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Evolution of Translation Surfaces in Euclidean 3-Space E^3

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Abstract: Geometry and kinematics have been intimately connected in their historical evolution and, although it is currently less fashionable, the further development of such connections is crucial to many computer-aided design and manufacturing. In this paper, the evolution of the translation surfaces and their generating curves in E^3 are investigated. Integrability conditions of the Gauss-Weingarten equations are obtained. Kinematics of moving frame fields associated to these surfaces are described. The evolution equations of the Christoffel symbols, the second fundamental quantities and Gauss-Codazzi equations for the motion are established. Thus, the evolution equations of the curvatures in terms of their intrinsic geometric formulas are derived. Two examples of translation surfaces and their motions are considered and plotted.

Keywords: Kinematics of surfaces, Evolution Surfaces.
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1 Introduction

The evolution of curves and surfaces has significant applications in computer vision and image processing [1], such as scale space by linear and nonlinear diffusions [2] and [3] image enhancement through anisotropic diffusions [4,5] and [6]and image segmentation by active contours [7,8] and [9].

The evolution of curves has been researched and studied extensively in various manifolds and homogeneous spaces. The connection between integrable equations (soliton equations) and the geometric motions of curves in spaces has been known for a long time. In fact, many integrable equations have been shown to describe the evolution invariants associated with certain movements of curves in particular geometric settings.

The dynamics of shapes in physics, chemistry and biology are modelled in terms of motion of surfaces and interfaces, and some dynamics of shapes are reduced to motion of plane curves. These models are specified by velocity fields, which are local or nonlocal functionals of the intrinsic quantities of curves [10].

Evolution of surfaces accompanies manifold physical phenomena: propagation of wave fronts [11], motion of interfaces, growth of crystals [12], geometry [13,14] and [15]. Most often one needs to consider the related three-dimensional problem for the domain the surface encloses to find out how it evolves in time.

Hence, geometry of curves and surfaces evolution is a quite recent field of the differential geometry which deals with curves and surfaces where the time plays the fundamental rule. Geometrically, curves and surfaces evolution means that deforming a curve and surface into another curve and surface in a continuous manner, respectively. This evolution process can be understood through the answer of the following geometrical questions:

• What is the final shape of the evolving curves and surfaces ?

• What is the invariant geometrical property during the evolution process ?

The main goal of this paper is to try to answer the above questions, in order to illustrate the translation surfaces behavior during the evolution process. Consequently, it gives a classification of the geometrical flows according to the resulting geometrical information which characterizes and recognizes each flow in a class of geometric flows.

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2 Geometric preliminaries

Here, and in the sequel, we assume that the indices $\{i,j,k,e\}$ and $\{l,m\}$ run over the ranges $\{1,2\}$ and $\{1,2,3\}$, respectively.

Let $C_1 : \alpha = \alpha(s_1)$ and $C_2 : \beta = \beta(s_2)$ be two curves parametrized by the arc lengths s_i in E^3. Consider the Frenet frame $\{\mathbf{t}_i(s_i),\ \mathbf{n}_i(s_i),\ \mathbf{b}_i(s_i)\}$ associated with the curves C_i. The derivatives of the vectors $\mathbf{t}_i(s_i)$ and $\mathbf{b}_i(s_i)$, when expressed in the basis $\{\mathbf{t}_i, \mathbf{n}_i, \mathbf{b}_i\}$, yield geometrical entities, the natural curvatures $\kappa_i(s_i)$ and torsions $\tau_i(s_i)$, which give us information about the behavior of the curves α and β in the neighborhood of s_i, respectively. Then the Frenet formulas of the curves C_i are defined by [10]:

$$\frac{d}{ds_i}\begin{pmatrix}\mathbf{t}_i(s_i)\\ \mathbf{n}_i(s_i)\\ \mathbf{b}_i(s_i)\end{pmatrix} = \begin{pmatrix}0 & \kappa_i(s_i) & 0\\ -\kappa_i(s_i) & 0 & \tau_i(s_i)\\ 0 & -\tau_i(s_i) & 0\end{pmatrix}\begin{pmatrix}\mathbf{t}_i(s_i)\\ \mathbf{n}_i(s_i)\\ \mathbf{b}_i(s_i)\end{pmatrix}. \tag{1}$$

