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Analysis of Two Generalized Exponential Populations Under Joint Type-I Progressive Hybrid Censoring Scheme

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Abstract: This paper discussed inference for two generalized exponential using the joint type-I progressively hybrid censoring (JPHC-I) scheme. It assumed that the lifetime distribution of the items from the two populations follow generalized exponential distribution. Based on the JPHC-I scheme, we first consider the maximum likelihood estimators of the unknown parameters along with thier asymptotic confidence intervals. Next, we provide the Bayesian inferences of the unknown parameters under the assumptions of independent gamma priors on the scale parameters using squared error (SE) and linear-exponential (LINEX) loss functions. Markov Chain Monte Carlo (MCMC) techniques is applied to carry out the Bayesian estimation procedure and in turn calculate the credible intervals. To evaluate the performance of the estimators, numerical example is carried out.

Keywords: Generalized exponential distribution, Joint type-I progressive hybrid censoring scheme, Maximum likelihood estimation, Bayesian estimation, MCMC, loss functions

1 Introduction

Censoring schemes are used to make life testing experiments time and cost effective, and give impetus to the performance of the design. Various types of censored data are available in the literature for analysis of lifetime experiments (see [1]). Most of these censored data deals with one-sample problem, however, there are situations in which the experimenter may compare two different populations. In such cases, the joint censoring scheme has been suggested in literature. As suggested by [2], [3], and [4], the joint censoring scheme is quite useful to compare the lifetime distribution of products coming from different units which are manufactured by two different lines in the same facility.

To describe this scheme, suppose that two samples of products of sizes m and n, respectively, are selected from these two lines of operation (say lines 1 and 2), and they are placed on a life testing experiment simultaneously. Based on cost considerations and time restrictions for completion of the test, the researcher may choose to terminate the life testing experiment as soon as a pre-specified number of failures are observed.

Several authors have addressed inferential issues based on joint progressive type-II censored (JPC–II) scheme. Notable among them are: [5,19,7,8,9,10], [11,12], and the references cited therein. [4], obtained both Bayesian and frequents estimators for two exponential populations under both joint progressive type-I censored (JPC-I) and joint type-I censored schemes.

[13] introduced type-I progressive hybrid censoring (PHC-I) scheme to overcome the drawback of type-II progressive censoring scheme, where the experiment length can be quite large to get the desirable number of failures. The PHC-I scheme provides flexibility to terminate the experiment at a prefixed time as well as withdrawal of items during the life test. They investigated the maximum likelihood and Bayes estimators of the unknown parameter of the exponential distribution, as well as different confidence intervals. [14] used PHC-I scheme to estimate the unknown parameters of the Weibull distribution. See also [15], [16], [17], and for more details see [18].

At the present time, the reliability tests should be performed with severe time limitations because of the short product development times, which make the usual joint progressive type-II censoring scheme proposed by [3] no longer

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appropriate in many field products. Therefore, [19] introduced a new joint type-I progressively hybrid censoring (JPHC-I) scheme. They investigated the maximum likelihood and Bayes estimators of the unknown parameter of the exponential distribution, as well as different confidence intervals.

According to JPHC-I scheme the experiment is terminated when the test reaches a predetermined time, or a predetermined number of failures has occurred. The experimental time is unbounded in the case of the usual joint progressive type-II censoring scheme, but for the JPHC-I scheme, the experimenter can determine it according to the maximum experimental time that can afford to continue. Using JPHC-I, [20] investigated the maximum likelihood and Bayes estimators of the unknown parameters of Weibull distribution.Based on the JPHC-I scheme, we analyze estimation problems of two generalized exponential populations. The maximum likelihood estimators (MLEs) and approximate confidence intervals (ACIs) of the unknown parameters are constructed. Also Bayes estimators (BEs) of the unknown parameters under the assumptions of independent gamma priors on the scale parameters are obtained using SE and LINEX loss functions and credible intervals depending on MCMC techniques. Further, both frequents and Bayesian confidence intervals (BCIs) are obtained. Some simulation experiments are performed to compare the performances of the estimators based on JPHC-I. One data set analysis has been performed for illustrative purposes.

The rest of this paper is organized as follows. In Section 2, we introduce the model and provide the necessary assumptions. The MLEs and asymptotic confidence intervals will be discussed in Section 3. In Section 4, BEs under SE and LINEX loss functions and Bayesian credible intervals for the parameters are given using JPHC-I scheme. Numircal results and the analysis of one data set are provided in Section 5. Finally a conclusion is given in Section 6.

2 Model description

Suppose we consider products from two different populations. we draw a random sample of size m from population (1) with distribution function $F(x)$ and density function $f(x)$, and a random sample of size n from population (2) with distribution function $G(y)$ and density function $g(y)$. The two independent samples are placed simultaneously in a life testing experiment. Further, $W_{(1)} \leqslant W_{(2)} \leqslant \cdots \leqslant W_{(N)}$ denote the order statistics of $N = m+n$ random variables$(X_1,\ldots,X_m,Y_1,\ldots,Y_n)$. The proposed JPHC-I can be described as follows. The integer $r<N$ is fixed at the beginning of the experiment, $R_1,\ldots,R_r$ are r pre-fixed integers satisfying $R_1+\cdots+R_r+r=N$ and the time point $T$is also fixed beforehand. At the time of the first failure (that may be from either X or Y), R_1 units are randomly withdrawn from the remaining $N-1$ surviving units. Similarly, at the time of the second failure (that may be from either X or Y), R_2 units are randomly withdrawn from the remaining $N-R_1-2$ surviving units, and so on. If the r^{th}failure occurs before the time point T, the experiment stops at the time point $w_{(r)}$. On the other hand, suppose the r^{th}failure does not occur before time point $T$and only Jfailures occur before the time point T, where $0 \leq J < r$. Then, at the time point T all the remaining R_J^*, $R_J^* = N-(R_1+\cdots+R_J)-J$units are removed and the experiment terminates at time point T. We denote the two cases as Case I and Case II, respectively.

Let $R_i = s_i + q_i,\ i=1,\ldots,D$ (where $D<N$ being a prefixed integer, $D=r$ for case I and $D=J$ for case II) and $s_i(q_i)$ is the number of units withdrawn at the time of the i^{th} failure that belongs to X (Y) sample and these are unknown and random variables. The data observed in this form will consist of (Z,W,S), where $W=(w_{(1)},\ldots,w_{(D)})$, $Z=(z_1,\ldots,z_D)$ with $z_i = 1$ or 0 according as whether $w_{(i)}$ is either X- or Y-failure, respectively, $S=(s_1,\ldots,s_D)$ and $R=(R_1,R_2,\ldots,R_D)$ has the decomposition $S+Q=(s_1,\ldots,s_D)+(q,\ldots,q_D)$.

The likelihood function (without the constant term) of (z,w,s) can be written as

$$L \propto \begin{cases} \prod\limits_{i=1}^{r} \left(f(w_{(i)})^{z_i} g(w_{(i)})^{1-z_i}\right) \left(\bar{F}(w_{(i)})\right)^{s_i} \left(\bar{G}(w_{(i)})\right)^{q_i} \text{ for Case I} \\ \prod\limits_{i=1}^{J} \left(f(w_{(i)})^{z_i} g(w_{(i)})^{1-z_i}\right) \left(\bar{F}(w_{(i)})\right)^{s_i} \left(\bar{G}(w_{(i)})\right)^{q_i} \left(\bar{F}(T)\right)^{s_J^*} \left(\bar{G}(T)\right)^{q_J^*}, \quad \text{for Case II,} \end{cases} \tag{1}$$

where $s_J^* = m-(s_1+\cdots+s_J)-m_J$, $q_J^* = n-(q_1+\cdots+q_J)-n_J$, (m_J,n_J) number of failure units in the (first and second) sample respectively, $\bar{F}(\cdot) = 1-F(\cdot)$ and $\bar{G}(\cdot)=1-G(\cdot)$ are the survival function of the two populations.

3 Maximum likelihood estimation

Suppose the lifetimes of m units of population (1), $X_1,\ldots,X_m$, are independent and identically distributed (iid) random variables from generalized exponential (GE) (α_1,θ_1) population with density and distribution functions as

$$
\begin{aligned}
&f(x)=\alpha_1\theta_1(1-e^{-\theta_1 x})^{\alpha_1-1}e^{-\theta_1 x}\\
&\text{and}\\
&F(x)=(1-e^{-\theta_1 x})^{\alpha_1}\ \text{for}\ x>0,\ \alpha_1,\theta_1>0,
\end{aligned}
\tag{2}
$$

respectively.

Similarly, let the lifetimes of n units of population (2), $Y_1,\ldots,Y_n$ be iid random variables from GE (α_2,θ_2) population with density and distribution functions as

$$
\begin{aligned}
&g(y)=\alpha_2\theta_2(1-e^{-\theta_2 x})^{\alpha_2-1}e^{-\theta_2 x}\\
&\text{and}\\
&G(x)=(1-e^{-\theta_2 x})^{\alpha_2}\ \text{for}\ y>0,\ \alpha_2,\theta_2>0.
\end{aligned}
\tag{3}
$$

respectively. The log-likelihood function corresponding to Equations (2) and (3) is given by

$$
\ln L=\begin{cases}
m_r\ln\alpha_1+m_r\ln\theta_1+n_r\ln\alpha_2+n_r\ln\theta_2+\sum_{i=1}^{r}z_i\ln(1-U_i)^{\alpha_1-1}+\sum_{i=1}^{r}z_i\ln U_i\\
+\sum_{i=1}^{r}(1-z_i)\ln(1-P_i)^{\alpha_2-1}+\sum_{i=1}^{r}(1-z_i)\ln P_i+\sum_{i=1}^{r}s_i\ln(1-\lambda_1^{\alpha_1})+\sum_{i=1}^{r}q_i\ln(1-\lambda_2^{\alpha_2}),\ \text{Case I}\\
m_J\ln\alpha_1+m_J\ln\theta_1+n_J\ln\alpha_2+n_J\ln\theta_2+\sum_{i=1}^{J}z_i\ln(1-U_i)^{\alpha_1-1}+\sum_{i=1}^{J}z_i\ln U_i\\
+\sum_{i=1}^{J}(1-z_i)\ln(1-P_i)^{\alpha_2-1}+\sum_{i=1}^{J}(1-z_i)\ln P_i+\sum_{i=1}^{J}s_i\ln(1-\lambda_1^{\alpha_1})+\sum_{i=1}^{J}q_i\ln(1-{\lambda_2}^{\alpha_2})\\
+s_J^*\ln(1-\gamma_1^{\alpha_1})+q_J^*\ln(1-\gamma_2^{\alpha_2}),\ \text{Case II,}
\end{cases}
\tag{4}
$$

where $U_i=e^{-\theta_1 w_{(i)}}$, $P_i=e^{-\theta_2 w_{(i)}}$, $V_1=e^{-\theta_1 w_{(J)}}$, $V_2=e^{-\theta_2 w_{(J)}}$, $\lambda_1=(1-V_1)$, $\lambda_2=(1-V_2)$, $A_1=e^{-\theta_1 T}$, $A_2=e^{-\theta_2 T}$, $\gamma_1=(1-A_1)$ and $\gamma_2=(1-A_2)$.

Differentiating partially (4) with respect to α_1, α_2, θ_1 and θ_2 and equating them to zero, we get the following two equations
Case I:

$$
\begin{aligned}
&\frac{m_r}{\hat\alpha_1}+\sum_{i=1}^{r}z_i\ln(1-\hat U_i)-\frac{\sum_{i=1}^{r}s_i\hat\lambda_1^{\hat\alpha_1}\ln\hat\lambda_1}{1-\hat\lambda_1^{\hat\alpha_1}}=0\\
&\frac{n_r}{\hat\alpha_2}+\sum_{i=1}^{r}(1-z_i)\ln(1-\hat P_i)-\frac{\sum_{i=1}^{r}q_i\hat\lambda_2^{\hat\alpha_2}\ln\hat\lambda_2}{1-\hat\lambda_2^{\hat\alpha_2}}=0\\
&\frac{m_r}{\hat\theta_1}+(\hat\alpha_1-1)\sum_{i=1}^{r}\frac{z_i w_{(i)}\hat U_i}{1-\hat U_i}-\sum_{i=1}^{r}z_i w_{(i)}-\frac{\sum_{i=1}^{r}s_i\hat\alpha_1 w_{(r)}\hat\lambda_1^{\hat\alpha_1-1}\hat V_1}{1-\hat\lambda_1^{\hat\alpha_1}}=0\\
&\frac{n_r}{\hat\theta_2}+(\hat\alpha_2-1)\sum_{i=1}^{r}\frac{(1-z_i)w_{(i)}\hat P_i}{1-\hat P_i}-\sum_{i=1}^{r}(1-z_i)w_{(i)}-\frac{\sum_{i=1}^{r}q_i\hat\alpha_2 w_{(r)}\hat\lambda_2^{\hat\alpha_2-1}\hat V_2}{1-\hat\lambda_2^{\hat\alpha_2}}=0,
\end{aligned}
\tag{5}
$$

where $\hat U_i=e^{-\hat\theta_1 w_{(i)}}$, $\hat P_i=e^{-\hat\theta_2 w_{(i)}}$, $\hat V_1=e^{-\hat\theta_1 w_{(r)}}$, $\hat V_2=e^{-\hat\theta_2 w_{(r)}}$, $\hat\lambda_1=(1-\hat V_1)$ and $\hat\lambda_2=(1-\hat V_2)$.

