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Hardy–Leindler Type Inequalities on Time Scales

S. H. Saker[*]

Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

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Abstract: In this paper, we will prove some new dynamic inequalities on a time scale $\mathbb{T}$. These inequalities, as special cases, when $\mathbb{T}=\mathbb{R}$ contain some integral inequalities and when $\mathbb{T}=\mathbb{N}$ contain the discrete inequalities due to Leindler. The main results will be proved by using the Hölder inequality and a simple consequence of Keller's chain rule on time scales. From our results, as applications, we will derive some new continuous and discrete Wirtinger type inequalities. The technique in this paper is completely different from the technique used by Leindler to prove his main results.

Keywords: Hardy's inequality, Leindler's inequality, time scales

1 Introduction

Since the discovery of the classical Hardy inequalities (continuous or discrete) much work has been done, and many papers which deal with new proofs, various generalizations and extensions have appeared in the literature. We refer the reader to the books [14,15,21] and the papers [2,5,11,12,13,16,18,19,20,23] and the references cited therein. The classical Hardy inequality states that for $f\geq 0$ and integrable over any finite interval $(0,x)$ and f^k is integrable and convergent over $(0,\infty)$ and $k>1$, then

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^k dx \leq \left(\frac{k}{k-1}\right)^k \int_0^\infty f^k(x)dx. \quad (1)$$

The constant $(k/(k-1))^k$ is the best possible. The classical discrete Hardy inequality is given by

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}a(i)\right)^k \leq \left(\frac{k}{k-1}\right)^k \sum_{n=1}^{\infty} a^k(n), \quad k>1. \quad (2)$$

Some of the generalizations of the discrete Hardy inequality (2) are due to Leindler [16,17]. In particular, Leindler in [16] proved that if $p>1$, $\lambda(n), g(n)>0$, then

$$\sum_{n=1}^{\infty}\lambda(n)\left(\sum_{s=1}^{n}g(s)\right)^p \leq p^p\sum_{n=1}^{\infty}\lambda^{1-p}(n)\left(\sum_{s=n}^{\infty}\lambda(s)\right)^p g^p(n), \quad (3)$$

and

$$\sum_{n=1}^{\infty}\lambda(n)\left(\sum_{k=n}^{\infty}g(k)\right)^p \leq p^p\sum_{n=1}^{\infty}\lambda^{1-p}(n)\left(\sum_{k=1}^{n}\lambda(k)\right)^p g^p(n). \quad (4)$$

The converses of (3) and (4) are proved by Leindler in [17]. He proved that if $0<p\leq 1$, then

$$\sum_{n=1}^{\infty}\lambda(n)\left(\sum_{k=1}^{n}g(k)\right)^p \geq p^p\sum_{n=1}^{\infty}\lambda^{1-p}(n)\left(\sum_{k=n}^{\infty}\lambda(k)\right)^p g^p(n), \quad (5)$$

and

$$\sum_{n=1}^{\infty}\lambda(n)\left(\sum_{k=n}^{\infty}g(k)\right)^p \geq p^p\sum_{n=1}^{\infty}\lambda^{1-p}(n)\left(\sum_{k=1}^{n}\lambda(p)\right)^p g^p(n). \quad (6)$$

Dynamic inequalities of Hardy type were established in [22,25,26,27,28,29,30,31] on a time scale $\mathbb{T}$, which is an arbitrary closed subset of the real numbers $\mathbb{R}$. The cases when the time scale is equal to the reals or to the integers represent the classical theories of integral and of discrete inequalities. In this paper, without loss of generality, we assume that $\sup\mathbb{T}=\infty$, and define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}} := [t_0,\infty)\cap\mathbb{T}$. For more details of time scale analysis, we refer the reader to the two books by Bohner and Peterson [3], [4] which summarize and organize much of the time scale calculus.

The natural question now is: if it is possible to prove some new dynamic inequalities on time scales which as

[*] Corresponding author e-mail: shsaker@mans.edu.eg

special cases contain the inequalities (3)-(6)? The main aim of this paper, in Section 2, is to give an affirmative answer to this question. The main results will be proved by making use of Hölder's inequality and a simple consequence of Keller's chain rule on time scales. From our results, for the sake of applications, we will derive some new continuous and discrete Wirtinger type inequalities (see [1]). It is worth to mention here that the technique that we will apply in this paper is completely different from the technique used by Leindler to prove his main results.