We denote a surface M in E^3 by

$$\mathbf{X}(s_i) = (x_l(s_i)).$$

Let $\mathbf{N}$ be the standard unit normal vector field on the surface M defined by $\mathbf{N} = \frac{\mathbf{X}_1\times\mathbf{X}_2}{\|\mathbf{X}_1\times\mathbf{X}_2\|}$, where, $\mathbf{X}_i = \frac{\partial\mathbf{X}}{\partial s_i}$. Thus, we have the metric g_{ij} and the coefficients of the second fundamental form h_{ij},

$$g_{ij} = <\mathbf{X}_i, \mathbf{X}_j> , \quad h_{ij} = <\mathbf{X}_{ij}, \mathbf{N}>, \tag{2}$$

where $\langle\,,\rangle$ is the Euclidean inner product.
Thus, the Gaussian curvature G and the mean curvature H are given by

$$G = Det(h_{ij})/Det(g_{ij}) \quad \text{and} \quad H = \frac{1}{2}tr\left(g^{ij}\,h_{jk}\right), \tag{3}$$

respectively, where, (g^{ij}) is the associated contravariant metric tensor field of the covariant metric tensor field (g_{ij}), i.e., $g^{ik}g_{jk} = \delta^i_j$.
As one moves along the surface (at a fixed time), the tangent and normal vectors change according to the Gauss-Weingarten equations (GWE),

$$\frac{\partial}{\partial s_j}\begin{bmatrix}\mathbf{X}_1\\ \mathbf{X}_2\\ \cdots\\ \mathbf{N}\end{bmatrix} = \begin{bmatrix}\Gamma^k_{ij} & \vdots & h_{ij}\\ \cdots\cdots & \vdots & \cdots\cdots\\ h^k_j & \vdots & 0\end{bmatrix}\begin{bmatrix}\mathbf{X}_1\\ \mathbf{X}_2\\ \cdots\\ \mathbf{N}\end{bmatrix}, \quad h^k_j = g^{ki}\,h_{ij} \tag{4}$$

where Γ^k_{ij} are called the Christoffel symbols of the $2\underline{nd}$ kind, which are given as

$$\Gamma^k_{ij} = \frac{1}{2}\sum_e g^{ke}\left(\frac{\partial g_{je}}{\partial u^i} + \frac{\partial g_{ie}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^e}\right). \tag{5}$$

The norm of the $2\underline{nd}$ fundamental form is given by:

$$S^2 = \sum_{i,j} h^i_j\, h^j_i \ , \tag{6}$$

where h^j_i are given from (4).

From the compatibility conditions of (4), we get the Gauss-Codazzi equations,

$$R_{ijke} = h_{ik}\,h_{je} - h_{ie}h_{jk}, \tag{7}$$

$$\nabla_i\,h_{jk} = \nabla_j\,h_{ik} \ , \tag{8}$$

where R_{ijke} is the Riemann tensor and ∇_i is the covariant derivative.

The Riemann curvature tensor is defined by

$$R^e_{ijk} = \Gamma^e_{ik,j} - \Gamma^e_{ij,k} + \sum_h\left(\Gamma^e_{jh}\Gamma^h_{ik} - \Gamma^e_{kh}\Gamma^h_{ij}\right). \tag{9}$$

where the comma denotes the partial derivative.