Case II:

$$\frac{m_J}{\hat{\alpha}_1}+\sum_{i=1}^{J} z_i \ln(1-\hat{U}_i)-\frac{\sum_{i=1}^{J} s_i \hat{\lambda}_1^{\hat{\alpha}_1}\ln\hat{\lambda}_1}{1-\hat{\lambda}_1^{\hat{\alpha}_1}}-\frac{s_J^*\hat{\gamma}_1^{\hat{\alpha}_1}\ln\hat{\gamma}_1}{1-\hat{\gamma}_1^{\hat{\alpha}_1}}=0$$

$$\frac{n_J}{\hat{\alpha}_2}+\sum_{i=1}^{J}(1-z_i)\ln(1-\hat{P}_i)-\frac{\sum_{i=1}^{J} q_i \hat{\lambda}_2^{\hat{\alpha}_2}\ln\hat{\lambda}_2}{1-\hat{\lambda}_2^{\hat{\alpha}_2}}-\frac{q_J^*\hat{\gamma}_2^{\hat{\alpha}_2}\ln\hat{\gamma}_2}{1-\hat{\gamma}_2^{\hat{\alpha}_2}}=0$$

$$\frac{m_J}{\hat{\theta}_1}+(\hat{\alpha}_1-1)\sum_{i=1}^{J}\frac{z_i w_{(i)}\hat{U}_i}{1-\hat{U}_i}-\sum_{i=1}^{J} z_i w_{(i)}-\frac{\sum_{i=1}^{J} s_i\hat{\alpha}_1 w_{(J)}\hat{\lambda}_1^{\hat{\alpha}_1-1}\hat{V}_1}{1-\hat{\lambda}_1^{\hat{\alpha}_1}}-\frac{s_J^*\hat{\alpha}_1 T\hat{\gamma}_1^{\hat{\alpha}_1-1}\hat{A}_1}{1-\hat{\gamma}_1^{\hat{\alpha}_1}}=0$$

$$\frac{n_J}{\hat{\theta}_2}+(\hat{\alpha}_2-1)\sum_{i=1}^{J}\frac{(1-z_i)w_{(i)}\hat{P}_i}{1-\hat{P}_i}-\sum_{i=1}^{J}(1-z_i)w_{(i)}-\frac{\sum_{i=1}^{J} q_i\hat{\alpha}_2 w_{(J)}\hat{\lambda}_2^{\hat{\alpha}_2-1}\hat{V}_2}{1-\hat{\lambda}_2^{\hat{\alpha}_2}}-\frac{q_J^*\hat{\alpha}_2 T\hat{\gamma}_2^{\hat{\alpha}_2-1}\hat{A}_2}{1-\hat{\gamma}_2^{\hat{\alpha}_2}}=0, \tag{6}$$

where $\hat{U}_i=e^{-\hat{\theta}_1 w_{(i)}}$, $\hat{P}_i=e^{-\hat{\theta}_2 w_{(i)}}$, $\hat{V}_1=e^{-\hat{\theta}_1 w_{(J)}}$, $\hat{V}_2=e^{-\hat{\theta}_2 w_{(J)}}$, $\hat{\lambda}_1=(1-\hat{V}_1)$, $\hat{\lambda}_2=(1-\hat{V}_2)$, $\hat{A}_1=e^{-\hat{\theta}_1 T}$, $\hat{A}_2=e^{-\hat{\theta}_2 T}$, $\hat{\gamma}_1=(1-\hat{A}_1)$ and $\hat{\gamma}_2=(1-\hat{A}_2)$.

Then the maximum likelihood estimates of the parameters α_1, α_2, θ_1 and θ_2 can be obtained by solving system of equations (5) and (6). No explicit form for these estimates, a numerical technique using R statistical programming language is considered to obtain $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\theta}_1$ and $\hat{\theta}_2$.

The variance-covariance matrix for the maximum likelihood estimators of α_1, α_2, θ_1 and θ_2 can be obtained by inverting the information matrix with the elements that are negative of the expected values of the second order derivatives of logarithms of the likelihood functions. [21] concluded that the approximate variance covariance matrix may be obtained by replacing expected values (say $\vartheta=(\alpha_1,\alpha_2,\theta_1,\theta_2)$) by their MLEs $\hat{\vartheta}=(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\theta}_1,\hat{\theta}_2)$. Now the Fisher information matrix associated with $\hat{\vartheta}$ is defined as

$$I(\hat{\vartheta})\cong\begin{bmatrix}-\frac{\partial^2\ln L}{\partial\alpha_1^2} & 0 & -\frac{\partial^2\ln L}{\partial\alpha_1\partial\theta_1} & 0\\ 0 & -\frac{\partial^2\ln L}{\partial\alpha_2^2} & 0 & -\frac{\partial^2\ln L}{\partial\alpha_2\partial\theta_2}\\ -\frac{\partial^2\ln L}{\partial\alpha_1\partial\theta_1} & 0 & -\frac{\partial^2\ln L}{\partial\theta_1^2} & 0\\ 0 & -\frac{\partial^2\ln L}{\partial\alpha_2\partial\theta_2} & 0 & -\frac{\partial^2\ln L}{\partial\theta_2^2}\end{bmatrix},$$

where
Case I:

$$-\frac{\partial^2\ln L}{\partial\alpha_1^2}=\frac{m_r}{\alpha_1^2}+\frac{\sum_{i=1}^{r} s_i\lambda_1^{\alpha_1}(\ln\lambda_1)^2}{(1-\lambda_1^{\alpha_1})^2},$$

$$-\frac{\partial^2\ln L}{\partial\alpha_2^2}=\frac{n_r}{\alpha_2^2}+\frac{\sum_{i=1}^{r} q_i\lambda_2^{\alpha_2}(\ln\lambda_2)^2}{(1-\lambda_2^{\alpha_2})^2},$$

$$\begin{aligned}-\frac{\partial^2\ln L}{\partial\theta_1^2}=&\frac{m_r}{\theta_1^2}+(\alpha_1-1)\sum_{i=1}^{r}\frac{z_i(w_{(i)})^2U_i}{(1-U_i)^2}+\sum_{i=1}^{r} s_i\alpha_1 w_{(r)}(1-\lambda_1^{\alpha_1})^{-2}\left[(1-\lambda_1^{\alpha_1})V_1 w_{(r)}\lambda_1^{\alpha_1-1}\right]\\ &\times\left[\left((\alpha_1-1)V_1\lambda_1^{-1}\right)-1\right]+\left[\alpha_1 V_1^2 w_{(r)}\lambda_1^{2(\alpha_1-1)}\right]+\sum_{i=1}^{r} s_i\alpha_1 w_{(r)}(1-\lambda_1^{\alpha_1})^{-2}\\ &\times\left[(1-\lambda_1^{\alpha_1})V_1 w_{(r)}\lambda_1^{\alpha_1-1}\right]\left[\left((\alpha_1-1)V_1\lambda_1^{-1}\right)-1\right]+\left[\alpha_1 V_1^2 w_{(r)}\lambda_1^{2(\alpha_1-1)}\right],\end{aligned}$$

$$-\frac{\partial^2 \ln L}{\partial \theta_2{}^2} = \frac{n_r}{\theta_2{}^2} + (\alpha_2 - 1)\sum_{i=1}^{r} \frac{z_i (w_{(i)})^2 U_i}{(1-U_i)^2} + \sum_{i=1}^{r} q_i \alpha_2 w_{(r)} (1-\lambda_2{}^{\alpha_2})^{-2} \left[(1-\lambda_2{}^{\alpha_2}) V_2 w_{(r)} \lambda_2{}^{\alpha_2 - 1}\right]$$
$$\times \left[\left((\alpha_2-1)V_2\lambda_2{}^{-1}\right)-1\right] + \left[\alpha_2 V_2{}^2 w_{(r)} \lambda_2{}^{2(\alpha_2-1)}\right],$$

$$-\frac{\partial^2 \ln L}{\partial \alpha_1 \partial \theta_1} = -\sum\nolimits_{i=1}^{r} \frac{z_i w_{(i)} U_i}{1-U_i} + \sum\nolimits_{i=1}^{r} s_i V_1 w_{(r)} (1-\lambda_1^{\alpha_1})^{-2} \left[(1-\lambda_1^{\alpha_1})\left(\alpha_1 \lambda_1^{\alpha_1-1} \ln\lambda_1 + \lambda_1^{\alpha_1-1}\right)\right] + \left[\alpha_1 \lambda_1^{2\alpha_1-1}\ln\lambda_1\right],$$

and

$$-\frac{\partial^2 \ln L}{\partial \alpha_2 \partial \theta_2} = -\sum\nolimits_{i=1}^{r} \frac{z_i w_{(i)} U_i}{1-U_i} + \sum\nolimits_{i=1}^{r} q_i V_2 w_{(r)} (1-\lambda_2^{\alpha_2})^{-2} \left[(1-\lambda_2^{\alpha_2})\left(\alpha_2 \lambda_2^{\alpha_2-1} \ln\lambda_2 + \lambda_2^{\alpha_2-1}\right)\right] + \left[\alpha_2 \lambda_2^{2\alpha_2-1}\ln\lambda_2\right]. \tag{7}$$

Case II:

$$-\frac{\partial^2 \ln L}{\partial \alpha_1{}^2} = \frac{m_J}{\alpha_1{}^2} + \frac{\sum_{i=1}^{J} s_i \lambda_1{}^{\alpha_1} (\ln\lambda_1)^2}{(1-\lambda_1{}^{\alpha_1})^2} + \frac{s_J^* \beta_1{}^{\alpha_1}(\ln\beta_1)^2}{(1-\beta_1{}^{\alpha_1})^2},$$

$$-\frac{\partial^2 \ln L}{\partial \alpha_2{}^2} = \frac{n_J}{\alpha_2{}^2} + \frac{\sum_{i=1}^{J} q_i \lambda_2{}^{\alpha_2} (\ln\lambda_2)^2}{(1-\lambda_2{}^{\alpha_2})^2} + \frac{q_J^* \beta_2{}^{\alpha_2}(\ln\beta_2)^2}{(1-\beta_2{}^{\alpha_2})^2},$$

$$-\frac{\partial^2 \ln L}{\partial \theta_1{}^2} = \frac{m_J}{\theta_1{}^2} + (\alpha_1 - 1)\sum\nolimits_{i=1}^{J} \frac{z_i (w_{(i)})^2 U_i}{(1-U_i)^2} + \sum\nolimits_{i=1}^{J} s_i \alpha_1 w_{(J)} (1-\lambda_1{}^{\alpha_1})^{-2} \left[(1-\lambda_1{}^{\alpha_1}) V_1 w_{(J)} \lambda_1{}^{\alpha_1 - 1}\right]$$
$$\times \left[\left((\alpha_1-1)V_1\lambda_1{}^{-1}\right)-1\right] + \left[\alpha_1 V_1{}^2 w_{(J)} \lambda_1{}^{2(\alpha_1-1)}\right] + s_J^* \alpha_1 T (1-\gamma_1{}^{\alpha_1})^{-2}\left[(1-\gamma_1{}^{\alpha_1}) A_1 T \gamma_1{}^{\alpha_1-1}\right]$$
$$\times \left[\left((\alpha_1-1)A_1\gamma_1{}^{-1}\right)-1\right]\left[\alpha_1 A_1{}^2 T \gamma_1{}^{2(\alpha_1-1)}\right],$$

$$-\frac{\partial^2 \ln L}{\partial \theta_2{}^2} = \frac{n}{\theta_2{}^2} + (\alpha_2 - 1)\sum\nolimits_{i=1}^{J} \frac{z_i (w_{(i)})^2 U_i}{(1-U_i)^2} + \sum\nolimits_{i=1}^{J} q_i \alpha_2 w_{(J)} (1-\lambda_2{}^{\alpha_2})^{-2} \left[(1-\lambda_2{}^{\alpha_2}) V_{21} w_{(J)} \lambda_2{}^{\alpha_2 - 1}\right]$$
$$\times \left[\left((\alpha_2-1)V_2\lambda_2{}^{-1}\right)-1\right] + \left[\alpha_2 V_2{}^2 w_{(J)} \lambda_2{}^{2(\alpha_2-1)}\right] + q_J^* \alpha_2 T (1-\gamma_2{}^{\alpha_2})^{-2}\left[(1-\gamma_2{}^{\alpha_2}) A_2 T \gamma_2{}^{\alpha_2-1}\right]$$
$$\times \left[\left((\alpha_2-1)A_2\gamma_2{}^{-1}\right)-1\right]\left[\alpha_2 A_2{}^2 T \gamma_2{}^{2(\alpha_2-1)}\right],$$

$$-\frac{\partial^2 \ln L}{\partial \alpha_1 \partial \theta_1} = -\sum\nolimits_{i=1}^{J} \frac{z_i w_{(i)} U_i}{1-U_i} + \sum\nolimits_{i=1}^{J} s_i V_1 w_{(J)} (1-\lambda_1{}^{\alpha_1})^{-2} \left[(1-\lambda_1{}^{\alpha_1})\left(\alpha_1 \lambda_1{}^{\alpha_1-1} \ln\lambda_1 + \lambda_1{}^{\alpha_1-1}\right)\right]$$
$$+ \left[\alpha_1 \lambda_1{}^{2\alpha_1-1}\ln\lambda_1\right] + s_J^* A_1 T (1-\gamma_1{}^{\alpha_1})^{-2}\left[(1-\gamma_1{}^{\alpha_1})\left(\alpha_1\gamma_1{}^{\alpha_1-1}\ln\gamma_1 + \gamma_1{}^{\alpha_1-1}\right)\right] + \left[\alpha_1 \gamma_1{}^{2\alpha_1-1}\ln\gamma_1\right],$$

and

$$-\frac{\partial^2 \ln L}{\partial \alpha_2 \partial \theta_2} = -\sum\nolimits_{i=1}^{J} \frac{z_i w_{(i)} U_i}{1-U_i} + \sum\nolimits_{i=1}^{J} q_i V_2 w_{(J)} (1-\lambda_2{}^{\alpha_2})^{-2} \left[(1-\lambda_2{}^{\alpha_2})\left(\alpha_2 \lambda_2{}^{\alpha_2-1} \ln\lambda_2 + \lambda_2{}^{\alpha_2-1}\right)\right]$$
$$+ \left[\alpha_2 \lambda_2{}^{2\alpha_2-1}\ln\lambda_2\right] + q_J^* A_2 T (1-\gamma_2{}^{\alpha_2})^{-2}\left[(1-\gamma_2{}^{\alpha_2})\left(\alpha_2\gamma_2{}^{\alpha_2-1}\ln\gamma_2 + \gamma_2{}^{\alpha_2-1}\right)\right] + \left[\alpha_2 \gamma_2{}^{2\alpha_2-1}\ln\gamma_2\right]. \tag{8}$$

Using the asymptotic normality of the MLEs, we can express the approximate $100(1-\varphi)\%$ confidence intervals for α_1, α_2, θ_1 and θ_2. Suppose that $\hat{\delta}$ is the MLE of the parameter vector $\delta = (\alpha_1, \alpha_2, \theta_1, \theta_2)$. Denote the Fisher information matrix corresponding to δ by I_δ and $\phi = \lim_{n\to\infty} n I_\delta^{-1}$. Then, $\hat{\delta}$ is asymptotically normal distributed, i.e., $\sqrt{n}(\hat{\delta}-\delta) \sim N(0,\phi)$ (see [22]). In particular, let $\left(\hat{S}_{\hat{\alpha}_i}\right)^2 = \phi_{(i,i)}/n$ $i = 1,2$ are the (i,i) elements in the matrix $\hat{\phi} = n\hat{I}_\delta^{-1}$ and $\hat{I}_\delta$ is the estimator of I_δ. Therefore, asymptotic normality confidence intervals of δ_i $i = 1,2$, with confidence level $100(1-\varphi)\%$ are given by

$$\hat{\alpha}_i \pm z_{(1-\varphi/2)} \hat{S}_{\hat{\alpha}_i},\ i = 1,2 \quad \text{and} \quad \hat{\theta}_i \pm z_{(1-\varphi/2)} \hat{S}_{\hat{\theta}_i}\ i = 1,2,$$

where $z_{(1-\varphi)}/2$ denotes the upper $(1-\varphi)/2$ percentage point of the standard normal distribution.