2 Main Results

For completeness, before we prove the main results, we recall the following concepts related to the notion of time scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Without loss of generality, we assume that $\sup\mathbb{T}=\infty$, and define the time scale interval $[a,b]_{\mathbb{T}}$ by $[a,b]_{\mathbb{T}}:=[a,b]\cap\mathbb{T}$. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e, when $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{N}$ and $\mathbb{T}=q^{\mathbb{N}_0}=\{q^t : t\in\mathbb{N}_0\}$ where $q>1$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The forward jump operator and the backward jump operator are defined by:

$$\sigma(t):=\inf\{s\in\mathbb{T}:\ s>t\},\quad \rho(t):=\sup\{s\in\mathbb{T}:\ s<t\},$$

where $\sup\emptyset=\inf\mathbb{T}$. A point $t\in\mathbb{T}$, is said to be left–dense if $\rho(t)=t$ and $t>\inf\mathbb{T}$, is right–dense if $\sigma(t)=t$, is left–scattered if $\rho(t)<t$ and right–scattered if $\sigma(t)>t$. A function $g:\mathbb{T}\to\mathbb{R}$ is said to be right–dense continuous (rd–continuous) provided g is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd–continuous functions is denoted by $C_{rd}(\mathbb{T})$.

The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t):=\sigma(t)-t$, and for any function $f:\mathbb{T}\to\mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. Fix $t\in\mathbb{T}$ and let $x:\mathbb{T}\to\mathbb{R}$. Define $x^{\Delta}(t)$ to be the number (if it exists) with the property that given any $\varepsilon>0$ there is a neighborhood U of t with

$$|[x(\sigma(t))-x(s)]-x^{\Delta}(t)[\sigma(t)-s]|\leq\varepsilon|\sigma(t)-s|,$$

for all $s\in U$. In this case, we say $x^{\Delta}(t)$ is the **(**delta) derivative of x at t and that x is (delta) differentiable at t. We will frequently use the following results due to Hilger [10]. Throughout the paper will assume that $g:\mathbb{T}\to\mathbb{R}$ and let $t\in\mathbb{T}$.

(i)If g is differentiable at t, then g is continuous at t.
(ii)If g is continuous at t and t is right-scattered, then g is differentiable at t with $g^{\Delta}(t)=\frac{g(\sigma(t))-g(t)}{\mu(t)}$.
(iii)If g is differentiable and t is right-dense, then $g^{\Delta}(t)=\lim_{s\to t}\frac{g(t)-g(s)}{t-s}$.
(iv)If g is differentiable at t, then $g(\sigma(t))=g(t)+\mu(t)g^{\Delta}(t)$.

Note that if $\mathbb{T}=\mathbb{R}$ then

$$\sigma(t)=t,\ \mu(t)=0,\ f^{\Delta}(t)=f'(t),\ \int_a^b f(t)\Delta t=\int_a^b f(t)dt$$

if $\mathbb{T}=\mathbb{Z}$, then

$$\sigma(t)=t+1,\ \mu(t)=1,\ f^{\Delta}(t)=\Delta f(t),$$

and $\int_a^b f(t)\Delta t=\sum_{t=a}^{b-1}f(t)$, if $\mathbb{T}=h\mathbb{Z}$, $h>0$, then $\sigma(t)=t+h$, $\mu(t)=h$, and

$$y^{\Delta}(t)=\Delta_h y(t):=\frac{y(t+h)-y(t)}{h},$$

$$\int_a^b f(t)\Delta t=\sum_{k=0}^{\frac{b-a-h}{h}}f(a+kh)h,$$

and if $\mathbb{T}=\{t: t=q^k,\ k\in\mathbb{N}_0,\ q>1\}$, then $\sigma(t)=qt$, $\mu(t)=(q-1)t$,

$$x^{\Delta}(t)=\Delta_q x(t)=\frac{(x(qt)-x(t))}{(q-1)t},$$

$$\int_{t_0}^{\infty}f(t)\Delta t=\sum_{k=n_0}^{\infty}f(q^k)\mu(q^k),$$

where $t_0=q^{n_0}$, and if $\mathbb{T}=\mathbb{N}_0^2:=\{n^2:n\in\mathbb{N}_0\}$, then $\sigma(t)=(\sqrt{t}+1)^2$,

$$\mu(t)=1+2\sqrt{t},\ \Delta_{\mathcal{N}}y(t)=\frac{y((\sqrt{t}+1)^2)-y(t)}{1+2\sqrt{t}}.$$