Nakayama, et al. [16] and [17] discovered connections between integrable evolution and the motion of curves in a 3-dimensional Euclidean space. They considered that $\alpha = \alpha(s_1,t)$ denote a point on a space curve at the time t.The conventional geometrical model is specified by velocity fields

$$\frac{\partial\alpha}{\partial t} = v^1_1\mathbf{t}_1 + v^2_1\mathbf{n}_1 + v^3_1\mathbf{b}_1, \tag{10}$$

similarly for a space curve $\beta = \beta(s_2,t)$, we have

$$\frac{\partial\beta}{\partial t} = v^1_2\mathbf{t}_2 + v^2_2\mathbf{n}_2 + v^3_2\mathbf{b}_2, \tag{11}$$

where $\mathbf{t}_i, \mathbf{n}_i$ and $\mathbf{b}_i$ are the unit tangent, normal and binormal vectors along the curves, and v^1_i, v^2_i and v^3_i are the tangential, normal and binormal velocities, respectively. Velocity fields are functionals of the intrinsic quantities of curves, for example, curvature, k_i, torsion τ_i and their s_i derivatives.
The time evolution equations for $\mathbf{t}_i, \mathbf{n}_i$ and $\mathbf{b}_i$ are given by

$$\frac{\partial}{\partial t}\begin{bmatrix}\mathbf{t}_i(s_i)\\ \mathbf{n}_i(s_i)\\ \mathbf{b}_i(s_i)\end{bmatrix} = \begin{bmatrix}0 & \alpha_i & \beta_i\\ -\alpha_i & 0 & \gamma_i\\ -\beta_i & -\gamma_i & 0\end{bmatrix}\begin{bmatrix}\mathbf{t}_i(s_i)\\ \mathbf{n}_i(s_i)\\ \mathbf{b}_i(s_i)\end{bmatrix}, \tag{12}$$

where,

$$\begin{aligned}\alpha_i &= \left(\frac{\partial v^2_i}{\partial s_i} - \tau_i v^3_i + k_i v^1_i\right),\ \beta_i = \left(\frac{\partial v^3_i}{\partial s_i} + \tau_i v^2_i\right),\\ \gamma_i &= \left(\frac{1}{k_i}\frac{\partial\beta_i}{\partial s_i} + \frac{\tau_i}{k_i}\alpha_i\right),\end{aligned} \tag{13}$$

Thus, Nakayama, et al. [16] and [17] obtained the time evolution equations for curvature and torsion to motion of space curve.

Also, Nakayama, et al [18,19] and [20] introduced the dynamics of the surface. They considered the velocity of the surface is expressed by

$$\frac{\partial\mathbf{X}}{\partial t} = V^i\mathbf{X}_i + V^3\mathbf{N},\ \ V^l = V^l\ (s_i\ ,t), \tag{14}$$

where V^i and V^3 are the tangential and the normal velocities, respectively.
Using (4) and (14), one can obtain the time evolution equations for the local frame,

$$\frac{\partial}{\partial t}\begin{bmatrix}\mathbf{X}_1\\ \mathbf{X}_2\\ \cdots\\ \mathbf{N}\end{bmatrix}=\left[\begin{array}{c|c}-V^3h_i^k+\nabla_iV^k & V^3_{,i}+V^jh_{ij}\\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\\ -g^{ki}\left(V^3_{,i}+V^jh_{ij}\right) & 0\end{array}\right]\begin{bmatrix}\mathbf{X}_1\\ \mathbf{X}_2\\ \cdots\\ \mathbf{N}\end{bmatrix} \tag{15}$$

Thus, and using the compatibility conditions, one can get the time evolution equations for g_{ij} and h_{ij}. See [18,19] and [20].

3 Intrinsic geometry of the translation surfaces in $\mathbf{E}^3$

In this section the translation surface in Euclidean 3-space is considered.The geometric invariants on the translation surfaces are derived .