4 Bayes estimation

In this section, Bayesian method is used to obtain the estimators for the unknown parameters $\alpha_i,\ i=1,2$ and $\theta_i,\ i=1,2$ using symmetric squared error loss function and asymmetric LINEX loss functions for Case I and Case II. Consider that α_1, α_2, θ_1 and θ_2 have the following independent gamma prior distributions

$$\pi(\alpha_k) \propto \frac{b_k^{a_k}}{\Gamma(a_k)}\alpha_k^{a_k-1}e^{-b_k\alpha_k},\ a_k,b_k,\alpha_k>0,$$

and

$$\pi(\theta_k) \propto \frac{b_k^{a_k}}{\Gamma(a_k)}\theta_k^{a_k-1}e^{-b_k\theta_k},\theta_k>0,\ k=1,2. \tag{9}$$

Here all the hyper parameters a_k and b_k are assumed to be known and non-negative. Combining (9) with equation (1) and using Bayes theorem, the joint posterior density function of α_1, α_2, θ_1 and θ_2 can be written as:

$$\pi(data\backslash\alpha_1,\alpha_2,\theta_1,\theta_2) = \frac{1}{\Psi}L(\alpha_1,\alpha_2,\theta_1,\theta_2,w,z)\pi(\alpha_k)\pi(\theta_k),$$

where
$\Psi = \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty L(\alpha_1,\alpha_2,\theta_1,\theta_2,w,z)\pi(\alpha_k)\pi(\theta_k)d\alpha_k d\theta_k,\ k=1,2.$

Therefore, the Bayes estimator of any function of α_1, α_2, θ_1 and θ_2, say $\delta(\alpha_1,\alpha_2,\theta_1,\theta_2)$ under the squared error (SE) loss function is

$$\begin{aligned}\hat{\delta}_{SE} &= E_{\alpha_1,\alpha_2,\theta_1,\theta_2,data}\left(\delta(\alpha_1,\alpha_2,\theta_1,\theta_2)\right)\\ &= \frac{1}{\Psi}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \delta(\alpha_1,\alpha_2,\theta_1,\theta_2)\\ &\times L(\alpha_1,\alpha_2,\theta_1,\theta_2,w,z)\pi(\alpha_k)\pi(\theta_k)d\alpha_k d\theta_k,\ k=1,2.\end{aligned} \tag{10}$$

Under a LINEX loss function the Bayes estimate of a function $\delta(\alpha_1,\alpha_2,\theta_1,\theta_2)$ is given by

$$\hat{\delta}_{LIN} = -\frac{1}{c}\ln E(e^{-c\delta}),\ c\neq 0, \tag{11}$$

where, for $k=1,2$,

$$\begin{aligned}E(e^{-c\delta}) &= \frac{1}{\Psi}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-c\delta}\\ &\times L(\alpha_1,\alpha_2,\theta_1,\theta_2,w,z)\pi(\alpha_k)\pi(\theta_k)d\alpha_k d\theta_k.\end{aligned}$$

Equations (5), (7), (10) and (11) in Case I and equations (6), (8), (10) and (11) in Case II are difficult to obtain. An iterative procedure is applied to solve these equations numerically. Normally, the ratio of four integrals given by equations (10) and (11) can not to be obtained in a closed form. In this case, one may utilize the MCMC technique to generate samples from the posterior distributions, after that, compute the Bayes estimators of the individual parameters.

In Bayesian inference, compute the full posterior joint distribution over a set of random variables will be desired. Most often posterior distributions will not be standard distributions, because, this often requires calculating intractable integrals, therefore, the analytical equations can be performed with sampling techniques based upon Markov chain Monte Carlo (MCMC) methods using simulated samples from posterior distribution. MCMC is an increasingly popular method for obtaining information about distributions, especially for estimating posterior distributions in Bayesian inference. It is a computer-driven sampling method and allows one to characterize a distribution without knowing all of the distributions mathematical properties by randomly sampling values out of the distribution. Several standard approaches to define such Markov chains exist, namely: Metropolis-Hastings (M-H) algorithm (see [23]) and Gibbs sampling (see [24]). Using these algorithms it is possible to implement posterior simulation in essentially any problem. The Gibbs sampling algorithm is one of the simplest MCMC algorithms. It was introduced by [25] and discussed for Bayesian computations by [24].

5 Numerical illustration

In this section, we carry out some simulation studies to illustrate the finite sample performance of the proposed method under choices of sample sizes and choices of censoring schemes. The ML estimates and the Bayes estimates based on SE and LINEX loss functions are all compared by means of a simulation study, and a numerical example is finally presented to illustrate all the inferential results established in the preceding sections.

5.1 Simulation study

In this subsection, a simulation study is performed to evaluate the behaviour of the results obtained in the previous sections, including the Maximum Likelihood estimates, Bayes estimates and the corresponding confidence/credible intervals by considering choices of values of the parameters.

To run the experiment according to a jointly Type-I progressively hybrid censored sampling from two GE populations, we propose the following algorithm:

Step 1: Set the parameter values of α_1, α_2, θ_1 and θ_2.
Step 2: Generate X and Y independent observations of sizes m and n from $GE_1(\alpha_1,\theta_1)$ $GE_2(\alpha_2,\theta_2)$, respectively.
Step 3: Combine the two generated samples and rearrange them in ascending order.
Step 4: For a specific values of m, n, r and R, generate an ordinary JPCS-I sample using the algorithm proposed by [6].

A large number (1,000) of the JPHC-I samples are generated from both (GE) populations when the true values of the parameters α_i and θ_i for $i = 1,2$ are taken as $(\alpha_1,\alpha_2,\theta_1,\theta_2) = (1.5,2,0.5,0.75)$ for some different combinations of m, n, r, T and R. The sample sizes of both populations are taken as $m = n = 40$(small), 60 (moderate) and 80 (large) for each specified time $T = (1.25,2.5)$. When the number of failed subjects achieves or exceeds a certain value r, where the percentages of failure information $(r/N)\%$ are considered as 25, 50 and 75%, the test is terminated. Also, for each N and r, different censoring schemes (CSs) to remove survival units during the lifetime experiment are assumed as given in Table 1.

The selection value of the hyper-parameters, when an informative prior of the density parameter is taken into account, is the main issue in Bayesian procedure. [26] and [27] suggested ways to overcome this problem. If the proper prior information, i.e., $a_{1i},a_{2i},b_{1i},b_{2i} = 0,\ i = 1,2$ is available, the joint posterior distribution of α_i and $\theta_i,\ i = 1,2$ reduced proportional to the likelihood function (3), therefore, if one does not have prior information on the unknown parameters of interest, it is preferable to use the MLEs instead of the BEs because the latter are computationally more expensive.We used two informative priors of α_i and $\theta_i,\ i = 1,2$ called; prior (1): $(a_{11},a_{12},b_{11},b_{12}) = (1.2,1.5,1.5,2)$, prior (2): $(a_{21},a_{22},b_{21},b_{22}) = (1.5,1.5,1.2,1.8)$. The values of hyper-parameters of the unknown parameters α_i and $\theta_i,\ i = 1,2$ are chosen in such way that the prior mean become the expected value of the corresponding parameter, initial value of LINEX loss function is $(c = 2)$ see [26].

To run the Gibbs sampler algorithm, we started with the MLEs, and then draw samples from various full conditionals, in turn, using the most recent values of all other conditioning variables unless some systematic pattern of convergence was achieved. The algorithm Gibbs sampling can be described as follows:

Step 1: Start with an initial guess $\left(\alpha_1^{(0)} = \alpha_1, \alpha_2^{(0)} = \alpha_2, \theta_1^{(0)} = \theta_1, \text{ and } \theta_2^{(0)} = \theta_2\right)$.
Step 2: Set $t = 1$.
Step 3: Generate $\alpha_1^{(t)}$ using M-H algorithm with normal proposal distribution $q(\alpha_1) = N(\hat{\alpha}_1, \mathrm{var}(\hat{\alpha}_1))$ as follows:
– (a) Let $\varepsilon = \alpha_1^{(t-1)}$. We use $\hat{\alpha}_1$ as $\alpha_1^{(0)}$.
– (b) Generate ω from the proposal distribution.
– (c) Obtain $P(\varepsilon,\omega) = \min\left(1, \frac{q(\varepsilon)g(\omega|w_{(r)})}{q(\omega)g(\varepsilon|w_{(r)})}\right)$.
– (d) Accept ω with probability $P(\varepsilon,\omega)$, or accept ε with probability $1-P(\varepsilon,\omega)$.
Step 4: Generate $\alpha_2^{(t)}$, $\theta_1^{(t)}$ and $\theta_2^{(t)}$.
Step 5: Set $t = t+1$.
Step 6: Repeat steps 2–4, M times, and obtain $U^{(t)} = \left(\alpha_1^{(t)}, \alpha_2^{(t)}, \theta_1^{(t)}, \theta_2^{(t)}\right)$, $t = 1,\ldots,M$.

In order to guarantee the convergence, the first simulated varieties, M_0, of the algorithm may be biased by the initial value, therefore, usually discarded in the beginning of the analysis implementation (burn-in period). Then the selected samples are $U^{(t)} = \left(\alpha_1^{(t)}, \alpha_2^{(t)}, \theta_1^{(t)}, \theta_2^{(t)}\right)$ for $t = M_0+1,\ldots,M$, are sufficiently large M.

Hence, the BE of $U=(\alpha_1,\alpha_2,\theta_1,\theta_2)$ based on SE loss is given by

$$\tilde{U}_{SE}=\sum_{t=M_0+1}^{M}U^{(t)}/(M-M_0),$$

where M_0 is burn-in.

To construct the symmetric BCIs of α_1, α_2, θ_1 and θ_2, sort the generated MCMC samples (after burn-in-period) in ascending order for $U^{(t)}=\left(\alpha_1^{(t)},\alpha_2^{(t)},\theta_1^{(t)},\theta_2^{(t)}\right)$ for $t=M_0+1,\ldots,M$ as $U_{(M_o+1)},U_{(M_o+2)},\ldots,U_{(M)}$, then the $100(1-2\varphi)\%$ two-sided BCIs of $U=(\alpha_1,\alpha_2,\theta_1,\theta_2)$ is given by

$$\left(U_{(\varphi M)},U_{((1-\varphi)M)}\right).$$

Hence, (12,000) MCMC samples are generated and discard the first (2,000) values as 'burn-in'. Hence, the average Bayes MCMC estimates and 95% two-sided credible intervals are computed based on (10,000) MCMC samples. The average values of all the estimates, the mean squared error (MSE), the confidence lengths and the coverage probabilities (CPs) are obtained.

The average point values of the MLEs and BEs under the SE and LINEX loss functions of α_i and θ_i for $i=1,2$ with their MSEs are computed and reported in Tables 2-5. In addition, the associated average confidence lengths (ACLs) of 95% ACIs/BCIs also are summarized in Tables 6-9. All required computational algorithms are coded in R statistical programming language software version 3.6.1 with 'maxLik' package proposed by [28]. Also, the computations of Bayesian MCMC estimates were performed using 'CODA' package proposed by [29].

From the numerical results established in Tables 2-9, MSEs and ACLs (with their CPs) of the proposed estimators are used from which we can make the following observations.

It is clear that the MLEs and BEs of α_i and θ_i for $i=1,2$ are good estimators in term of minimum MSEs, as expect, using gamma informative prior in this example, the BEs are better (as they include prior information) than MLEs in respect of their MSE values. As N increases, MSEs of $\alpha_i,\theta_i,\ i=1,2$ reduced significantly while that associated with $\alpha_i,\theta_i,\ i=1,2$ increase. Further, when T increases, the MSEs associated with scale parameters $\theta_i,\ i=1,2$ increase while associated with shape parameter $\alpha_i,\ i=1,2$ decrease. In most cases, when the failure percent $(r/N)\%$ increases, the point estimates become even better as expected.

As the effective increase in sample sizes, the ACLs associated with 95% ACIs/BCIs narrow down as expected. Also, when T increases, the ACLs of both ACIs and BCIs tend to decrease. It is also observed that as the failure proportion increases, the ACLs for scale parameters $\theta_i,\ i=1,2$ increase while decrease for the shape parameter $\alpha_i,\ i=1,2$. Moreover, when the failure proportion increases, it is observed that the performance of the scale parameters $\theta_i,\ i=1,2$ become better in contrast to the shape parameter $\alpha_i,\ i=1,2$ in terms of their ACLs values.

The credible intervals estimates of the unknown parameters are better than asymptotic confidenceintervals in terms of their ACLs due to they include prior information. Since the variance of prior (2) is smaller than prior (1), it can be seen that the Bayes (point and interval) estimates based on prior (2) has perform better than prior (1) in terms of minimum MSEs and ACLs for each setting.

Comparing the four different CSs which is better, it is clear that the MSEs associated with all estimates for the unknown parameters α_i and θ_i for $i=1,2$ are greater based on the CS-IV than the other censoring schemes. Because the expected duration of the experiments for CS-I (where remaining $N-r$ units are withdrawn in first stage) and CS-II (where remaining $N-r$ units are withdrawn After each stages), and CS-III (where remaining $N-r$ units are withdrawn in both first and last stages), are greater than the CS-IV (where remaining $N-r$ units are withdrawn at the $r-th$ failure occur).Therefore, we recommend the Bayesian MCMC estimation of the GE population parameters using the hybrid Gibbs within M-H algorithm sampler.

5.2 Real-life data analysis

In this subsection we analyze a set of data that arose in tests on the endurance of deep grove ball bearings. They were discussed by [30] and [31]. The data are the number of revolutions in millions before failure for each of the 23 ball bearings in the life test and they are: 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04 and 173.40.

