\alpha_1} \\ \\ \mathrm{d}\chi^{(m)}_{\alpha_m} \end{pmatrix}. \tag{48}$$

4.3 Invariant integration measure

Invariant measure, or Haar measure, on the $SU(N)$ manifold can be obtained by tacking the wedge product of the left or right invariant one-forms. The result is, of course, equivalent by tacking the determinant of the matrix A (or matrix $\widetilde{A}$ because of the fact that left and right invariant measures on $SU(N)$ are equal), where we get

$$\mathrm{d}\mu[SU(N)] = \mathrm{Det}(A) \prod_{\alpha_1=1}^{N-1} \mathrm{d}\eta_{\alpha_1} \prod_{m=2}^{N} \prod_{\alpha_m=1}^{m-1} \mathrm{d}\gamma_{\alpha_m} \mathrm{d}\xi_{\alpha_m}.$$

5 An example: The $SU(2)$ group

In this section we consider for more illustration the case of $N=2$ explicitly. In this particular case we have three parameters η_1, $\gamma^{(2)}_1$ and $\xi^{(2)}_1$, where for the sake of simplicity we set them as η, γ and ξ respectively. In this case we see that

$$\Omega^{(2;2)} = \begin{pmatrix} \cos\gamma & \mathrm{e}^{i\xi}\sin\gamma \\ -\mathrm{e}^{-i\xi}\sin\gamma & \cos\gamma \end{pmatrix},$$

$$\Omega^{(1;2)} = \begin{pmatrix} \mathrm{e}^{i\eta/2} & 0 \\ 0 & \mathrm{e}^{-i\eta/2} \end{pmatrix},$$

and therefore

$$W^{(2,2)} = \begin{pmatrix} \cos\gamma \mathrm{e}^{i\eta/2} & \mathrm{e}^{i(\xi-\eta/2)}\sin\gamma \\ -\mathrm{e}^{-i(\xi-\eta/2)}\sin\gamma & \cos\gamma \mathrm{e}^{-i\eta/2} \end{pmatrix},$$

We also find

$$A^{(2)}_1 = \begin{pmatrix} 0 & \mathrm{e}^{i(\xi-\eta)} \\ -\mathrm{e}^{-i(\xi-\eta)} & 0 \end{pmatrix},$$

$$A^{(2)}_2 = \begin{pmatrix} -i\sin^2\gamma & i\sin\gamma\cos\gamma \mathrm{e}^{i(\xi-\eta)} \\ i\sin\gamma\cos\gamma \mathrm{e}^{-i(\xi-\eta)} & i\sin^2\gamma \end{pmatrix}.$$

where can be used to write the matrix A as

$$A = \left(\begin{array}{c|cc} i & 0 & 0 \\ \hline 0 & 2i\sin\zeta & 2i\cos\zeta \\ -i(1-\cos 2\gamma) & i\sin 2\gamma\cos\zeta & -i\sin 2\gamma\sin\zeta \end{array} \right),$$

with the following inverse

$$A^{-1} = \left(\begin{array}{c|cc} -i & 0 & 0 \\ \hline -i\cos\zeta\tan\gamma & -\frac{i}{2}\sin\zeta & -i\cos\zeta\csc 2\gamma \\ i\sin\zeta\tan\gamma & -\frac{i}{2}\cos\zeta & i\sin\zeta\csc 2\gamma \end{array} \right),$$

where we have used $\zeta = \xi - \eta$. These two matrices can be used to obtain the left invariant vector fields as

$$\Lambda^{(1)}_1 = -i\frac{\partial}{\partial\eta},$$

$$\Lambda^{(2)}_1 = -i\cos\zeta\tan\gamma\frac{\partial}{\partial\eta} - \frac{i}{2}\sin\zeta\frac{\partial}{\partial\gamma} - i\cos\zeta\csc 2\gamma\frac{\partial}{\partial\xi}$$

$$\Lambda^{(2)}_2 = i\sin\zeta\tan\gamma\frac{\partial}{\partial\eta} - \frac{i}{2}\cos\zeta\frac{\partial}{\partial\gamma} + i\sin\zeta\csc 2\gamma\frac{\partial}{\partial\xi},$$

and the left invariant one-forms as follows

$$\omega_1^{(1)} = i\mathrm{d}\eta - i(1-\cos 2\gamma)\mathrm{d}\xi,$$
$$\omega_1^{(2)} = 2i\sin\zeta\mathrm{d}\gamma + i\sin 2\gamma\cos\zeta\mathrm{d}\xi,$$
$$\omega_2^{(2)} = 2i\cos\zeta\mathrm{d}\gamma - i\sin 2\gamma\sin\zeta\mathrm{d}\xi.$$