In this paper, we will refer to the (delta) integral which we can define as follows. If $G^{\Delta}(t)=g(t)$, then the Cauchy (delta) integral of g is defined by $\int_a^t g(s)\Delta s:=G(t)-G(a)$. It can be shown (see [3]) that if $g\in C_{rd}(\mathbb{T})$, then the Cauchy integral $G(t):=\int_{t_0}^t g(s)\Delta s$ exists, $t_0\in\mathbb{T}$, and satisfies $G^{\Delta}(t)=g(t)$, $t\in\mathbb{T}$. An infinite integral is defined as $\int_a^{\infty}f(t)\Delta t=\lim_{b\to\infty}\int_a^b f(t)\Delta t$. We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma}\neq 0$, here $g^{\sigma}=g\circ\sigma$) of two differentiable function f and g

$$(fg)^{\Delta}=f^{\Delta}g+f^{\sigma}g^{\Delta}=fg^{\Delta}+f^{\Delta}g^{\sigma} \tag{7}$$

and

$$\left(\frac{f}{g}\right)^{\Delta}=\frac{f^{\Delta}g-fg^{\Delta}}{gg^{\sigma}}. \tag{8}$$

We say that a function $p:\mathbb{T}\to\mathbb{R}$ is regressive provided $1+\mu(t)p(t)\neq 0$, $t\in\mathbb{T}$. The chain rule formula that we will use in this paper is

$$(x^{\gamma}(t))^{\Delta}=\gamma\int_{0}^{1}\left[hx^{\sigma}+(1-h)x\right]^{\gamma-1}dhx^{\Delta}(t), \qquad (9)$$

which is a simple consequence of Keller's chain rule [3, Theorem 1.90]. The integration by parts formula is given by

$$\int_a^b u(t)v^{\Delta}(t)\Delta t=[u(t)v(t)]_a^b-\int_a^b u^{\Delta}(t)v^{\sigma}(t)\Delta t. \qquad (10)$$

To prove the main results, we will use the following Hölder inequality [3, Theorem 6.13]. Let $a,\ b\in\mathbb{T}$. For $u,v\in C_{rd}(\mathbb{T},\mathbb{R})$, we have

$$\int_a^b |u(t)v(t)|\Delta t\le\left[\int_a^b |u(t)|^q\Delta t\right]^{\frac{1}{q}}\left[\int_a^b |v(t)|^p\Delta t\right]^{\frac{1}{p}}, \qquad (11)$$

where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. Throughout the paper, we will assume that the functions are nonnegative rd-continuous functions, $\Delta-$differentiable, locally delta integrable and the left hand sides of the inequalities exists if the right hand side exists. In the following theorem, we will prove the time scale version of Leindler's inequality (3) on time scales.

Theorem 2.1. *Let $\mathbb{T}$ be a time scale and $p>1$. Let*

$$\Lambda(t):=\int_t^{\infty}\lambda(s)\Delta s, \qquad (12)$$

and

$$\Phi(t):=\int_a^t g(s)\Delta s,\quad \textit{for any}\quad t\in[a,\infty)_{\mathbb{T}}. \qquad (13)$$

Then

$$\int_a^{\infty}\lambda(t)(\Phi^{\sigma}(t))^p\Delta t\le p^p\int_a^{\infty}(\lambda(t))^{1-p}\Lambda^p(t)g^p(t)\Delta t. \qquad (14)$$

Proof. Integrating the left hand side of (14) by parts formula (10) with $u^{\Delta}(t)=\lambda(t)$, $v^{\sigma}(t)=(\Phi^{\sigma}(t))^p$, to obtain

$$\int_a^{\infty}\lambda(t)(\Phi^{\sigma}(t))^p\Delta t=u(t)\Phi^p(t)|_a^{\infty}+\int_a^{\infty}(-u(t))(\Phi^p(t))^{\Delta}\Delta t,$$

where $u(t)=-\int_t^{\infty}\lambda(s)\Delta s=-\Lambda(t)$. This and the facts that $\Phi(a)=0$ and $\Lambda(\infty)=0$, imply that

$$\int_a^{\infty}\lambda(t)(\Phi^{\sigma}(t))^p\Delta t=\int_a^{\infty}(\Lambda(t))(\Phi^p(t))^{\Delta}\Delta t. \qquad (15)$$

Applying the chain rule ([3, Theorem 1.87]) $f^{\Delta}(\delta(t))=f^{'}(\delta(d))\delta^{\Delta}(t)$, where $d\in[t,\sigma(t)]$, we see that there exists $d\in[t,\sigma(t)]$ such that

$$(\Phi^p(t))^{\Delta}=p\Phi^{p-1}(d)(\Phi^{\Delta}(t))=p\Phi^{p-1}(d)g(t). \qquad (16)$$

Since $\Phi^{\Delta}(t)=g(t)>0$, and $\sigma(t)\ge d$, we see that $\Phi(d)\le\Phi^{\sigma}(t)$. This and (16) imply that

$$(\Phi^p(t))^{\Delta}(\Lambda(t))\le pg(t)(\Lambda(t))(\Phi^{\sigma}(t))^{p-1}. \qquad (17)$$