To illustrate the findings of the paper, the data are divided into two samples by randomly sampling $m=13$ observations and considering these observations as the group X, and the remaining $n=10$ observations are taken as the group Y, (see Table 10). Firstly, before analyzing the datasets of Table 10, one question arises about whether the data sets fit the GE distribution or not. Thus, we use the MLEs to obtain the Kolmogorov-Smirnov (K-S) distance and the corresponding p-value for each group. Using the failure times of group X, the MLEs of the model parameters α_1 and θ_1 become 3.9201 and 0.0355, respectively. Similarly, using the failure times of group Y, the MLEs of the model parameters α_2 and θ_2 become 5.0678 and 0.0305, respectively. Hence, the K-S distance (with associate p-value) for each group X and Y is 0.3072(0.1718) and 0.2255(0.0545), respectively. These results indicate that the GE distribution fits Gupta's and Kundu's datasets quite well(see Table 11).

For fitting the assumed two data sets graphically, we plot the empirical cdfs and the corresponding fitted cdfs for the GE distribution, also, we plot the histogram and the corresponding fitted pdf lines for same distribution. Figure 1 showed the fitted lines for the cdfs and pdfs for the given two data sets and corresponding GE distribution. The figures also indicate that the GE distribution provide better fit at least for this data set.

From the original data set given in Table 10, one can generate, e.g., three JPHC-I samples with different number of stages $r = (10,15,23)$ at time censoring $T = 54$ and removed items R_j are assumed as given in Table 12.

Because no any prior information is available about the model parameters, the BEs are developed with a non-informative prior, i.e., $a_{1i}, a_{2i}, b_{1i}, b_{2i} = 0,\ i = 1,2$. Here, we have used two informative priors of α_i and $\theta_i,\ i = 1,2$ called prior (1): $(a_{11}, a_{12}, b_{11}, b_{12}) = (1.2, 1.5, 1.5, 2)$, prior (2): $(a_{21}, a_{22}, b_{21}, b_{22}) = (1.5, 1.5, 1.2, 1.8)$. Initial value of LINEX loss function with $(c = 2)$.Using the hybrid Gibbs within M-H sampler algorithm described in Section 4, (12,000) MCMC samples are generated and discard the first (2,000) values as 'burn-in'. Hence, the average Bayes MCMC estimates and 95% two-sided credible intervals are computed based on (10,000) MCMC samples. The MLEs, BEs (with their standard errors) and two-sided 95% ACIs/BCIs of the unknown parameters α_i and $\theta_i,\ i = 1,2$ are computed and reported in Table 13 and 14.

It is evident from the estimates that the generated posteriors of the unknown GE population parameters α_i and $\theta_i,\ i = 1,2$ are fairly symmetric and corresponds well to the theoretical posterior density functions. The analysis of these real data sets shows the flexibility of the proposed JPHC-I scheme to remove reliable items at any time and terminate the experiment at $min(X_{(r)}, T)$, and furthermore, compromise between reduce the experiment time and the observation of at least some extreme lifetimes sought.

For each data set, the marginal posterior density estimates of α_i and $\theta_i,\ i = 1,2$ based on (10,000) chain values using the Gaussian kernel are plotted in Figure 2. Similarly, in each histogram plot, it is evident from the estimates that the generated posteriors of the unknown GE population parameters α_i and $\theta_i,\ i = 1,2$ are fairly symmetric and corresponds well to the theoretical posterior density functions. To assess the convergence of 10000 MCMC outputs, trace plots of the conditional posterior distributions of α_i and $\theta_i,\ i = 1,2$ for each data set are shown in Figure 2. It indicates that the MCMC procedure converges very well.

6 Conclusions

In this article, the maximum likelihood and Bayesian estimation based on SE and LINEX loss functions for the unknown parameters of two GE distributions has been discussed based on a new scheme called the JPHC-I scheme. The ML estimates and the Bayesian estimates have then been compared through a simulation study, and a numerical example has also been presented to illustrate all the inferential results established here. The computational results show that the Bayesian estimation based on the SE and LINEX loss functions is more precise than the ML estimation. As N increases, the MSEs of α_i and $\theta_i,\ i = 1,2$ reduced significantly while that associated with α_i and $\theta_i,\ i = 1,2$ increase. Further, when T increases, the MSEs associated with scale parameters $\theta_i,\ i = 1,2$ increase while associated with shape parameter $\alpha_i,\ i = 1,2$ decrease. In most cases, when the failure percent $(r/N)\%$ increases, the point estimates become even better as expected.

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Table 1: Removal patterns of units in various censoring schemes.

(m,n)	r	Censoring Schemes			
		I	II	III	IV
(40,40)	20	$(60, 0^{*19})$	(3^{*20})	$(30, 0^{*18}, 30)$	$(0^{*19}, 60)$
	40	$(40, 0^{*39})$	(1^{*40})	$(20, 0^{*38}, 20)$	$(0^{*39}, 40)$
	60	$(20, 0^{*59})$	$(1^{*20}, 0^{*40})$	$(10, 0^{*58}, 10)$	$(0^{*59}, 20)$
(60,60)	30	$(90, 0^{*29})$	(3^{*30})	$(45, 0^{*28}, 45)$	$(0^{*29}, 90)$
	60	$(60, 0^{*59})$	(1^{*60})	$(30, 0^{*58}, 30)$	$(0^{*59}, 60)$
	90	$(30, 0^{*89})$	$(1^{*30}, 0^{*60})$	$(15, 0^{*88}, 15)$	$(0^{*89}, 30)$
(80,80)	40	$(120, 0^{*39})$	(3^{*40})	$(60, 0^{*38}, 60)$	$(0^{*39}, 120)$
	80	$(80, 0^{*79})$	(1^{*80})	$(40, 0^{*88}, 40)$	$(0^{*79}, 80)$
	120	$(40, 0^{*119})$	$(1^{*40}, 0^{*80})$	$(20, 0^{*118}, 20)$	$(0^{*119}, 40)$

Here, $(1^{*3},0)$, for example, means that the censoring scheme employed is (1,1,1,0).

Table 2: The average estimates (MSEs) of $\alpha_1 = 1.5$ under choices of censoring scheme and choices of T's.

(m,n)	r	Scheme	MLEs		BEs			
					SE		LINEX	
$T \rightarrow$			1.25	2.5	1.25	2.5	1.25	2.5
(40,40)	20	I	6.0948 (3.9210)	6.1129 (3.9653)	2.3873 (0.5916)	2.3874 (0.5883)	1.7048 (0.2570)	1.7100 (0.2561)
		II	4.5692 (2.1645)	4.6193 (2.1945)	1.6978 (0.0812)	1.7102 (0.0814)	1.3973 (0.0338)	1.4184 (0.0327)
		III	3.7916 (1.4377)	3.8136 (1.4489)	1.5427 (0.0418)	1.5479 (0.0423)	1.3139 (0.0266)	1.3175 (0.0267)
		IV	2.2993 (0.3913)	2.3030 (0.3915)	1.2453 (0.0214)	1.2477 (0.0212)	1.0943 (0.0261)	1.0956 (0.0260)
	40	I	2.9844 (0.6187)	3.8290 (1.1181)	1.6920 (0.0594)	2.2069 (0.1385)	1.4338 (0.0649)	1.9201 (0.0743)
		II	2.1975 (0.1870)	3.8451 (1.1287)	1.4696 (0.0531)	2.0871 (0.0922)	1.2131 (0.0352)	1.7801 (0.0373)
		III	2.1322 (0.1404)	3.9881 (1.2729)	1.4269 (0.0183)	2.0557 (0.0736)	1.2679 (0.0166)	1.7604 (0.0336)
		IV	1.8890 (0.1037)	3.8280 (1.1728)	1.3298 (0.0191)	1.8566 (0.0402)	1.1892 (0.0198)	1.5860 (0.0178)
	60	I	2.1469 (0.1625)	2.2829 (0.1585)	1.4321 (0.0176)	1.7207 (0.0754)	1.2768 (0.0164)	1.3460 (0.0736)
		II	2.2671 (0.2144)	2.3295 (0.1608)	1.4810 (0.0201)	1.9466 (0.3874)	1.3167 (0.0164)	1.2797 (0.0872)
		III	1.9424 (0.1231)	1.9035 (0.0711)	1.3574 (0.0180)	1.5264 (0.0191)	1.2135 (0.0183)	1.3864 (0.0151)
		IV	1.8128 (0.0700)	1.7560 (0.0502)	1.2990 (0.0162)	1.4359 (0.0163)	1.1689 (0.0188)	1.3150 (0.0142)
(60,60)	30	I	5.3515 (2.7578)	5.3578 (2.7608)	2.6940 (0.5539)	2.6908 (0.5504)	2.1046 (0.3029)	2.1127 (0.3020)
		II	4.3172 (1.6049)	4.3330 (1.6161)	2.0015 (0.0866)	2.0037 (0.0867)	1.7115 (0.0363)	1.7150 (0.0363)
		III	3.2042 (0.7086)	3.2108 (0.7100)	1.7036 (0.0321)	1.7059 (0.0321)	1.4943 (0.0163)	1.4960 (0.0162)
		IV	2.0557 (0.1604)	2.0568 (0.1605)	1.3716 (0.0158)	1.3719 (0.0158)	1.2263 (0.0166)	1.2264 (0.0166)
	60	I	2.6296 (0.2738)	3.4699 (0.6459)	1.7986 (0.0556)	2.4080 (0.1573)	1.5806 (0.0373)	2.1589 (0.0969)
		II	2.0421 (0.0948)	3.5277 (0.6884)	1.5574 (0.0580)	2.3429 (0.1183)	1.3571 (0.0182)	2.0677 (0.0600)
		III	1.9333 (0.0731)	3.3682 (0.5966)	1.4827 (0.0149)	2.1903 (0.0832)	1.3549 (0.0128)	1.9382 (0.0407)
		IV	1.7375 (0.0456)	3.3754 (0.6514)	1.3754 (0.0142)	2.0813 (0.0636)	1.2622 (0.0148)	1.8263 (0.0297)
	90	I	2.0228 (0.0828)	2.1636 (0.0867)	1.5461 (0.0175)	1.8239 (0.0865)	1.4107 (0.0129)	1.4807 (0.0433)
		II	2.0242 (0.0795)	2.1502 (0.0839)	1.5506 (0.0171)	1.9098 (0.1878)	1.4150 (0.0126)	1.4108 (0.1338)
		III	1.8279 (0.0590)	1.8315 (0.0365)	1.4234 (0.0145)	1.5943 (0.0148)	1.3023 (0.0135)	1.4898 (0.0108)
		IV	1.7157 (0.0386)	1.6441 (0.0212)	1.3753 (0.0129)	1.4527 (0.0106)	1.2632 (0.0136)	1.3637 (0.0098)
(80,80)	40	I	4.7079 (1.8418)	4.7075 (1.8416)	2.8193 (0.5204)	2.8187 (0.5199)	2.3714 (0.3146)	2.3729 (0.3145)
		II	3.8249 (0.9892)	3.8320 (0.9923)	2.1810 (0.0993)	2.1839 (0.0991)	1.9082 (0.0491)	1.9123 (0.0489)
		III	2.8161 (0.3847)	2.8169 (0.3848)	1.7782 (0.0332)	1.7784 (0.0332)	1.6007 (0.0182)	1.6009 (0.0182)
		IV	1.8791 (0.0600)	1.8793 (0.0600)	1.4258 (0.0126)	1.4258 (0.0126)	1.2997 (0.0123)	1.2998 (0.0123)
	80	I	2.4008 (0.1536)	3.1733 (0.3965)	1.8505 (0.0435)	2.4608 (0.1450)	1.6767 (0.0307)	2.2550 (0.0974)
		II	1.9330 (0.0524)	3.2908 (0.4614)	1.5940 (0.0195)	2.4635 (0.1326)	1.4621 (0.0144)	2.2389 (0.0819)
		III	1.8607 (0.0478)	3.1838 (0.4553)	1.5255 (0.0133)	2.3025 (0.0989)	1.4172 (0.0105)	2.0830 (0.0580)
		IV	1.6561 (0.0249)	3.1099 (0.4064)	1.4068 (0.0111)	2.2045 (0.0810)	1.3119 (0.0112)	1.9786 (0.0443)
	120	I	1.9050 (0.0544)	2.0668 (0.0635)	1.5543 (0.0157)	1.8538 (0.0766)	1.4450 (0.0120)	1.6145 (0.0429)
		II	1.9331 (0.0477)	2.0687 (0.0550)	1.5940 (0.0144)	1.9048 (0.1520)	1.4794 (0.0103)	1.5622 (0.0660)
		III	1.7361 (0.0286)	1.7612 (0.0228)	1.4535 (0.0097)	1.5851 (0.0108)	1.3546 (0.0093)	1.5031 (0.0080)
		IV	1.6750 (0.0276)	1.5984 (0.0130)	1.4095 (0.0110)	1.4607 (0.0080)	1.3156 (0.0110)	1.3911 (0.0075)

Table 3: The average estimates (MSEs) of $\theta_1 = 0.5$ under choices of censoring scheme and choices of T's.