For calculation of the right invariant vector fields we note that in this particular case we have $\widetilde{W}^{(1;2)} = \Omega^{(2;2)}$ and $\widetilde{W}^{(2;2)} = \mathbb{I}$, and therefore we find

$$\widetilde{A}_1^{(1)} = \begin{pmatrix} \frac{i}{2}\cos 2\gamma & \frac{-i}{2}\sin 2\gamma e^{i\xi} \\ \frac{-i}{2}\sin 2\gamma e^{-i\xi} & \frac{-i}{2}\cos 2\gamma \end{pmatrix},$$
$$\widetilde{A}_1^{(2)} = \begin{pmatrix} 0 & e^{i\xi} \\ -e^{-i\xi} & 0 \end{pmatrix},$$
$$\widetilde{A}_2^{(2)} = \begin{pmatrix} i\sin^2\gamma & i\sin\gamma\cos\gamma e^{i\xi} \\ -i\sin\gamma\cos\gamma e^{-i\xi} & -i\sin^2\gamma \end{pmatrix}.$$

We then obtain the matrix $\widetilde{A}$ and its inverse $\widetilde{A}^{-1}$ as

$$\widetilde{A} = \left(\begin{array}{c|cc} i\cos 2\gamma & -i\sin 2\gamma\cos\xi & i\sin 2\gamma\sin\xi \\ \hline 0 & 2i\sin\xi & 2i\cos\xi \\ i(1-\cos 2\gamma) & i\sin 2\gamma\cos\xi & -i\sin 2\gamma\sin\xi \end{array}\right),$$

and

$$\widetilde{A}^{-1} = \left(\begin{array}{c|cc} -i & 0 & -i \\ \hline i\cos\xi\tan\gamma & -\frac{i}{2}\sin\xi & -i\cos\xi\cot 2\gamma \\ -i\sin\xi\tan\gamma & -\frac{i}{2}\cos\xi & i\sin\xi\cot 2\gamma \end{array}\right),$$

where can be used to obtain the right invariant vector fields

$$\widetilde{\Lambda}_1^{(1)} = -i\frac{\partial}{\partial\eta} - i\frac{\partial}{\partial\xi},$$
$$\widetilde{\Lambda}_1^{(2)} = i\cos\xi\tan\gamma\frac{\partial}{\partial\eta} - \frac{i}{2}\sin\xi\frac{\partial}{\partial\gamma} - i\cos\xi\cot 2\gamma\frac{\partial}{\partial\xi},$$
$$\widetilde{\Lambda}_2^{(2)} = -i\sin\xi\tan\gamma\frac{\partial}{\partial\eta} - \frac{i}{2}\cos\xi\frac{\partial}{\partial\gamma} + i\sin\xi\cot 2\gamma\frac{\partial}{\partial\xi},$$

and the right invariant one-forms

$$\widetilde{\omega}_1^{(1)} = i\cos 2\gamma\mathrm{d}\eta + i(1-\cos 2\gamma)\mathrm{d}\xi,$$
$$\widetilde{\omega}_1^{(2)} = -i\sin 2\gamma\cos\xi\mathrm{d}\eta + 2i\sin\xi\mathrm{d}\gamma + i\sin 2\gamma\cos\xi\mathrm{d}\xi,$$
$$\widetilde{\omega}_2^{(2)} = i\sin 2\gamma\sin\xi\mathrm{d}\eta + 2i\cos\xi\mathrm{d}\gamma - i\sin 2\gamma\sin\xi\mathrm{d}\xi.$$

Finally, the invariant measure of this group are also obtained as

$$\mathrm{d}\mu[SU(2)] = 2\sin 2\gamma\mathrm{d}\eta\mathrm{d}\gamma\mathrm{d}\xi.$$

6 Conclusion

In this paper we present a method for explicit calculation of the left and right invariant vector fields and one-forms on $SU(N)$ manifold. The calculations are based on the coset parametrization of the unitary group $SU(N)$. It is shown that in the canonical coset parametrization, the invariant measure on the group manifold is decomposed as the product of the invariant measure on the constitute cosets. The advantage of this approach is the possibility of calculating explicitly the differential geometry on $SU(N)$ for every N, in the sense that by knowing N, it is enough to construct the matrix A and then taking the transpose and inverse of this matrix, which are not difficult task to handle. As an illustrative example we calculate explicitly, the differential structure on $SU(2)$ group.

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