When a space curve is translated over another space curve, the resulting surface can be considered as the most general appearance of a translation surface. Consequently, this surface can be parameterized as the sum of two space curves. Quite often, the class of translation surfaces is restricted to those that can be parameterized as the sum of two plane curves. So it can be parameterized by a patch [21]:

$$M:\mathbf{X}\;(s_i)=\alpha\,(s_1)+\beta\,(s_2)\,,\quad s_i\in I, \tag{16}$$

where s_i are the parameters of the arc lengths of the curves α and β, respectively.
Thus, the two tangent vectors to the surface (16) are given by

$$\mathbf{X}_i=\mathbf{t}_i\;,\quad \mathbf{X}_i=\frac{\partial\mathbf{X}}{\partial s_i}, \tag{17}$$

where $\mathbf{t}_i$ denote the tangents of the curves α and β, respectively.
Using (16) and (17), it is easily checked that the metric of M is given by

$$(g_{ij})=\begin{cases}1, & i=j\\ cos\;\theta, & i\neq j\end{cases},\; det\;(g_{ij})=sin^2\theta,\qquad \theta\neq n\pi,\; n=0,1,2,... \tag{18}$$

The unit normal vector of the surface M is given by

$$\mathbf{N}\,(s_i)=(\mathbf{t}_1\times\mathbf{t}_2)\;csc\;\theta, \tag{19}$$

This leads to the coefficients of the second fundamental form h_{ij} where

$$(h_{ij})=\begin{cases}\kappa_i\;csc\;\theta\,[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_j]\;, & i=j\\ 0, & i\neq j\end{cases},\qquad det\;(h_{ij})=csc^2\theta\prod_{i\neq j}\kappa_i\;[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_j], \tag{20}$$

where κ_i denote the curvatures of the curves α and β, respectively and $[t_1t_2n_j]$ denotes the triple scalar product to the vectors t_1,t_2 and n_j.
Thus , one can get the following:

Corollary 1. *The Gaussian curvature is given by*

$$G=csc^4\theta\prod_{i\neq j}\kappa_i\;[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_j]\,. \tag{21}$$

Corollary 2. *The mean curvature is given by*

$$H=\frac{1}{2}csc^3\theta\sum_i\kappa_i\;[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_i]\,. \tag{22}$$

Corollary 3.

$$h^i_j=\begin{cases}\kappa_i\;csc^3\theta\,[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_i]\,, & i=j\\ -\kappa_j\;cot\theta csc^2\theta\,[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_j]\,, & i\neq j\end{cases} \tag{23}$$

Corollary 4. *The norm of the 2nd fundamental form is given by:*

$$S^2=4H^2-2G \tag{24}$$

Corollary 5. *The Christoffel symbols are given as follows*

$$\left.\begin{array}{ll}\Gamma^1_{11}=-\kappa_1\,cot^2\theta, & \Gamma^2_{22}=-\kappa_2\,cot^2\theta\;,\\ \Gamma^2_{11}=\kappa_1 cos\theta csc^2\theta, & \Gamma^1_{22}=\kappa_2 cos\theta csc^2\theta,\end{array}\right\} \tag{25}$$

and the other components are zero.

Corollary 6. *The Riemann tensor R_{ijke} and the Riemann curvature tensor R^e_{ijk} are given by*

$$\begin{aligned}R_{1212}=R_{2121}=-R_{1221}=-R_{2112}\\ =csc^2\theta\prod_{i\neq j}\kappa_i\;[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_j],\end{aligned} \tag{26}$$

$$\begin{aligned}R^2_{121}=R^2_{122}=R^1_{221}=-R^2_{112}=-R^1_{212}\\ =\kappa_1\kappa_2 cos\;\theta csc^2\theta,\end{aligned} \tag{27}$$

and the other components are zero.