(m,n)	r	Scheme	MLEs		BEs			
					SE		LINEX	
$T \rightarrow$			1.25	2.5	1.25	2.5	1.25	2.5
(40,40)	20	I	4.0707 (1.4158)	4.0815 (1.4206)	1.8004 (0.2790)	1.7972 (0.2775)	1.3907 (0.1684)	1.3923 (0.1682)
		II	2.2581 (0.4195)	2.2854 (0.4247)	0.9703 (0.0457)	0.9725 (0.0454)	0.8396 (0.0317)	0.8431 (0.0303)
		III	2.2413 (0.4067)	2.2559 (0.4089)	1.0663 (0.0535)	1.0734 (0.0538)	0.9196 (0.0336)	0.9257 (0.0338)
		IV	0.9018 (0.0406)	0.9055 (0.0406)	0.4817 (0.0044)	0.4848 (0.0043)	0.4340 (0.0040)	0.4368 (0.0039)
	40	I	1.5053 (0.1434)	1.9729 (0.2381)	0.9528 (0.0403)	1.4070 (0.0947)	0.8703 (0.0370)	1.3147 (0.0776)
		II	0.7915 (0.0214)	1.7415 (0.1778)	0.5216 (0.0149)	1.1787 (0.0543)	0.4873 (0.00940	1.0977 (0.0428)
		III	0.8470 (0.0244)	2.0785 (0.2756)	0.5702 (0.0059)	1.4086 (0.0903)	0.5301 (0.0049)	1.3027 (0.0711)
		IV	0.6004 (0.0098)	2.0339 (0.2608)	0.4096 (0.0041)	1.3509 (0.0784)	0.3847 (0.0043)	1.2524 (0.0622)
	60	I	0.8548 (0.0236)	0.9362 (0.0282)	0.5771 (0.0058)	0.8267 (0.0495)	0.5381 (0.0047)	0.7401 (0.0412)
		II	0.8002 (0.0205)	0.8601 (0.0210)	0.5291 (0.0048)	0.7846 (0.0546)	0.4954 (0.0042)	0.7157 (0.0466)
		III	0.6946 (0.0129)	0.7076 (0.0112)	0.4795 (0.0039)	0.5924 (0.0054)	0.4483 (0.0037)	0.5729 (0.0049)
		IV	0.5746 (0.0075)	0.5732 (0.0071)	0.3942 (0.0035)	0.4811 (0.0040)	0.3706 (0.0038)	0.4660 (0.0037)
(60,60)	30	I	3.8875 (1.2445)	3.8915 (1.2456)	2.2825 (0.4006)	2.2775 (0.3986)	1.8926 (0.2763)	1.8903 (0.2739)
		II	2.1913 (0.3578)	2.1987 (0.3588)	1.1624 (0.0604)	1.1642 (0.0603)	1.0442 (0.0434)	1.0462 (0.0434)
		III	2.0487 (0.2921)	2.0557 (0.2929)	1.2359 (0.0682)	1.2408 (0.0685)	1.1068 (0.0485)	1.1111 (0.0487)
		IV	0.8548 (0.0265)	0.8561 (0.0265)	0.5530 (0.0041)	0.5539 (0.0041)	0.5091 (0.0032)	0.5099 (0.0032)
	60	I	1.4306 (0.1122)	1.9504 (0.2253)	1.0529 (0.0475)	1.5555 (0.1206)	0.9723 (0.0417)	1.4856 (0.1060)
		II	0.7515 (0.0142)	1.6887 (0.1544)	0.5495 (0.0051)	1.3029 (0.0708)	0.5315 (0.0074)	1.2418 (0.0607)
		III	0.7886 (0.0151)	1.9617 (0.2276)	0.6016 (0.0050)	1.5149 (0.1090)	0.5701 (0.0041)	1.4411 (0.0963)
		IV	0.5591 (0.0044)	1.9582 (0.2265)	0.4280 (0.0028)	1.4896 (0.1033)	0.4077 (0.0029)	1.4085 (0.0874)
	90	I	0.8336 (0.0184)	0.9163 (0.0230)	0.6376 (0.0061)	0.8280 (0.0343)	0.6052 (0.0049)	0.7831 (0.0280)
		II	0.7508 (0.0128)	0.8188 (0.0153)	0.5682 (0.0043)	0.8176 (0.0676)	0.5401 (0.0036)	0.6948 (0.0336)
		III	0.6564 (0.0080)	0.6780 (0.0064)	0.4994 (0.0030)	0.6036 (0.0036)	0.4746 (0.0028)	0.5904 (0.0032)
		IV	0.5531 (0.0040)	0.5328 (0.0029)	0.4274 (0.0025)	0.4755 (0.0021)	0.4074 (0.0027)	0.4651 (0.0021)
(80,80)	40	I	3.7489 (1.1322)	3.7484 (1.1320)	2.5706 (0.5109)	2.5697 (0.5105)	2.3026 (0.4062)	2.3040 (0.4063)
		II	2.0637 (0.2867)	2.0673 (0.2873)	1.3120 (0.0805)	1.3130 (0.0804)	1.1995 (0.0619)	1.2024 (0.0619)
		III	1.9485 (0.2445)	1.9494 (0.2446)	1.3345 (0.0825)	1.3351 (0.0825)	1.2230 (0.0631)	1.2236 (0.0631)
		IV	0.7993 (0.0179)	0.7997 (0.0179)	0.5808 (0.0038)	0.5811 (0.0038)	0.5436 (0.0030)	0.5439 (0.0029)
	80	I	1.3640 (0.0933)	1.8994 (0.2051)	1.0970 (0.0512)	1.6210 (0.1333)	1.0297 (0.0409)	1.5673 (0.1214)
		II	0.7264 (0.0103)	1.6642 (0.1449)	0.5898 (0.0073)	1.3768 (0.0823)	0.5628 (0.0042)	1.3307 (0.0742)
		III	0.7737 (0.0129)	1.9324 (0.2154)	0.6283 (0.0052)	1.5894 (0.1238)	0.6024 (0.0044)	1.5270 (0.1102)
		IV	0.5396 (0.0030)	1.9132 (0.2097)	0.4438 (0.0021)	1.5596 (0.1169)	0.4270 (0.0022)	1.4931 (0.1028)
	120	I	0.7987 (0.0147)	0.8903 (0.0193)	0.6495 (0.0061)	0.8720 (0.0479)	0.6237 (0.0051)	0.7849 (0.0203)
		II	0.7281 (0.0095)	0.8006 (0.0120)	0.5882 (0.0037)	0.7686 (0.0288)	0.5648 (0.0031)	0.7136 (0.0232)
		III	0.6364 (0.0054)	0.6571 (0.0047)	0.5208 (0.0022)	0.6000 (0.0028)	0.5005 (0.0020)	0.5891 (0.0026)
		IV	0.5483 (0.0031)	0.5205 (0.0016)	0.4473 (0.0021)	0.4778 (0.0013)	0.4308 (0.0022)	0.4698 (0.0013)

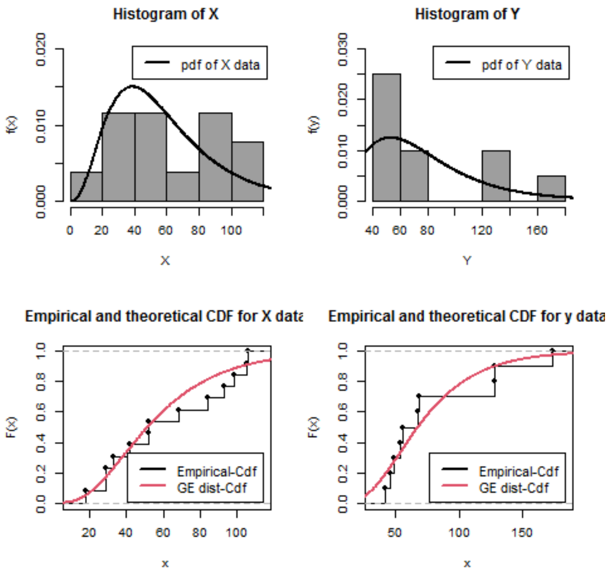

Fig. 1: Estimated pdf and cdf for the given two data set with corresponding GE distribution.

Table 4: The average estimates (MSEs) of $\alpha_2 = 2$ under choices of censoring scheme and choices of T's.

(m,n)	r	Scheme	MLEs		BEs			
					SE		LINEX	
$T \rightarrow$			1.25	2.5	1.25	2.5	1.25	2.5
(40,40)	20	I	7.4853 (5.8901)	7.4816 (5.8806)	2.7789 (0.9344)	2.7876 (0.9312)	1.8068 (0.3145)	1.8186 (0.3147)
		II	4.3574 (1.5845)	4.3169 (1.5472)	1.9980 (0.1006)	2.0244 (0.1003)	1.6540 (0.0669)	1.6869 (0.0628)
		III	2.8359 (0.4312)	2.8305 (0.4283)	1.6222 (0.0730)	1.6205 (0.0728)	1.4147 (0.0667)	1.4148 (0.0665)
		IV	2.4079 (0.2098)	2.4058 (0.2089)	1.4636 (0.0525)	1.4667 (0.0523)	1.2929 (0.0669)	1.2956 (0.0666)
	40	I	4.4667 (1.6270)	4.6843 (1.5147)	2.0619 (0.1113)	2.5755 (0.1339)	1.6321 (0.0593)	2.1720 (0.0575)
		II	3.3734 (0.5802)	3.2867 (0.3861)	1.7675 (0.0462)	2.2531 (0.0603)	1.4715 (0.0569)	1.9937 (0.0382)
		III	2.7098 (0.2243)	2.2278 (0.0648)	1.6649 (0.0339)	1.6950 (0.0336)	1.4515 (0.0434)	1.5370 (0.0381)
		IV	2.3188 (0.1456)	1.8762 (0.0376)	1.5250 (0.0414)	1.5438 (0.0393)	1.3392 (0.0551)	1.4255 (0.0462)
	60	I	2.7940 (0.2613)	3.4638 (0.6172)	1.7108 (0.0332)	2.0745 (0.0757)	1.4923 (0.0410)	1.5876 (0.0809)
		II	3.1642 (0.4263)	4.3611 (1.1379)	1.8521 (0.0335)	2.3706 (0.1431)	1.6040 (0.0362)	1.6457 (0.1763)
		III	2.4471 (0.1818)	2.5395 (0.1244)	1.5729 (0.0384)	1.9082 (0.0244)	1.3842 (0.0509)	1.6982 (0.0252)
		IV	2.2477 (0.1128)	2.1037 (0.0396)	1.5017 (0.0420)	1.6821 (0.0252)	1.3225 (0.0570)	1.5263 (0.0325)
(60,60)	30	I	6.8532 (4.9402)	6.8359 (4.9143)	3.0888 (0.7303)	3.0842 (0.7252)	2.2963 (0.3746)	2.3070 (0.3730)
		II	3.7601 (0.7365)	3.7454 (0.7182)	2.2110 (0.0830)	2.2117 (0.0823)	1.9197 (0.0408)	1.9246 (0.0399)
		III	2.5148 (0.1927)	2.5133 (0.1925)	1.7088 (0.0382)	1.7080 (0.0383)	1.5305 (0.0393)	1.5303 (0.0393)
		IV	2.1932 (0.0829)	2.1924 (0.0828)	1.5643 (0.0370)	1.5650 (0.0371)	1.4121 (0.0470)	1.4125 (0.0470)
	60	I	3.9440 (0.8664)	4.3325 (0.9800)	2.3231 (0.0804)	2.9319 (0.1961)	1.9802 (0.0481)	2.5786 (0.1070)
		II	3.0408 (0.2691)	3.0692 (0.2016)	2.0283 (0.0340)	2.4055 (0.0550)	1.7677 (0.0648)	2.1847 (0.0308)
		III	2.4602 (0.1052)	2.0916 (0.0324)	1.8007 (0.0249)	1.7456 (0.0250)	1.6111 (0.0292)	1.6246 (0.0289)
		IV	2.1728 (0.0507)	1.7916 (0.0210)	1.6515 (0.0288)	1.5742 (0.0302)	1.4877 (0.0372)	1.4905 (0.0358)
	90	I	2.6103 (0.1523)	3.2395 (0.3085)	1.8748 (0.0300)	2.3309 (0.1009)	1.6719 (0.0276)	1.9619 (0.1487)
		II	2.9201 (0.2357)	4.0288 (0.6639)	2.0343 (0.0303)	2.6893 (0.2509)	1.8056 (0.0230)	1.9815 (0.0940)
		III	2.2921 (0.0692)	2.4292 (0.0606)	1.7053 (0.0260)	2.0229 (0.0194)	1.5334 (0.0331)	1.8569 (0.0159)
		IV	2.1208 (0.0567)	2.0458 (0.0238)	1.6149 (0.0305)	1.7662 (0.0181)	1.4621 (0.0396)	1.6392 (0.0220)
(80,80)	40	I	5.9498 (3.0998)	5.9500 (3.0998)	3.2565 (0.5523)	3.2582 (0.5519)	2.6674 (0.3050)	2.6663 (0.3048)
		II	3.6117 (0.5944)	3.6046 (0.5907)	2.3969 (0.0877)	2.3990 (0.0872)	2.1291 (0.0444)	2.1307 (0.0441)
		III	2.3132 (0.0747)	2.3128 (0.0746)	1.7588 (0.0272)	1.7591 (0.0272)	1.6098 (0.0304)	1.6101 (0.0304)
		IV	2.0910 (0.0506)	2.0910 (0.0506)	1.6248 (0.0297)	1.6249 (0.0297)	1.4888 (0.0372)	1.4889 (0.0372)
	80	I	3.4227 (0.4443)	3.8749 (0.5473)	2.3743 (0.0639)	2.9576 (0.1748)	2.0891 (0.0438)	2.6809 (0.1095)
		II	2.9120 (0.1753)	3.0175 (0.1644)	2.1814 (0.0658)	2.5133 (0.0602)	1.9464 (0.0581)	2.3423 (0.0390)
		III	2.3933 (0.0663)	2.0465 (0.0196)	1.8812 (0.0211)	1.8001 (0.0178)	1.7158 (0.0215)	1.7009 (0.0200)
		IV	2.0763 (0.0384)	1.7448 (0.0187)	1.6920 (0.0250)	1.5855 (0.0274)	1.5563 (0.0307)	1.5198 (0.0315)
	120	I	2.5102 (0.0862)	3.1679 (0.2330)	1.9582 (0.0217)	2.5539 (0.1589)	1.7868 (0.0202)	2.2022 (0.0503)
		II	2.7203 (0.1248)	3.8976 (0.5482)	2.1089 (0.0275)	2.9686 (0.3972)	1.9107 (0.0190)	2.3376 (0.1209)
		III	2.1953 (0.0386)	2.3932 (0.0433)	1.7658 (0.0196)	2.0806 (0.0162)	1.6206 (0.0247)	1.9428 (0.0123)
		IV	2.1437 (0.0445)	2.0697 (0.0204)	1.7289 (0.0238)	1.8443 (0.0146)	1.5867 (0.0285)	1.7373 (0.0162)

Table 5: The average estimates (MSEs) of $\theta_2 = 0.75$ under choices of censoring scheme and choices of T's.