Substituting (17) into (15), we have

$$\int_a^{\infty}\lambda(t)(\Phi^{\sigma}(t))^p\Delta t\le p\int_a^{\infty}\frac{g(t)\Lambda(t)}{(\lambda(t))^{\frac{p-1}{p}}}(\lambda(t))^{\frac{p-1}{p}}(\Phi^{\sigma}(t))^{p-1}\Delta t. \qquad (18)$$

Applying the Hölder inequality (11) on the right hand side of (18) with indices $p,\ p/p-1$, we see that

$$\int_a^{\infty}\frac{g(t)\Lambda(t)}{(\lambda(t))^{\frac{p-1}{p}}}\left[(\lambda(t))^{\frac{p-1}{p}}(\Phi^{\sigma}(t))^{p-1}\right]\Delta t\le\left[\int_a^{\infty}\left[\frac{\Lambda(t)g(t)}{(\lambda(t))^{\frac{p-1}{p}}}\right]^p\Delta t\right]^{1/p}\times\left[\int_a^{\infty}\lambda(t)(\Phi^{\sigma}(t))^p\Delta t\right]^{1-1/p}. \qquad (19)$$

Substituting (20) into (18), we have

$$\int_a^{\infty}\lambda(t)(\Phi^{\sigma}(t))^p\Delta t\le p\left[\int_a^{\infty}\frac{(\Lambda(t)g(t))^p}{(\lambda(t))^{p-1}}\Delta t\right]\times^{1/p}\left[\int_a^{\infty}\lambda(t)(\Phi^{\sigma}(t))^p\Delta t\right]^{1-\frac{1}{p}}. \qquad (20)$$

This implies that

$$\int_a^{\infty}\lambda(t)(\Phi^{\sigma}(t))^p\Delta t\le p^p\int_a^{\infty}(\lambda(t))^{1-p}\Lambda^p(t)g^p(t)\Delta t,$$

which is the desired inequality (14). The proof is complete.

Remark. As a special case of Theorem 2.1 when $\mathbb{T}=\mathbb{R}$, we have the following integral inequality of Leindler's type (note that when $\mathbb{T}=\mathbb{R}$, we have $\sigma(t)=t$)

$$\int_a^{\infty}\lambda(t)\left(\int_a^t g(s)ds\right)^p dt\le p^p\int_a^{\infty}\lambda^{1-p}(t)\left(\int_t^{\infty}\lambda(s)ds\right)^p g^p(t)dt,\ p>1.$$

From this inequality, we have the following Wirtinger type inequality

$$\int_a^{\infty}\lambda^{1-p}(t)\left(\int_t^{\infty}\lambda(s)ds\right)^p(G^{'}(t))^p dt\ge\frac{1}{p^p}\int_a^{\infty}\lambda(t)G^p(t)dt,\ p>1,$$

where $G(t)$ is continuous and differentiable function with $G(a)=0$. As a special case if $\lambda(t)=1/t^2$ and replace ∞ by 1, we get the well-known inequality due to Hardy

$$\int_0^1\left(U^{'}(t)\right)^2dt\geq\frac{1}{4}\int_0^1\frac{1}{t^2}U^2(t),\text{ with }U(0)=0,$$

with the best constant $1/4$.

The Wirtinger type inequalities have extensive applications on partial differential and difference equations, harmonic analysis, approximations, number theory, optimization, convex geometry, spectral theory of differential and difference operators, and others (see [24]).

Remark. Assume that $\mathbb{T}=\mathbb{N}$ in Theorem 2.1, $p>1$, $a=1$. Furthermore assume that

$$\sum_{n=1}^{\infty}\lambda^{1-p}(n)\left(\sum_{s=n}^{\infty}\lambda(s)\right)^p g^p(n),$$

is convergent. In this case the inequality (14) becomes the following discrete Leindler's inequality

$$\sum_{n=1}^{\infty}\lambda(n)\left(\sum_{s=1}^{n}g(s)\right)^p \leq p^p\sum_{n=1}^{\infty}\lambda^{1-p}(n)\left(\sum_{s=n}^{\infty}\lambda(s)\right)^p g^p(n),\ \ p>1.$$

From this inequality, we have the following discrete Wirtinger type inequality

$$\sum_{n=1}^{\infty}\lambda^{1-p}(n)\left(\sum_{s=n}^{\infty}\lambda(s)\right)^p(\Delta G(n))^p \geq\frac{1}{p^p}\sum_{n=1}^{\infty}\lambda(n)\left(G(n)\right)^p,\ \ p>1,$$

where $G(n)$ is a positive sequence with $G(1)=0$.

In the following theorem, we will prove a time scale version of Leindler's inequality (4) on time scales.