4 Evolution of the translation surfaces

In the study of the motion of curves, the author seeks to get the partial differential equations (PDEs), which describe the evolution of the curvature and torsion of the evolving curve. Similarly, the study of the motion of surfaces seeks to get (PDEs),which describe the evolution of the coefficients of the 1st and 2nd fundamental forms because the surface can be derived from them. For this purpose, let a translation surface, moving in 3-dimensional Euclidean space E^3, be given at the time t by the position vector

$$M:\mathbf{X}\;(s_i,t)=\alpha\,(s_1,t)+\beta\,(s_2,t)\,,\quad s_i\in I, \tag{28}$$

where, $\alpha\,(s_1,0)=\alpha\,(s_1)\,,\quad\beta\,(s_1,0)=\beta\,(s_1)\,.$

4.1 Evolution of the curves α and β

Nakayama, et al. [16] and [17] studied the evolution and the motion of curves as mentioned in section 2 of this paper. Also, Mukhergee and Balakrishnan [22] studied the motion of curves which have been specified via the evolution of the frame field $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$.

In this section we apply the same method of Mukhergee and Balakrishnan[22]. In our case we study the evolution of curves α and β which have been specified via the evolution of the frame fields $\{\mathbf{t}_i,\mathbf{n}_i,\mathbf{b}_i\}$.

The Serret-Frenet equations (1) and the equations of evolution (12) can be written concisely in the following form

$$\frac{\partial F}{\partial s_i}=A_i\,.F\ ,\quad \frac{\partial F}{\partial t}=B_i\,.F\ \ . \tag{29}$$

Applying the compatibility conditions, we have

$$\frac{\partial}{\partial t}\frac{\partial F}{\partial s_i}=\frac{\partial}{\partial s_i}\frac{\partial F}{\partial t} \tag{30}$$

After some calculation using (1) and (12), we have

$$\frac{\partial A_i}{\partial t}-\frac{\partial B_i}{\partial s_i}+[A_i\ ,B_i]=O_{3\times 3}, \tag{31}$$

where $[A_i\ ,B_i]=A_i\,B_i-B_iA_i$ is the Lie bracket of A_i and B_i , thus we have

$$\begin{pmatrix} 0 & \alpha_{i,i}-\kappa_{i,t}-\tau_i\,\beta_i & \beta_{i,i}-\kappa_i\gamma_i+\tau_i\,\alpha_i \\ -(\alpha_{i,i}-\kappa_{i,t}-\tau_i\,\beta_i) & 0 & \gamma_{i,i}-\tau_{i,t}-\kappa_i\,\beta_i \\ -(\beta_{i,i}-\kappa_i\gamma_i+\tau_i\,\alpha_i) & -(\gamma_{i,i}-\tau_{i,t}-\kappa_i\,\beta_i) & 0 \end{pmatrix}=O_{3\times 3}, \tag{32}$$

where, $f_{i,i}=\frac{\partial f_i}{\partial s_i}$ and $f_{i,t}=\frac{\partial f_i}{\partial t}$. Thus the compatibility conditions give the temporal evolution of the curvatures κ_i and torsons τ_i of the curves α and β in terms of the velocities $\{\alpha_i\ ,\beta_i\ ,\gamma_i\}$, which can be written as coupled nonlinear partial differential equations (CNLPDEs) as follows:

Theorem 1.

$$\frac{\partial\kappa_i}{\partial t}=\frac{\partial\alpha_i}{\partial s_i}-\tau_i\,\beta_i\ ,\quad \frac{\partial\tau_i}{\partial t}=\frac{\partial}{\partial s_i}\left(\frac{\partial\beta_i}{\partial s_i}+\tau_i\,\alpha_i\right)+\kappa_i\,\beta_i \tag{33}$$

The above theorem is considered the main result of this section. For a given $\{\alpha_i\ ,\beta_i\ ,\gamma_i\}$, the motion of the curves are determined from these equations. These equations were solved using different methods, see for example [23, 24] and [25], which apply the tanh and sech methods.