(m,n)	r	Scheme	MLEs		BEs			
					SE		LINEX	
$T \rightarrow$			1.25	2.5	1.25	2.5	1.25	2.5
(40,40)	20	I	4.6174 (1.7316)	4.6081 (1.7248)	2.0554 (0.3199)	2.0558 (0.3187)	1.5177 (0.1671)	1.5224 (0.1670)
		II	1.6156 (0.1235)	1.5972 (0.1193)	0.8247 (0.0220)	0.8215 (0.0149)	0.7329 (0.0126)	0.7427 (0.0120)
		III	1.0661 (0.0341)	1.0619 (0.0338)	0.6105 (0.0098)	0.6078 (0.0099)	0.5628 (0.0101)	0.5608 (0.0102)
		IV	0.7528 (0.0145)	0.7506 (0.0145)	0.4292 (0.0143)	0.4286 (0.0143)	0.3961 (0.0160)	0.3956 (0.0161)
	40	I	2.2390 (0.3165)	2.1998 (0.2338)	1.2779 (0.0589)	1.5929 (0.0853)	1.1255 (0.0388)	1.4916 (0.0674)
		II	1.3287 (0.0666)	1.1250 (0.0223)	0.7462 (0.0086)	0.8489 (0.0059)	0.7219 (0.0238)	0.8172 (0.0058)
		III	1.1536 (0.0323)	0.7716 (0.0038)	0.7705 (0.0072)	0.6111 (0.0047)	0.7150 (0.0064)	0.5886 (0.0052)
		IV	0.8084 (0.0089)	0.4793 (0.0096)	0.5449 (0.0078)	0.3940 (0.0149)	0.5094 (0.0090)	0.3807 (0.0152)
	60	I	1.2043 (0.0376)	1.4608 (0.0733)	0.8144 (0.0078)	1.0684 (0.0593)	0.7570 (0.0066)	0.9967 (0.0501)
		II	1.2297 (0.0428)	1.6229 (0.1057)	0.8005 (0.0087)	1.1068 (0.0785)	0.7449 (0.0076)	0.9880 (0.0625)
		III	0.9462 (0.0151)	1.0011 (0.0133)	0.6382 (0.0064)	0.8241 (0.0042)	0.5953 (0.0070)	0.7989 (0.0047)
		IV	0.8046 (0.0090)	0.7615 (0.0029)	0.5475 (0.0081)	0.6424 (0.0031)	0.5114 (0.0093)	0.6226 (0.0035)
(60,60)	30	I	4.2805 (1.3912)	4.2634 (1.3713)	2.4478 (0.3809)	2.4474 (0.3797)	2.0380 (0.2538)	2.0439 (0.2543)
		II	1.4755 (0.0820)	1.4706 (0.0811)	0.9224 (0.0152)	0.9187 (0.0146)	0.8522 (0.0111)	0.8522 (0.0110)
		III	0.9808 (0.0183)	0.9785 (0.0183)	0.6616 (0.0062)	0.6601 (0.0062)	0.6195 (0.0064)	0.6182 (0.0065)
		IV	0.7006 (0.0085)	0.6996 (0.0085)	0.4710 (0.0110)	0.4705 (0.0110)	0.4421 (0.0123)	0.4416 (0.0123)
	60	I	2.0834 (0.2273)	2.1344 (0.2062)	1.4555 (0.0749)	1.7220 (0.1044)	1.3264 (0.0563)	1.6528 (0.0909)
		II	1.2478 (0.0419)	1.0833 (0.0159)	0.8707 (0.0088)	0.8977 (0.0056)	0.8286 (0.0118)	0.8723 (0.0048)
		III	1.0938 (0.0219)	0.7379 (0.0024)	0.8349 (0.0066)	0.6270 (0.0035)	0.7897 (0.0055)	0.6106 (0.0038)
		IV	0.7956 (0.0056)	0.4640 (0.0095)	0.6059 (0.0052)	0.4027 (0.0132)	0.5761 (0.0059)	0.3953 (0.0137)
	90	I	1.1518 (0.0265)	1.4233 (0.0583)	0.8793 (0.0078)	1.1789 (0.0632)	0.8332 (0.0063)	1.1307 (0.0548)
		II	1.1850 (0.0307)	1.5845 (0.0863)	0.8798 (0.0082)	1.2445 (0.0851)	0.8342 (0.0067)	1.1499 (0.0776)
		III	0.9194 (0.0099)	0.9901 (0.0095)	0.6989 (0.0043)	0.8732 (0.0042)	0.6632 (0.0045)	0.8526 (0.0036)
		IV	0.7775 (0.0053)	0.7566 (0.0019)	0.5937 (0.0054)	0.6754 (0.0020)	0.5648 (0.0062)	0.6605 (0.0022)
(80,80)	40	I	4.0569 (1.1824)	4.0574 (1.1825)	2.7533 (0.4747)	2.7551 (0.4748)	2.4524 (0.3592)	2.4517 (0.3590)
		II	1.4510 (0.0698)	1.4479 (0.0692)	1.0095 (0.0170)	1.0093 (0.0169)	0.9576 (0.0154)	0.9524 (0.0131)
		III	0.9268 (0.0113)	0.9266 (0.0113)	0.6885 (0.0045)	0.6885 (0.0045)	0.6539 (0.0047)	0.6539 (0.0047)
		IV	0.6609 (0.0062)	0.6607 (0.0062	0.4865 (0.0094)	0.4864 (0.0094)	0.4622 (0.0105)	0.4621 (0.0105)
	80	I	1.9535 (0.1754)	2.0590 (0.1805)	1.5131 (0.0736)	1.7657 (0.1108)	1.4328 (0.0677)	1.7168 (0.1011)
		II	1.2114 (0.0333)	1.0705 (0.0136)	0.9320 (0.0098)	0.9294 (0.00590	0.8923 (0.0097)	0.9109 (0.0052)
		III	1.0851 (0.0184)	0.7301 (0.0017)	0.8791 (0.0066)	0.6490 (0.0025)	0.8418 (0.0055)	0.6363 (0.0027)
		IV	0.7616 (0.0044)	0.4504 (0.0099)	0.6206 (0.0044)	0.4051 (0.0128)	0.5971 (0.0050)	0.3994 (0.0132)
	120	I	1.1238 (0.0216)	1.4039 (0.0528)	0.9121 (0.0077)	1.2103 (0.0343)	0.8751 (0.0064)	1.1846 (0.0363)
		II	1.1397 (0.0238)	1.5751 (0.0814)	0.9155 (0.0084)	1.3222 (0.0722)	0.8771 (0.0070)	1.2259 (0.0664)
		III	0.8966 (0.0072)	0.9929 (0.0092)	0.7293 (0.0034)	0.9013 (0.0048)	0.7002 (0.0034)	0.8835 (0.0043)
		IV	0.7842 (0.0045)	0.7625 (0.0015)	0.6347 (0.0041)	0.6974 (0.0015)	0.6102 (0.0045)	0.6856 (0.0016)

Table 6: The ACLs (CPs) for 95% asymptotic/credible intervals of $(\alpha_1, \alpha_2) = (1.5, 2)$ under choices of censoring scheme and $T = 1.25$.

(m,n)	r	Scheme	ACI		BCI			
					SE		LINEX	
$T \rightarrow$			$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_1$	$\hat{\alpha}_2$
(40,40)	20	I	14.438 (93.42)	18.012 (93.65)	5.8231 (95.06)	8.2673 (95.05)	3.3760 (95.05)	3.6260 (95.24)
		II	11.426 (94.04)	10.647 (95.13)	2.6454 (96.06)	3.2040 (95.44)	2.0579 (96.53)	2.5538 (96.64)
		III	9.7153 (94.79)	6.5635 (95.68)	1.9810 (96.06)	2.1328 (95.57)	1.4295 (96.48)	1.7035 (95.17)
		IV	5.8473 (96.60)	5.1329 (95.90)	1.3087 (95.80)	1.6218 (96.10)	1.0371 (96.30)	1.3440 (96.30)
	40	I	6.8985 (95.69)	10.725 (95.67)	2.5668 (97.64)	2.9812 (96.59)	2.4233 (95.07)	2.7244 (95.06)
		II	4.5036 (95.80)	7.2539 (94.89)	1.7690 (96.38)	2.2730 (97.51)	1.9362 (95.03)	2.3427 (97.20)
		III	3.9306 (95.49)	5.1729 (95.10)	1.5556 (95.60)	1.7124 (96.00)	1.1902 (97.10)	1.4064 (95.50)
		IV	3.6908 (96.30)	4.5644 (96.90)	1.4482 (95.80)	1.5689 (95.80)	1.2096 (97.20)	1.2551 (95.60)
	60	I	4.3012 (96.40)	5.5221 (95.50)	1.4702 (96.60)	1.7244 (96.10)	1.2363 (96.10)	1.4140 (96.10)
		II	4.7126 (95.90)	6.5078 (96.00)	1.6332 (95.80)	2.0033 (95.70)	1.3145 (96.80)	1.6642 (96.40)
		III	3.9378 (97.09)	4.9413 (96.99)	1.4678 (96.19)	1.5682 (95.49)	1.2045 (96.19)	1.2540 (95.99)
		IV	3.0429 (94.29)	4.0505 (96.10)	1.2665 (96.89)	1.4845 (96.49)	1.0234 (96.69)	1.1945 (96.70)
(60,60)	30	I	12.352 (94.16)	16.823 (94.50)	5.5002 (95.97)	6.7768 (95.42)	4.4111 (95.03)	4.6362 (95.10)
		II	9.9024 (94.92)	7.8106 (94.77)	2.4597 (95.34)	2.8735 (95.65)	1.9127 (95.67)	2.3135 (95.77)
		III	7.2141 (94.98)	5.0425 (97.19)	1.8333 (95.79)	1.8412 (95.69)	1.4829 (95.79)	1.5913 (95.59)
		IV	4.2870 (96.50)	3.4895 (95.40)	1.3822 (95.80)	1.5736 (96.60)	1.1320 (96.30)	1.2958 (95.90)
	60	I	4.7416 (94.79)	8.2779 (95.29)	2.4267 (97.24)	3.1117 (97.12)	2.4937 (98.17)	3.0415 (98.25)
		II	3.1733 (95.90)	4.9734 (96.20)	1.5987 (96.82)	2.0923 (96.88)	1.7307 (98.50)	2.2050 (98.69)
		III	2.8911 (95.60)	3.5952 (95.90)	1.4769 (95.40)	1.6560 (95.90)	1.2171 (96.00)	1.3852 (96.30)
		IV	2.4787 (95.80)	2.7092 (94.60)	1.3142 (96.20)	1.5969 (96.10)	1.1321 (96.20)	1.2804 (95.30)
	90	I	2.9203 (94.90)	4.2076 (96.40)	1.4513 (95.50)	1.8004 (96.20)	1.2598 (95.60)	1.4323 (96.40)
		II	2.8299 (95.50)	4.8202 (95.60)	1.3956 (96.00)	2.0284 (95.40)	1.2379 (96.00)	1.5840 (95.70)
		III	2.7245 (96.10)	3.0553 (95.50)	1.3565 (95.80)	1.4864 (95.80)	1.1314 (96.50)	1.2197 (96.70)
		IV	2.2858 (95.30)	2.9154 (96.80)	1.2452 (95.60)	1.4080 (95.20)	1.0535 (95.80)	1.1771 (95.20)
(80,80)	40	I	10.298 (95.61)	13.645 (94.69)	5.2702 (95.60)	6.1253 (95.19)	4.3994 (96.13)	4.4551 (95.43)
		II	7.9786 (96.30)	7.1743 (96.90)	2.3969 (95.99)	2.7218 (95.59)	1.9240 (96.10)	2.1354 (95.49)
		III	5.6681 (95.10)	3.1591 (96.10)	1.8287 (96.20)	1.6200 (95.40)	1.5304 (95.90)	1.3960 (95.70)
		IV	2.6490 (94.89)	2.7669 (95.80)	1.3028 (96.90)	1.4580 (95.70)	1.0888 (96.50)	1.2706 (95.70)
	80	I	3.3382 (94.89)	6.0994 (96.10)	2.1296 (96.80)	2.5145 (95.76)	2.0846 (98.19)	2.6707 (98.29)
		II	2.2739 (95.69)	3.7653 (95.59)	1.4537 (96.20)	1.9686 (95.99)	1.4313 (97.59)	1.7787 (98.09)
		III	2.3135 (95.90)	2.7974 (95.40)	1.3654 (96.60)	1.6039 (96.00)	1.1372 (96.00)	1.3392 (96.30)
		IV	1.8584 (95.70)	2.4133 (95.30)	1.1730 (95.90)	1.4012 (95.50)	1.0154 (95.70)	1.1979 (95.50)
	120	I	2.4176 (95.70)	3.0429 (96.10)	1.3982 (95.70)	1.7150 (95.90)	1.2242 (95.90)	1.4348 (96.30)
		II	2.1086 (95.60)	3.3488 (95.30)	1.3262 (96.30)	1.8685 (96.50)	1.1854 (96.00)	1.5751 (96.30)
		III	1.8812 (95.10)	2.3139 (96.20)	1.1630 (96.80)	1.3974 (97.30)	0.9697 (96.20)	1.1704 (96.90)
		IV	1.9423 (95.90)	2.5546 (94.60)	1.1924 (96.60)	1.4821 (96.30)	1.0236 (96.80)	1.2649 (96.20)

Table 7: The ACLs (CPs) for 95% asymptotic/credible intervals of $(\alpha_1,\alpha_2) = (1.5,2)$ under choices of censoring scheme and $T = 2.5$.