Theorem 2.2. *Let $\mathbb{T}$ be a time scale and $p>1$. Let*

$$\overline{\Lambda}(t)=\int_a^t\lambda(s)\Delta s, \tag{21}$$

and

$$\overline{\Phi}(t):=\int_t^{\infty}g(s)\Delta s,\quad \textit{for any}\quad t\in[a,\infty)_{\mathbb{T}}. \tag{22}$$

Then

$$\int_a^{\infty}\lambda(t)(\overline{\Phi}(t))^p\Delta t\leq p^p\int_a^{\infty}\lambda^{1-p}(t)\left(\overline{\Lambda}^{\sigma}(t)\right)^p(g(t))^p\Delta t. \tag{23}$$

Proof. To prove the inequality (23), we integrate the left hand side by parts formula (10) with $v^{\Delta}(t)=\lambda(t)$, and $u(t)=\overline{\Phi}^p(t)$, to obtain

$$\int_a^{\infty}\lambda(t)(\overline{\Phi}(t))^p\Delta t=\overline{\Lambda}(t)\overline{\Phi}^p(t)\Big|_a^{\infty} +\int_a^{\infty}\left(\overline{\Lambda}^{\sigma}(t)\right)\left(-\overline{\Phi}^p(t)\right)^{\Delta}\Delta t.$$

Using the facts that $\overline{\Phi}(\infty)=0$ and $\overline{\Lambda}(a)=0$, we get that

$$\int_a^{\infty}\lambda(t)\overline{\Phi}^p(t)\Delta t=\int_a^{\infty}\left(\overline{\Lambda}^{\sigma}(t)\right)\left(-\overline{\Phi}^p(t)\right)^{\Delta}\Delta t. \tag{24}$$

Applying the chain rule ([3, Theorem 1.87]) $f^{\Delta}(\delta(t))=f^{'}(\delta(d))\delta^{\Delta}(t)$, where $d\in[t,\sigma(t)]$, we see that there exists $d\in[t,\sigma(t)]$ such that

$$-\left(\overline{\Phi}^p(t)\right)^{\Delta}=-p\overline{\Phi}^{p-1}(d)(\overline{\Phi}^{\Delta}(t)). \tag{25}$$

Since $\overline{\Phi}^{\Delta}(t)=-g(t)\leq 0$, and $d\geq t$, we have

$$-\left(\overline{\Phi}^p(t)\right)^{\Delta}\left(\overline{\Lambda}^{\sigma}(t)\right)\leq pg(t)\overline{\Lambda}^{\sigma}(t)(\overline{\Phi}(t))^{p-1}. \tag{26}$$

Substituting (26) into (24), we have

$$\int_a^{\infty}\lambda(t)\overline{\Phi}^p(t)\Delta t\leq p\int_a^{\infty}(\overline{\Phi}(t))^{p-1}g(t)\overline{\Lambda}^{\sigma}(t)\Delta t.$$

This inequality can be written in the form

$$\int_a^{\infty}\lambda(t)\left(\overline{\Phi}(t)\right)^p\Delta t \leq p\int_a^{\infty}\left[\frac{\lambda(t)g(t)}{(\lambda(t))^{\frac{p-1}{p}}}\overline{\Lambda}^{\sigma}(t)\right] \times\left[(\lambda(t))^{\frac{p-1}{p}}(\overline{\Phi}(t))^{p-1}\right]\Delta t. \tag{27}$$

Applying the Hölder inequality on the right hand side with indices p and $p/p-1$, we see that

$$\int_a^{\infty}\left[\frac{g(t)\overline{\Lambda}^{\sigma}(t)}{(\lambda(t))^{\frac{p-1}{p}}}\right]\left[(\lambda(t))^{\frac{p-1}{p}}(\overline{\Phi}(t))^{p-1}\right]\Delta t \leq\left[\int_a^{\infty}\left[\frac{g(t)\overline{\Lambda}^{\sigma}(t)}{(\lambda(t))^{\frac{p-1}{p}}}\right]^p\Delta t\right]^{1/p} \times\left[\int_a^{\infty}\lambda(t)(\overline{\Phi}(t))^p\Delta t\right]^{1-1/p}. \tag{28}$$

Substituting (28) into (27), we have

$$\int_a^{\infty}\lambda(t)(\overline{\Phi}(t))^p\Delta t \leq p\left[\int_a^{\infty}\lambda^{1-p}(t)\left(\overline{\Lambda}^{\sigma}(t)g(t)\right)^p\Delta t\right]^{1/p} \times\left[\int_a^{\infty}\lambda(t)(\overline{\Phi}(t))^p\Delta t\right]^{1-\frac{1}{p}}. \tag{29}$$

This implies that

$$\int_a^\infty \lambda(t)(\overline{\Phi}(t))^p \Delta t \le p^p \int_a^\infty \lambda^{1-p}(t)\left(\overline{\Lambda}^\sigma(t)\right)^p (g(t))^p \Delta t, \tag{30}$$

which is the desired inequality (23). The proof is complete.