4.2 Integrability conditions of GWEs and time evoluton equations

The Gauss-Weingarten equations (4) and the equations of the evolution (15) can be written, respectively, as follows

$$\frac{\partial\psi}{\partial s_j}=C_j\ \psi\ ,\quad \frac{\partial\psi}{\partial t}=D\ \psi\ \ . \tag{34}$$

Thus, the compatibility conditions are given from

$$\frac{\partial}{\partial t}\frac{\partial\psi}{\partial s_i}=\frac{\partial}{\partial s_i}\frac{\partial\psi}{\partial t} \tag{35}$$

Therefore, using (4) and (15), we have the following

$$\frac{\partial C_i}{\partial t}-\frac{\partial D}{\partial s_j}+[C_j\ ,D]=O_{3\times 3}, \tag{36}$$

Thus, the compatibility conditions (35), splitted into two equations, which we called the $1\underline{st}$ and $2\underline{nd}$ compatibility conditions, where in each case we obtain evolution (PDEs) for invariant quantities related to each one.

4.3 The first compatibility conditions

Based on (35), we have

$$\frac{\partial}{\partial t}\frac{\partial\psi}{\partial s_1}=\frac{\partial}{\partial s_1}\frac{\partial\psi}{\partial t} \tag{37}$$

Thus, using (4), (15) and (34), one can get

$$\left.\begin{aligned}
\frac{\partial\Gamma^2_{11}}{\partial t}&=d_{12,1}+d_{12}\left(\Gamma^2_{12}-\Gamma^1_{11}\right)-\Gamma^2_{11}\left(d_{22}-d_{11}\right)\\
&\quad -d_{32}h_{11}-d_{13}h^2_1,\\
\frac{\partial\Gamma^1_{12}}{\partial t}&=d_{21,1}+d_{21}\left(\Gamma^1_{11}-\Gamma^2_{12}\right)+\Gamma^1_{12}\left(d_{22}-d_{11}\right)\\
&\quad -d_{31}h_{12}-d_{23}h^1_1,\\
\frac{\partial\Gamma^1_{11}}{\partial t}&=d_{11,1}+d_{12}\Gamma^1_{12}-d_{21}\Gamma^2_{11}-d_{31}h_{11}-d_{13}h^1_1,\\
\frac{\partial\Gamma^2_{12}}{\partial t}&=d_{22,1}+d_{21}\Gamma^2_{11}-d_{12}\Gamma^1_{12}-d_{32}h_{12}-d_{23}h^2_1,\\
\frac{\partial h_{11}}{\partial t}&=d_{13,1}-d_{13}\Gamma^1_{11}-d_{23}\Gamma^2_{11}+d_{11}h_{11}+d_{12}h_{12},\\
\frac{\partial h_{12}}{\partial t}&=d_{23,1}-d_{13}\Gamma^1_{12}-d_{23}\Gamma^2_{12}+d_{21}h_{11}+d_{22}h_{12},\\
\frac{\partial h^1_1}{\partial t}&=-d_{31,1}-d_{31}\Gamma^1_{11}-d_{32}\Gamma^1_{12}-d_{11}h^1_1-d_{21}h^2_1,\\
\frac{\partial h^2_1}{\partial t}&=-d_{32,1}-d_{31}\Gamma^2_{11}-d_{32}\Gamma^2_{12}-d_{12}h^1_1-d_{22}h^2_1,
\end{aligned}\right\} \tag{38}$$

where, d_{lm} are the elements of the matrix D which are given from (15), (34) and $d_{lm,1}=\frac{\partial d_{lm}}{\partial s_1}$ .Thus, using (20), (23) and (25), one can get

Theorem 2. *The evolution equations of the Christoffel symbols, h_{ij} and h_i^j for translation surfaces are given by*