(m,n)	r	Scheme	ACI		BCI			
					SE		LINEX	
$T \rightarrow$			$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_1$	$\hat{\alpha}_2$
(40,40)	20	I	14.452 (93.53)	17.998 (93.78)	5.8231 (95.09)	8.2673 (95.08)	3.3760 (95.06)	3.6605 (95.04)
		II	11.472 (94.14)	10.550 (95.23)	2.5707 (95.63)	3.1166 (95.12)	1.8757 (96.04)	2.2864 (95.45)
		III	9.7409 (94.89)	6.5474 (95.78)	1.9966 (96.17)	2.1268 (95.67)	1.4368 (96.48)	1.6766 (95.37)
		IV	5.8491 (96.60)	5.1259 (96.00)	1.3087 (95.80)	1.6186 (96.10)	1.0398 (96.20)	1.3440 (96.20)
	40	I	8.5341 (95.69)	10.210 (95.99)	2.8130 (95.47)	3.2964 (95.58)	2.3379 (95.99)	2.5573 (96.09)
		II	8.5628 (95.58)	5.8244 (95.40)	2.4020 (95.70)	2.4361 (95.60)	1.7977 (95.70)	2.0227 (95.90)
		III	9.0022 (95.78)	3.0287 (95.60)	2.1711 (96.29)	1.7843 (96.20)	1.6932 (97.19)	1.4852 (96.20)
		IV	8.7531 (95.57)	2.3546 (95.20)	1.9443 (96.29)	1.4991 (95.49)	1.4614 (95.59)	1.2877 (95.50)
	60	I	3.8664 (96.80)	7.4003 (96.80)	2.6770 (95.12)	2.6005 (98.72)	2.3141 (95.01)	2.7234 (95.11)
		II	3.7607 (96.30)	9.0855 (95.39)	3.3137 (95.09)	3.7368 (95.14)	2.4784 (95.04)	3.1240 (95.01)
		III	2.9028 (96.30)	3.8276 (96.50)	1.5148 (95.78)	1.7447 (96.06)	1.2854 (96.40)	1.4360 (96.39)
		IV	2.5919 (96.70)	2.4350 (95.30)	1.3619 (96.70)	1.4562 (95.80)	1.1463 (96.00)	1.2031 (95.90)
(60,60)	30	I	12.353 (94.16)	16.788 (94.51)	5.4627 (96.21)	6.7450 (95.34)	4.4111 (95.03)	4.5911 (96.56)
		II	9.9261 (94.82)	7.7334 (94.67)	2.4597 (95.37)	2.8644 (95.88)	1.9012 (95.68)	2.2878 (95.57)
		III	7.2169 (94.98)	5.0413 (97.19)	1.8328 (95.79)	1.8412 (95.69)	1.4829 (95.79)	1.5913 (95.59)
		IV	4.2883 (96.50)	3.4884 (95.40)	1.3917 (95.60)	1.5832 (96.10)	1.1334 (96.20)	1.2994 (95.90)
	60	I	6.2970 (94.30)	8.1885 (95.80)	3.0276 (95.40)	3.5708 (95.60)	2.4795 (95.40)	2.9608 (96.00)
		II	6.5307 (94.80)	3.6651 (95.60)	2.5906 (95.60)	2.2658 (95.30)	1.8847 (95.70)	1.9357 (95.80)
		III	6.1709 (95.19)	2.2018 (95.50)	2.1995 (96.19)	1.4846 (96.00)	1.7614 (95.69)	1.3307 (97.40)
		IV	6.7700 (96.09)	1.5995 (95.69)	2.0876 (95.59)	1.3100 (95.89)	1.5873 (95.99)	1.1660 (96.29)
	90	I	2.5627 (95.90)	4.8808 (96.10)	2.3099 (98.38)	2.4926 (97.55)	2.3720 (95.01)	3.0870 (95.04)
		II	2.5294 (95.10)	6.2291 (95.00)	2.9835 (95.04)	3.2286 (98.69)	2.4965 (95.06)	3.3797 (95.09)
		III	1.9805 (96.10)	2.5481 (95.10)	1.3785 (96.19)	1.7055 (95.79)	1.2295 (95.80)	1.4416 (95.60)
		IV	1.7146 (95.80)	1.9043 (95.60)	1.1675 (96.60)	1.3291 (96.10)	1.0052 (96.70)	1.1156 (95.90)
(80,80)	40	I	10.298 (95.61)	13.644 (94.69)	5.2702 (95.60)	5.9183 95.19)	4.3856 (95.73)	4.4551 (95.43)
		II	7.9847 (96.30)	7.1591 (96.90)	2.3886 (95.70)	2.7018 95.70)	1.9097 (96.10)	2.1244 (95.40)
		III	5.6682 (95.10)	3.1564 (96.10)	1.8287 (96.20)	1.6307 95.50)	1.5304 (95.90)	1.3960 (95.70)
		IV	2.6484 (94.89)	2.7669 (95.80)	1.3028 (96.90)	1.4580 95.70)	1.0888 (96.50)	1.2706 (95.70)
	80	I	4.2338 (95.40)	5.4877 (95.70)	2.5967 (95.50)	3.2139 95.70)	2.2465 (95.50)	2.7500 (95.40)
		II	4.6518 (96.20)	3.0605 (96.40)	2.3451 (96.70)	2.1942 95.50)	1.9567 (96.30)	1.9319 (96.00)
		III	5.1400 (96.00)	1.7256 (95.80)	2.1570 (97.30)	1.3704 96.20)	1.8192 (95.30)	1.2527 (96.00)
		IV	4.7588 (94.60)	1.3695 (95.60)	2.0635 (95.90)	1.1673 96.50)	1.6962 (95.40)	1.1064 (95.90)
	120	I	2.1978 (94.90)	3.8554 (95.50)	1.7728 (97.39)	2.3814 96.73)	2.4919 (95.06)	3.1413 (95.84)
		II	1.8661 (95.40)	5.3787 (95.20)	2.0252 (98.38)	2.6756 96.87)	2.3997 (95.04)	3.6684 (95.02)
		III	1.5668 (95.80)	2.0708 (96.20)	1.1469 (96.30)	1.5150 97.09)	1.0183 (96.50)	1.3353 (97.30)
		IV	1.3629 (95.40)	1.7484 (95.80)	1.0286 (95.50)	1.2679 95.80)	0.9438 (95.80)	1.1069 (95.90)

Table 8: The ACLs (CPs) for 95% asymptotic/credible intervals of $(\theta_1,\theta_2)=(0.5,0.75)$ under choices of censoring scheme and $T=1.25$.

(m,n)	r	Scheme	ACI		BCI			
					SE		LINEX	
$T\rightarrow$			$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_1$	$\hat{\alpha}_2$
(40,40)	20	I	4.6544 (95.58)	6.0238 (95.54)	3.9138 (95.09)	4.4374 (95.09)	2.9743 (95.26)	3.2763 (95.01)
		II	4.1204 (95.75)	2.7325 (95.86)	1.6748 (96.17)	1.4659 (96.07)	1.5338 (95.04)	1.3553 (95.93)
		III	3.9903 (95.17)	1.9242 (95.78)	1.5739 (95.67)	0.9968 (96.37)	1.3841 (95.87)	0.9182 (95.88)
		IV	1.8715 (95.40)	1.4939 (94.80)	0.7574 (95.90)	0.7087 (96.00)	0.6808 (95.70)	0.6575 (96.20)
	40	I	2.5515 (95.29)	3.8175 (95.49)	1.7952 (95.07)	1.7950 (98.19)	1.6868 (95.10)	1.9495 (95.05)
		II	1.4087 (96.30)	2.2565 (95.10)	0.9333 (96.31)	1.0969 (97.74)	0.9603 (95.05)	1.2569 (95.10)
		III	1.3794 (96.20)	1.5715 (96.50)	0.8287 (95.90)	1.0056 (96.80)	0.7842 (96.50)	0.9430 (96.10)
		IV	1.1630 (97.10)	1.1462 (96.90)	0.6585 (95.60)	0.7272 (96.20)	0.6243 (95.50)	0.6773 (96.60)
	60	I	1.3027 (97.20)	1.6154 (96.10)	0.8677 (96.80)	1.0062 (96.20)	0.8049 (96.50)	0.9463 (96.20)
		II	1.3265 (95.80)	1.7457 (96.10)	0.8464 (97.10)	1.1083 (96.60)	0.7929 (97.20)	1.0390 (96.60)
		III	1.1864 (96.39)	1.3172 (96.19)	0.7536 (96.69)	0.8442 (95.79)	0.6858 (97.60)	0.7981 (96.09)
		IV	1.0307 (96.80)	1.1555 (95.80)	0.5850 (96.29)	0.7498 (96.49)	0.5521 (96.60)	0.7136 (96.80)
(60,60)	30	I	3.8621 (95.07)	4.7186 (95.70)	3.5036 (97.14)	3.4671 (96.82)	3.6175 (97.33)	3.5740 (95.67)
		II	3.3217 (95.48)	2.1250 (95.680	1.5125 (96.36)	1.2510 (96.06)	1.3551 (96.48)	1.1403 (96.18)
		III	2.8350 (95.49)	1.4136 (96.09)	1.4326 (96.29)	0.8677 (95.39)	1.2977 (95.69)	0.8095 (95.39)
		IV	1.4615 (95.80)	1.1247 (95.90)	0.7555 (97.30)	0.6834 (97.20)	0.6879 (96.40)	0.6316 (96.60)
	60	I	1.9831 (96.19)	2.7595 (95.79)	1.4875 (97.03)	1.7582 (96.58)	1.5920 (97.88)	1.9984 (97.95)
		II	1.1011 (96.20)	1.6231 (95.90)	0.8022 (97.68)	0.9854 (96.36)	0.8497 (96.58)	1.0732 (97.17)
		III	1.0236 (96.70)	1.2471 (96.90)	0.7617 (96.80)	0.9066 (97.30)	0.7387 (96.60)	0.8619 (97.50)
		IV	0.7927 (96.00)	0.9092 (96.50)	0.6009 (95.80)	0.6678 (97.00)	0.5751 (97.00)	0.6404 (97.10)
	90	I	1.0600 (96.40)	1.2651 (96.80)	0.7819 (96.20)	0.9424 (96.60)	0.7514 (95.80)	0.9003 (96.50)
		II	0.9999 (96.50)	1.3468 (96.30)	0.7374 (96.000	0.9701 (97.90)	0.6995 (96.20)	0.9448 (97.90)
		III	0.9208 (96.10)	1.0436 (96.60)	0.6795 (96.60)	0.7500 (95.90)	0.6454 (98.10)	0.7187 (96.30)
		IV	0.7570 (96.00)	0.8978 (96.40)	0.5605 (96.90)	0.6702 (96.00)	0.5352 (96.40)	0.6352 (96.30)
(80,80)	40	I	3.4331 (96.14)	3.6966 (95.75)	3.5068 (97.04)	3.2599 (96.42)	3.6059 (96.04)	3.1669 (96.65)
		II	2.5482 (96.00)	1.7808 (96.10)	1.4283 (97.49)	1.1963 (95.49)	1.3429 (97.00)	1.1235 (96.00)
		III	2.3096 (95.70)	1.1203 (96.70)	1.3574 (96.70)	0.7916 (95.50)	1.2705 (96.60)	0.7505 (96.50)
		IV	1.1737 (96.70)	0.9122 (95.70)	0.6690 (97.00)	0.6017 (96.40)	0.6265 (97.00)	0.5696 (97.00)
	80	I	1.6939 (95.70)	2.1668 (96.70)	1.4158 (95.98)	1.4405 (96.80)	1.4045 (97.09)	1.5167 (98.39)
		II	0.8962 (96.69)	1.3612 (95.99)	0.7148 (96.21)	0.9742 (95.99)	0.7430 (97.79)	1.0001 (96.89)
		III	0.9134 (96.40)	1.0509 (96.60)	0.7120 (96.90)	0.8621 (97.40)	0.6879 (96.30)	0.8219 (96.40)
		IV	0.6659 (96.30)	0.8173 (96.10)	0.5318 (97.50)	0.6322 (96.60)	0.5082 (96.00)	0.6160 (96.10)
	120	I	0.9443 (97.30)	1.0853 (96.40)	0.7422 (97.00)	0.8415 (96.40)	0.7249 (96.70)	0.8272 (96.50)
		II	0.8132 (96.10)	1.1531 (95.90)	0.6614 (97.40)	0.8861 (96.00)	0.6350 (96.90)	0.8606 (96.10)
		III	0.7376 (95.90)	0.8783 (96.90)	0.5835 (96.30)	0.7021 (96.60)	0.5619 (96.30)	0.6763 (97.20)
		IV	0.6632 (96.10)	0.8228 (95.20)	0.5208 (97.00)	0.6376 (96.10)	0.5000 (97.10)	0.6265 (97.90)

Table 9: The ACLs (CPs) for 95% asymptotic/credible intervals of $(\theta_1,\theta_2)=(0.5,0.75)$ under choices of censoring scheme and $T=2.5$.