Remark. As a special case of Theorem 2.2 when $\mathbb{T}=\mathbb{R}$ and $p>1$, we have the following integral inequality of Leindler's type (note that when $\mathbb{T}=\mathbb{R}$, we have $\overline{\Phi}^\sigma(t)=\overline{\Phi}(t)$)

$$\int_a^\infty \lambda(t)\left(\int_t^\infty g(s)ds\right)^p dt \le p^p \int_a^\infty \lambda^{1-p}(t)\left(\int_a^t \lambda(s)ds\right)^p g^p(t)dt.$$

Remark. Assume that $\mathbb{T}=\mathbb{N}$ in Theorem 2.2, $p>1$ and $a=1$. Furthermore assume that $\sum_{s=1}^\infty \lambda(n)\left(\sum_{k=1}^n \lambda(k)\right)^p g^p(n)$ is convergent. In this case the inequality (23) becomes the following discrete Leindler's inequality

$$\sum_{n=1}^\infty \lambda(n)\left(\sum_{k=n}^\infty g(k)\right)^p \le p^p \sum_{n=1}^\infty \lambda^{1-p}(n)\left(\sum_{k=1}^n \lambda(k)\right)^p g^p(n), \quad p>1. \tag{31}$$

In the following theorem, we will prove a time scale version of Leindler's inequality (5) on time scales.

Theorem 2.3. *Let $\mathbb{T}$ be a time scale and $0<p\le 1$. Let*

$$\Omega(t)=\int_t^\infty \lambda(s)\Delta s, \tag{32}$$

and

$$\Psi(t)=\int_a^t g(s)\Delta s. \tag{33}$$

Then

$$\int_a^\infty \lambda(t)(\Psi^\sigma(t))^p\Delta t \ge p^p \int_a^\infty \lambda^{1-p}(t)\Omega^p(t)g^p(t)\Delta t. \tag{34}$$

Proof. Integrating the left hand side of (34) by parts formula (10) with $u^\Delta(t)=\lambda(t)$ and $v^\sigma(t)=(\Psi^\sigma(t))^p$, we obtain

$$\int_a^\infty \lambda(t)(\Psi^\sigma(t))^p\Delta t = u(t)\Psi^p(t)|_a^\infty + \int_a^\infty (-u(t))(\Psi^p(t))^\Delta \Delta t,$$

where $u(t)=-\int_t^\infty \lambda(s)\Delta s=-\Omega(t)$. This and the facts that $\Psi(a)=0$ and $u(\infty)=0$ imply that

$$\int_a^\infty \lambda(t)(\Psi^\sigma(t))^p\Delta t = \int_a^\infty \Omega(t)(\Psi^p(t))^\Delta \Delta t. \tag{35}$$

Applying the chain rule ([3, Theorem 1.87]) $f^\Delta(\delta(t))=f'(\delta(d))\delta^\Delta(t)$, where $d\in[t,\sigma(t)]$, we see that there exists $d\in[t,\sigma(t)]$ such that

$$(\Psi^p(t))^\Delta = \frac{p}{\Psi^{1-p}(d)}(\Psi^\Delta(t)) = \frac{p}{\Psi^{1-p}(d)}g(t). \tag{36}$$

Since $\Psi^\Delta(t)=g(t)\ge 0$, and $\sigma(t)\ge d$, we see that $(\Psi^\sigma)\ge \Psi(d)$, and then

$$\frac{p}{\Psi^{1-p}(d)} \ge \frac{p}{(\Psi^\sigma(t))^{1-p}}, \quad (\text{where } p\le 1). \tag{37}$$

Combining (36) and (37), we have that

$$(\Psi^p(t))^\Delta\,\Omega(t) \ge \frac{p\lambda(t)g(t)\Omega(t)}{(\Psi^\sigma(t))^{1-p}}. \tag{38}$$

Substituting (38) into (35), we have

$$\left(\int_a^\infty \lambda(t)(\Psi^\sigma(t))^p\Delta t\right)^p \ge p^p\left[\int_a^\infty\left(\frac{g^p(t)\Omega^p(t)}{(\Psi^\sigma(t))^{p(1-p)}}\right)^{1/p}\Delta t\right]^p. \tag{39}$$

Applying the Hölder inequality

$$\int_a^b F(t)G(t)\Delta t \le \left[\int_a^b F^q(t)\Delta t\right]^{\frac{1}{q}}\left[\int_a^b G^h(t)\Delta t\right]^{\frac{1}{h}},$$