$$\left.\begin{aligned}
\frac{\partial \Gamma_{11}^2}{\partial t} &= d_{12,1} + d_{12}\kappa_1 \cot^2\theta - (d_{22} - d_{11})\,\kappa_1 \cos\theta\csc^2\theta \\
&\quad + (d_{13}\cot\,\theta - d_{32})\,\kappa_1 \csc^2\theta\,[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_1]\,, \\
\frac{\partial \Gamma_{12}^1}{\partial t} &= d_{21,1} - d_{21}\kappa_1 \cot^2\theta - d_{23}\kappa_1\csc^2\theta\,[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_1]\,, \\
\frac{\partial \Gamma_{11}^1}{\partial t} &= d_{11,1} - d_{21}\kappa_1 \cos\theta\csc^2\theta \\
&\quad - d_{13}\kappa_1\csc^2\theta\,(1-\csc\theta)\,[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_1]\,, \\
\frac{\partial \Gamma_{12}^2}{\partial t} &= d_{22,1} + d_{21}\kappa_1 \cos\,\theta\csc^2\theta \\
&\quad + d_{23}\kappa_1\cot\theta\csc^2\theta\,[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_1]\,, \\
\frac{\partial h_{11}}{\partial t} &= d_{13,1} - d_{13}\kappa_1 \cot^2\theta - d_{23}\kappa_1 \cos\theta\csc^2\theta \\
&\quad + d_{11}\kappa_1\csc^2\theta\,[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_1]\,, \\
\frac{\partial h_{12}}{\partial t} &= d_{23,1} + d_{21}\kappa_1\csc^2\theta\,[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_1]\,, \\
\frac{\partial h_1^1}{\partial t} &= -d_{31,1} - d_{31}\kappa_1 \cot^2\theta \\
&\quad - \kappa_1\csc^3\theta\,(d_{21}\cos\theta - d_{11})\,[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_1]\,, \\
\frac{\partial h_1^2}{\partial t} &= -d_{32,1} - d_{31}\kappa_1\cos\theta\csc^2\theta \\
&\quad + \kappa_1\csc^3\theta\,(d_{22}\cos\theta - d_{12})\,[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_1]\,.
\end{aligned}\right\} \tag{39}$$

4.4 The second compatibility conditions

Adopting (35), we have

$$\frac{\partial}{\partial t}\frac{\partial \psi}{\partial s_2} = \frac{\partial}{\partial s_2}\frac{\partial \psi}{\partial t} \tag{40}$$

Thus, using (4), (15) and (34), one can get

$$\left.\begin{aligned}
\frac{\partial \Gamma_{22}^1}{\partial t} &= d_{21,2} + d_{21}\left(\Gamma_{12}^1 - \Gamma_{22}^2\right) + \Gamma_{22}^1\,(d_{22} - d_{11}) \\
&\quad - d_{31}h_{22} - d_{23}h_2^1, \\
\frac{\partial \Gamma_{12}^2}{\partial t} &= d_{12,2} + d_{12}\left(\Gamma_{22}^1 - \Gamma_{12}^1\right) - \Gamma_{12}^2\,(d_{22} - d_{11}) \\
&\quad - d_{32}h_{12} - d_{13}h_2^2, \\
\frac{\partial \Gamma_{22}^2}{\partial t} &= d_{22,2} + d_{21}\Gamma_{12}^2 - d_{12}\Gamma_{22}^1 - d_{32}h_{22} - d_{23}h_2^2, \\
\frac{\partial \Gamma_{12}^1}{\partial t} &= d_{11,2} + d_{12}\Gamma_{22}^1 - d_{21}\Gamma_{12}^2 - d_{31}h_{12} - d_{13}h_2^1, \\
\frac{\partial h_{22}}{\partial t} &= d_{23,2} - d_{23}\Gamma_{22}^2 - d_{13}\Gamma_{22}^1 + d_{22}h_{22} + d_{21}h_{12}, \\
\frac{\partial h_{12}}{\partial t} &= d_{13,2} - d_{13}\Gamma_{12}^1 - d_{23}\Gamma_{12}^2 + d_{12}h_{22} + d_{11}h_{12}, \\
\frac{\partial h_2^2}{\partial t} &= -d_{32,2} + d_{32}\Gamma_{22}^2 + d_{31}\Gamma_{12}^2 + d_{22}h_2^2 + d_{12}h_2^1, \\
\frac{\partial h_2^1}{\partial t} &= -d_{31,2} + d_{32}\Gamma_{22}^1 + d_{31}\Gamma_{12}^1 + d_{21}h_2^2 + d_{11}h_2^1,
\end{aligned}\right\} \tag{41}$$