(m,n)	r	Scheme	ACI		BCI			
					SE		LINEX	
$T\rightarrow$			$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_1$	$\hat{\alpha}_2$
(40,40)	20	I	4.6062 (95.47)	6.0289 (95.44)	3.8875 (95.02)	4.4277 (95.01)	2.9743 (95.28)	3.2763 (95.02)
		II	4.0371 (95.75)	2.7030 (95.96)	1.6233 (95.94)	1.3644 (95.63)	1.4275 (95.45)	1.2660 (95.96)
		III	3.9338 (95.07)	1.9259 (95.78)	1.5493 (96.57)	0.9968 (96.37)	1.3519 (96.78)	0.9182 (95.88)
		IV	1.8691 (95.40)	1.4947 (94.80)	0.7424 (96.00)	0.7087 (96.00)	0.6688 (95.60)	0.6575 (96.20)
	40	I	1.8036 (95.20)	1.9053 (96.00)	1.2827 (95.38)	1.3242 (96.88)	1.1964 (96.90)	1.2551 (96.89)
		II	1.9085 (95.60)	1.1234 (95.60)	1.0678 (97.30)	0.8396 (96.00)	0.9783 (96.30)	0.8101 (96.20)
		III	2.0151 (96.30)	0.7613 (95.60)	1.0732 (95.89)	0.6180 (96.69)	0.9932 (96.80)	0.6107 (96.60)
		IV	1.9823 (95.90)	0.5850 (94.99)	0.9198 (97.19)	0.4835 (96.99)	0.8544 (96.80)	0.4628 (96.30)
	60	I	1.1849 (95.50)	1.8715 (95.40)	1.3131 (95.09)	1.1026 (98.31)	1.3230 (95.06)	1.8143 (95.03)
		II	1.1152 (94.30)	2.1295 (95.60)	1.4020 (95.05)	1.5392 (95.14)	1.5285 (95.06)	1.9497 (95.09)
		III	1.0295 (94.20)	1.0362 (96.30)	0.8420 (95.48)	0.7393 (96.27)	0.8125 (95.60)	0.7345 (96.70)
		IV	1.0034 (95.10)	0.6632 (96.20)	0.7683 (95.20)	0.5424 (95.70)	0.7325 (95.50)	0.5200 (95.80)
(60,60)	30	I	3.8306 (94.87)	4.5879 (95.39)	3.5036 (97.16)	3.4197 (96.73)	3.5135 (96.93)	3.5740 (95.68)
		II	3.2865 (95.48)	2.1194 (95.78)	1.4726 (96.38)	1.2435 (96.08)	1.3282 (96.88)	1.1371 (95.98)
		III	2.7983 (95.49)	1.4187 (96.09)	1.4012 (96.59)	0.8677 (95.39)	1.2598 (95.69)	0.8095 (95.39)
		IV	1.4562 (95.70)	1.1256 (95.90)	0.7431 (97.20)	0.6834 (97.20)	0.6728 (96.50)	0.6316 (96.60)
	60	I	1.5154 (95.90)	1.4945 (95.90)	1.1891 (96.70)	1.1699 (96.30)	1.1294 (96.50)	1.1596 (96.00)
		II	1.4220 (96.00)	0.8601 (95.80)	0.9644 (95.90)	0.7099 (97.10)	0.9013 (96.20)	0.7026 (96.90)
		III	1.4635 (96.50)	0.6055 (96.60)	0.9231 (96.99)	0.5158 (97.40)	0.8766 (96.79)	0.5059 (97.10)
		IV	1.4592 (95.69)	0.4450 (96.09)	0.8800 (95.69)	0.4076 (97.19)	0.8248 (96.29)	0.3957 (96.59)
	90	I	0.9370 (95.60)	1.4137 (96.10)	1.2666 (95.08)	0.9127 (96.98)	1.3248 (95.04)	1.7607 (95.09)
		II	0.8888 (95.40)	1.6003 (95.10)	1.4191 (95.05)	1.3578 (97.52)	1.3080 (95.08)	1.9958 (95.11)
		III	0.7075 (96.20)	0.7531 (96.10)	0.5801 (95.79)	0.6163 (96.89)	0.5784 (95.70)	0.6020 (97.10)
		IV	0.6516 (98.00)	0.5415 (96.90)	0.4266 (96.50)	0.4658 (97.20)	0.4196 (96.70)	0.4581 (97.70)
(80,80)	40	I	3.4353 (96.14)	3.6926 (95.75)	3.5068 (97.04)	3.2161 (96.32)	3.5782 (96.04)	3.1669 (96.66)
		II	2.5325 (96.00)	1.7775 (96.10)	1.4226 (97.50)	1.1920 (95.50)	1.3248 (96.20)	1.1211 (96.10)
		III	2.3053 (95.70)	1.1200 (96.70)	1.3508 (96.80)	0.7916 (95.50)	1.2644 (96.80)	0.7505 (96.50)
		IV	1.1716 (96.70)	0.9131 (95.70)	0.6668 (96.80)	0.6017 (96.40)	0.6203 (96.90)	0.5703 (96.30)
	80	I	1.1901 (96.40)	1.1898 (96.00)	1.0734 (96.70)	1.0585 (96.10)	1.0666 (96.60)	1.0621 (96.10)
		II	1.1989 (96.40)	0.7142 (96.10)	0.8979 (95.60)	0.6347 (96.40)	0.8719 (96.90)	0.6188 (96.50)
		III	1.2544 (95.60)	0.5119 (96.20)	0.8642 (96.70)	0.4656 (97.10)	0.8455 (95.60)	0.4621 (97.20)
		IV	1.2383 (96.30)	0.3804 (96.10)	0.7953 (96.70)	0.3750 (96.30)	0.7567 (95.90)	0.3706 (97.60)
	120	I	0.7954 (96.40)	1.2428 (96.00)	1.0110 (96.63)	0.8357 (96.99)	1.1374 (97.08)	1.4225 (97.37)
		II	0.6800 (96.30)	1.4336 (95.10)	1.1655 (95.01)	1.0051 (96.70)	1.1313 (95.04)	1.7123 (95.07)
		III	0.5866 (97.20)	0.7128 (96.40)	0.4803 (96.30)	0.6242 (96.40)	0.4721 (96.10)	0.6088 (96.70)
		IV	0.4830 (98.40)	0.4838 (96.70)	0.3613 (96.50)	0.4166 (97.50)	0.3548 (96.70)	0.4113 (97.40)

Table 10: Failure times of ball bearings divided into two samples.

Data: X
17.88, 28.92, 28.92, 33.00, 41.52, 51.84, 51.96, 68.64, 84.12, 93.12, 98.64, 105.12, 105.84
Data: Y
42.12, 45.60, 48.80, 54.12, 55.56, 67.80, 68.88, 127.92, 128.04, 173.40

Table 11: The MLEs and fitting K-S of GE under ball bearings data.

Data	Parameter	MLE	K-S	
			Statistic	p-value
X	α_1	3.9201	0.3072	0.1718
	θ_1	0.0355		
Y	α_2	5.0678	0.2255	0.5045
	θ_2	0.0305		

Table 12: Removal patterns of units in various censoring schemes.

(m,n)	r	Censoring Schemes		
		I	II	III
(13,10)	10	$(13, 0^{*9})$	$(1^{*9}, 4)$	$(0^{*9}, 13)$
	15	$(8, 0^{*14})$	$(1^{*8}, 0^{**7})$	$(0^{*14}, 8)$
	20	$(3, 0^{*19})$	$(1^{*3}, 0^{*17})$	$(0^{*19}, 3)$

Here, $(1^{*3},0)$, for example, means that the censoring scheme employed is (1,1,1,0).

Table 13: MLEs, BEs, and standard errors (St.E) for real data set based on JPHC-I under choices of censoring schemes and $T = 54$.

r	Scheme	Parameter	MLEs		BEs			
					SEL		LINEX	
			Estimate	St.E	Estimate	St.E	Estimate	St.E
10	I	α_1	12.1870	8.2197	2.2697	0.9892	1.6444	1.0266
		θ_1	0.0780	0.0237	0.0315	0.0128	0.0313	0.0129
		α_2	$4.5168\times10^{+2}$	6.3504×10^{-8}	1.5933	0.9747	1.0735	0.9646
		θ_2	0.1954	0.0125	0.0260	0.0144	0.0258	0.0142
	II	α_1	13.4761	11.3817	1.9726	0.9769	1.3716	0.9984
		θ_1	0.0760	0.0269	0.0265	0.0133	0.0263	0.0133
		α_2	$2.4995\times10^{+2}$	5.8164×10^{-7}	1.8789	1.0515	1.2586	1.1006
		θ_2	0.1086	0.0067	0.0130	0.0081	0.0130	0.0084
	III	α_1	4.2806	3.1544	1.5749	0.6995	1.2300	0.7108
		θ_1	0.0349	0.0165	0.0177	0.0079	0.0177	0.0079
		α_2	59.4325	2.0289×10^{-6}	2.1577	1.0955	1.5146	1.0679
		θ_2	0.0738	0.0059	0.0150	0.0069	0.0149	0.0070
15	I	α_1	8.8980	7.0987	2.0594	1.0868	1.4102	1.0239
		θ_1	0.0662	0.0253	0.0288	0.0145	0.0286	0.0138
		α_2	$1.5038\times10^{+2}$	8.4845×10^{-7}	2.4109	1.1306	1.7068	1.1622
		θ_2	0.0956	0.0061	0.0185	0.0077	0.0185	0.0077
	II	α_1	12.7369	10.8033	2.0447	0.9539	1.5189	0.9309
		θ_1	0.0739	0.0268	0.0280	0.0125	0.0278	0.0125
		α_2	$4.4293\times10^{+2}$	3.3169×10^{-7}	2.1613	1.1576	1.4795	1.1908
		θ_2	0.1223	0.0068	0.0175	0.0080	0.0174	0.0081
	III	α_1	4.2721	3.1406	1.5902	0.8290	1.1925	0.8146
		θ_1	0.0349	0.0165	0.0174	0.0088	0.0173	0.0091
		α_2	60.7585	1.9871×10^{-6}	1.6784	0.9667	1.1391	1.0019
		θ_2	0.0743	0.0059	0.0112	0.0073	0.0111	0.0077
20	I	α_1	5.5655	4.1752	1.7117	0.8161	1.2801	0.7978
		θ_1	0.0455	0.0194	0.0218	0.0105	0.0217	0.0102
		α_2	$1.5282\times10^{+2}$	8.3504×10^{-7}	2.5995	1.1916	1.7993	1.2012
		θ_2	0.0959	0.0061	0.0196	0.0078	0.0195	0.0082
	II	α_1	6.8097	5.2861	1.7454	0.8323	1.3049	0.9528
		θ_1	0.0518	0.0213	0.0220	0.0110	0.0218	0.0116
		α_2	$1.5444\times10^{+2}$	8.2643×10^{-7}	2.3534	1.1992	1.5659	1.1720
		θ_2	0.0961	0.0061	0.0176	0.0083	0.0175	0.0084
	III	α_1	4.2854	3.1590	1.4898	0.8005	1.0713	0.8135
		θ_1	0.0350	0.0165	0.0159	0.0097	0.0158	0.0095
		α_2	59.5062	2.0265×10^{-6}	1.7252	0.9815	1.1885	0.9871
		θ_2	0.0739	0.0059	0.0111	0.0072	0.0111	0.0071

Table 14: Associated interval estimates for ML and Bayesian for real data set based on JPHC-I under choices of censoring schemes and $T = 54$.

r	Scheme	Parameter	ACI	BCI	
				SE	LINEX
10	I	α_1	(7.0924,10.2815)	(0.6623,4.2289)	(0.6641,4.3641)
		θ_1	(0.0633,0.0927)	(0.0093,0.0555)	(0.0100,0.0566)
		α_2	(6.0036,6.0036)	(0.1846,3.3927)	(0.1638,3.3260)
		θ_2	(0.1876,0.2032)	(0.0019,0.0520)	(0.0018,0.0508)
	II	α_1	(6.0402,20.9120)	(0.4161,3.8361)	(0.4283,3.9174)
		θ_1	(0.0584,0.0936)	(0.0023,0.0503)	(0.0024,0.0515)
		α_2	(8.6013,8.6013)	(0.3723,3.9009)	(0.3637,4.0865)
		θ_2	(0.10423,0.1130)	(0.0011,0.0286)	(0.0010,0.0292)
	III	α_1	(2.2197,6.3415)	(0.3475,2.9210)	(0.3380,2.9528)
		θ_1	(0.0241,0.0458)	(0.0034,0.0333)	(0.0023,0.0320)
		α_2	(5.9391,5.9391)	(0.5386,4.3820)	(0.5130,4.2956)
		θ_2	(0.0700,0.0777)	(0.0035,0.0291)	(0.0041,0.0300)
15	I	α_1	(4.4982,3.2977)	(0.4699,4.2509)	(0.4654,4.1038)
		θ_1	(0.0505,0.0819)	(0.0013,0.0558)	(0.0012,0.0535)
		α_2	(1.5325,1.5325)	(0.5487,4.7004)	(0.6558,4.7921)
		θ_2	(0.0917,0.0994)	(0.0046,0.0327)	(0.0050,0.0329)
	II	α_1	(6.0410,19.4327)	(0.5674,3.9556)	(0.5362,3.8695)
		θ_1	(0.0572,0.0905)	(0.0053,0.0518)	(0.0041,0.0504)
		α_2	(1.0997,1.0997)	(0.6376,4.6221)	(0.4851,4.6790)
		θ_2	(0.1181,0.1265)	(0.0040,0.0335)	(0.0042,0.0340)
	III	α_1	(2.2202,6.3239)	(0.2957,3.2538)	(0.2975,3.2332)
		θ_1	(0.0241,0.0457)	(0.0032,0.0362)	(0.0034,0.0373)
		α_2	(5.9348,5.9384)	(0.2174,3.6421)	(0.2317,3.8121)
		θ_2	(0.0782,0.0077)	(0.0003,0.0255)	(0.0002,0.0268)
20	I	α_1	(2.9777,8.1532)	(0.3754,3.3183)	(0.3754,3.2549)
		θ_1	(0.0335,0.0576)	(0.0055,0.0414)	(0.0054,0.0411)
		α_2	(1.5329,1.5329)	(0.5792,4.8930)	(0.2954,4.7841)
		θ_2	(0.0921,0.0997)	(0.0028,0.0336)	(0.0007,0.0334)
	II	α_1	(3.5334,10.0860)	(0.3654,3.3844)	(0.3677,3.7708)
		θ_1	(0.0385,0.0650)	(0.0035,0.0442)	(0.0037,0.0467)
		α_2	(1.5442,1.5442)	(0.4235,4.6209)	(0.4235,4.4110)
		θ_2	(0.0923,0.0999)	(0.0017,0.0333)	(0.0012,0.0318)
	III	α_1	(2.2215,6.3493)	(0.2301,3.1081)	(0.2794,3.2139)
		θ_1	(0.0241,0.0458)	(0.0003,0.0336)	(0.0002,0.0339)
		α_2	(5.9306,5.9306)	(0.4149,3.7068)	(0.4196,3.7037)
		θ_2	(0.0700,0.0778)	(0.0007,0.0252)	(0.0007,0.0251)

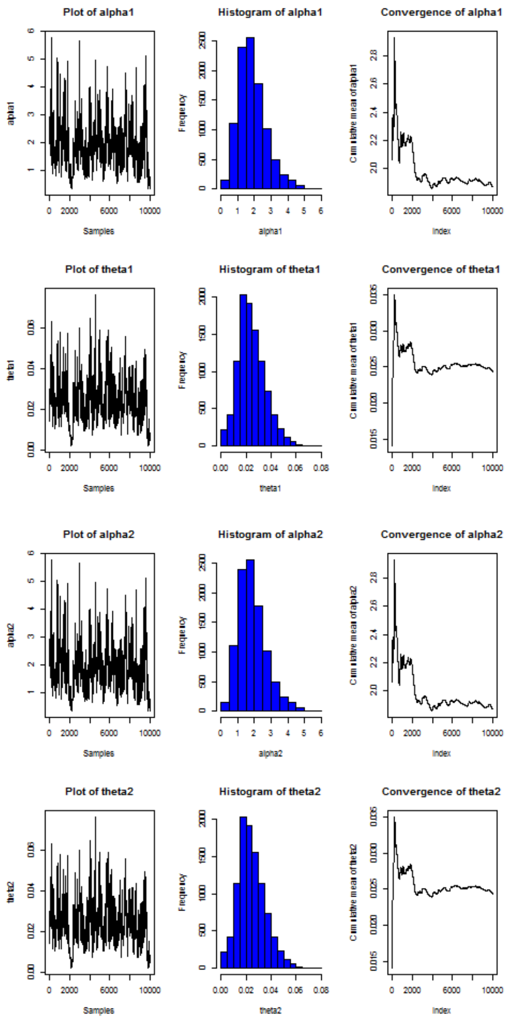

Fig. 2: MCMC trace plots, histograms, and convergence of α_i and $\theta_i,\ i=1,2$ from Gupta's and Kundu's datasets.

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