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$$\left[\int_a^\infty\left(\frac{g^p(t)\,(\Omega^p(t))}{(\Psi^\sigma(t))^{p(1-p)}}\right)^{1/p}\Delta t\right]^p,$$

with indices $q=1/p>1$ and $h=1/(1-p)$, and (note that $\frac{1}{q}+\frac{1}{h}=1$, where $q>1$)

$$F(t)=\frac{g^p(t)\Omega^p(t)}{(\Psi^\sigma(t))^{p(1-p)}}$$

and

$$G(t)=\lambda^{1-p}(t)(\Psi^\sigma(t))^{p(1-p)},$$

we see that

$$\begin{aligned}\left(\int_a^\infty F^{1/p}(t)\Delta t\right)^p &= \left[\int_a^\infty\left(\frac{g^p(t)\Omega^p(t)}{(\Psi^\sigma(t))^{p(1-p)}}\right)^{1/p}\Delta t\right]^p \\ &\ge \frac{\int_a^\infty F(t)G(t)\Delta t}{\left[\int_a^\infty (G(t))^{\frac{1}{1-p}}\right]^{1-p}} \\ &= \int_a^\infty \frac{g^p(t)\Omega^p(t)}{(\Psi^\sigma(t))^{p(1-p)}} \\ &\quad\times \frac{\lambda^{1-p}(t)(\Psi^\sigma(t))^{p(1-p)}}{\left[\int_a^\infty (\lambda^{1-p}(t)(\Psi^\sigma(t))^{p(1-p)})^{\frac{1}{1-p}}\Delta t\right]^{1-p}}\Delta t \\ &= \frac{\int_a^\infty g^p(t)\Omega^p(t)\lambda^{1-p}(t)\Delta t}{\left[\int_a^\infty \lambda(t)(\Psi^\sigma(t))^p\Delta t\right]^{1-p}}.\end{aligned}$$

This implies that

$$\left[\int_a^\infty \left(\frac{g^p(t)\Omega^p(t)}{(\Psi^\sigma(t))^{p(1-p)}}\right)^{1/p}\Delta t\right]^p \geq \frac{\int_a^\infty g^p(t)\Omega^p(t)\lambda^{1-p}(t)\Delta t}{\left[\int_a^\infty \lambda(t)(\Psi^\sigma(t))^p\right]^{1-p}}. \tag{40}$$

Substituting (40) into (39), we have

$$\left(\int_a^\infty \lambda(t)(\Psi^\sigma(t))^p\Delta t\right)^p \geq p^p \frac{\int_a^\infty g^p(t)(\Omega^p(t))\lambda^{1-p}(t)\Delta t}{\left[\int_a^\infty \lambda(t)(\Psi^\sigma(t))^p\Delta t\right]^{1-p}}.$$

This implies that

$$\int_a^\infty \lambda(t)(\Psi^\sigma(t))^p\Delta t \geq p^p\int_a^\infty \lambda^{1-p}(t)g^p(t)\Omega^p(t)\Delta t,$$

which is the desired inequality (34). The proof is complete.

Remark. As a special case of (34), when $\mathbb{T}=\mathbb{R}$ and $p<1$, we have the following integral inequality of Leindler's type (note that when $\mathbb{T}=\mathbb{R}$, we have $\Psi^\sigma(t)=\Psi(t)$)

$$\int_a^\infty \lambda(t)\left(\int_a^t g(s)ds\right)^p dt \geq p^p\int_a^\infty \lambda^{1-p}(t)\left(\int_t^\infty \lambda(s)\Delta s\right)^p g^p(t)dt.$$

*Remark.*Assume that $\mathbb{T}=\mathbb{N}$ in Theorem 2.3, $p\leq 1$ and $a=1$. Furthermore assume that $\sum_{n=1}^\infty \lambda^{1-p}(n)\Omega^p(n)a^p(n)$ is convergent and define. In this case the inequality (34) becomes the following discrete Leindler's inequality

$$\sum_{n=1}^\infty \lambda(n)\left(\sum_{k=1}^n g(k)\right)^p \geq p^p\sum_{n=1}^\infty \lambda^{1-p}(n)\left(\sum_{k=n}^\infty \lambda(k)\right)^p g^p(n).$$

In the following theorem, we will prove a new time scale version of Leindler's inequality (6) on time scales.

Theorem 2.4. *Let $\mathbb{T}$ be a time scale and $0<p\leq 1$. Let*

$$\overline{\Omega}(t)=\int_a^t \lambda(s)\Delta s, \tag{41}$$

and

$$\overline{\Psi}(t)=\int_t^\infty g(s)\Delta s. \tag{42}$$

Then

$$\int_a^\infty \lambda(t)(\overline{\Psi}(t))^p\Delta t \geq p^p\int_a^\infty \lambda^{1-p}(t)(\overline{\Omega}^\sigma(t))^p g^p(t)\Delta t. \tag{43}$$

Proof. Integrating the left hand side of (43) by parts formula (10) with $v^\Delta(t)=\lambda(t)$, and $u(t)=\left(\overline{\Psi}(t)\right)^p$, we obtain

$$\int_a^\infty \lambda(t)(\overline{\Psi}(t))^p\Delta t = v(t)\overline{\Psi}^p(t)\Big|_a^\infty + \int_a^\infty (v^\sigma(t))(-\overline{\Psi}^p(t))^\Delta\Delta t, \tag{44}$$

where $v(t)=\int_a^t \lambda(s)\Delta s=\overline{\Omega}(t)$. From the inequality (44) and the fact that $\overline{\Psi}(\infty)=\overline{\Omega}(a)=0$, we have

$$\int_a^\infty \lambda(t)(\overline{\Psi}^\sigma(t))^p\Delta t = \int_a^\infty \overline{\Omega}^\sigma(t)(-\overline{\Psi}^p(t))^\Delta\Delta t. \tag{45}$$

Applying the chain rule $f^\Delta(\delta(t))=f'(\delta(d))\delta^\Delta(t)$, where $d\in[t,\sigma(t)]$, we see that there exists $d\in[t,\sigma(t)]$ such that

$$\left(-\overline{\Psi}^p(t)\right)^\Delta = \frac{-p}{\overline{\Psi}^{1-p}(d)}(\overline{\Psi}^\Delta(t)) = \frac{p}{\overline{\Psi}^{1-p}(d)}g(t). \tag{46}$$

Since $\overline{\Psi}^\Delta(t)=-g(t)\leq 0$, and $d\geq t$, we see that $\overline{\Psi}(t)\geq\overline{\Psi}(d)$, and then

$$\frac{pg(t)}{\overline{\Psi}^{1-p}(d)} \geq \frac{pg(t)}{(\overline{\Psi}(t))^{1-p}}, \quad \text{(note that } p\leq 1).$$

This, (46) imply that

$$\left(-\overline{\Psi}^p(t)\right)^\Delta(\overline{\Omega}(\sigma(t))) \geq \frac{pg(t)\overline{\Omega}^\sigma(t)}{(\overline{\Psi}(t))^{1-p}}. \tag{47}$$

Substituting (38) into (35), we have

$$\left(\int_a^\infty \lambda(t)\left(\overline{\Psi}(t)\right)^p\Delta t\right)^p \geq p^p\left[\int_a^\infty\left(\frac{g^p(t)(\overline{\Omega}^\sigma(t))^p}{(\overline{\Psi}(t))^{p(1-p)}}\right)^{1/p}\Delta t\right]^p.$$

The rest of the proof is similar to the proof of Theorem 2.3 and hence is omitted. The proof is complete.

*Remark.*Assume that $\mathbb{T}=\mathbb{R}$ in Theorem 2.4 and $p\leq 1$. In this case, we have the following integral inequality of Leindler's type (note that when $\mathbb{T}=\mathbb{R}$, we have $\overline{\Omega}^\sigma(t)=\overline{\Omega}(t)$)

$$\int_a^\infty \lambda(t)\left(\int_t^\infty g(s)ds\right)^p dt \geq p^p\int_a^\infty \lambda^{1-p}(t)\left(\int_a^t \lambda(s)\Delta s\right)^p g^p(t)dt.$$

*Remark.*Assume that $\mathbb{T}=\mathbb{N}$ in Theorem 2.4, $p\leq 1$ and $a=1$. Furthermore assume that $\sum_{n=1}^\infty \lambda^{1-p}(n)\left(\sum_{k=1}^n \lambda(k)\right)^p a^p(n)$ is convergent. In this case the inequality (43) becomes the following discrete Leindler's type inequality

$$\sum_{n=1}^\infty \lambda(n)\left(\sum_{k=n}^\infty g(k)\right)^p \geq p^p\sum_{n=1}^\infty \lambda^{1-p}(n)\left(\sum_{k=1}^n \lambda(k)\right)^p g^p(n).$$

*Remark.*Some Wirtinger type inequalities can be derived form Theorem 2.2-4 as special cases. The details are left to the reader.

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Samir Saker is full professor of mathematics, Mansoura University, Egypt. His interest is qualitative analysis of dynamic equations, inequalities on time scales and qualitative behavior of delay models. He is an author of three books and more than 190 papers. He is the recipient of Shoman Aaward for Young Arab Scientists (2003, Jordan), Fulbright USA (2004 Trinity University), National State Prize (2005, Egypt) and Amin Lotfy in Mathematics (2009, Egypt). He have been selected as a Member of National Committee of Mathematics in Egypt.