As a consequence, using (20), (23) and (25), one can get

Theorem 3. *The evolution equations of the Christoffel symbols, h_{ij} and h_i^j for translation surfaces are given by*

$$\left.\begin{aligned}
\frac{\partial \Gamma_{22}^1}{\partial t} &= d_{21,2} + d_{21}\kappa_2\cot^2\theta + (d_{22} - d_{11})\,\kappa_2 \cos\theta\csc^2\theta \\
&\quad + (d_{23}\cot\theta - d_{31})\,\kappa_2 \csc^2\theta\,[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_2]\,, \\
\frac{\partial \Gamma_{12}^2}{\partial t} &= d_{12,2} + d_{12}\kappa_2 \cos\theta\csc^2\theta - d_{13}\kappa_2\csc^2\theta\,[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_2]\,, \\
\frac{\partial \Gamma_{22}^2}{\partial t} &= d_{22,2} - d_{12}\kappa_2 \cos\theta\csc^2\theta \\
&\quad - d_{23}\kappa_2\csc^2\theta\,(1-\csc\theta)\,[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_2]\,, \\
\frac{\partial \Gamma_{12}^1}{\partial t} &= d_{11,2} + d_{12}\kappa_2 \cos\theta\csc^2\theta + d_{13}\kappa_2\cot\theta\csc^2\theta\,[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_2]\,, \\
\frac{\partial h_{22}}{\partial t} &= d_{23,2} + d_{23}\kappa_2\cot^2\theta - d_{13}\kappa_2 \cos\theta\csc^2\theta \\
&\quad + d_{22}\kappa_2\csc^2\theta\,[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_2]\,, \\
\frac{\partial h_{12}}{\partial t} &= d_{13,2} + d_{12}\kappa_2\csc^2\theta\,[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_2]\,, \\
\frac{\partial h_2^2}{\partial t} &= -d_{32,2} + d_{32}\kappa_2\cot^2\theta \\
&\quad + \kappa_2\csc^3\theta\,(d_{12}\cos\theta - d_{22})\,[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_2]\,, \\
\frac{\partial h_2^1}{\partial t} &= -d_{31,2} - d_{32}\kappa_2 \cos\theta\csc^2\theta \\
&\quad + \kappa_2\csc^3\theta\,(d_{11}\cos\theta - d_{21})\,[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_2]\,.
\end{aligned}\right\} \tag{42}$$

It is very important to notice that the GWEs for surfaces are analogous to the Frenet equations for curves. Also, the first and second fundamental coefficients for surfaces are analogous to the curvature and torsion for curves.

4.5 Evolution of the curvatures

Here, (PDEs) which describe the evolution of the curvatures of evolving the translation surface in terms of curvatures and velocities of the surface and the curves are derived.

From the foregoing results, using (3),(6), (7) and (9), we get the proof of the very important theorems

Theorem 4. *Evolution of the Gaussian curvature is given by*

$$\left.\begin{aligned}
\frac{\partial G}{\partial t} &= G\,\left[\textstyle\sum_i d_{ii} - 2\cot^2\theta\sum_i(\alpha_i + \beta_i)\right] + 2H\,\textstyle\sum_i d_{i3,i} \\
&\quad - \kappa_1\kappa_2\cot\theta\,(d_{13} - d_{23}\cos\theta)\textstyle\sum_i\ [\mathbf{t}_1\mathbf{t}_2\mathbf{n}_i]\,.
\end{aligned}\right\} \tag{43}$$

Theorem 5. *Evolution of the mean curvature function is given by*

$$\left.\begin{aligned}
\frac{\partial H}{\partial t} &= H\,\left[\tfrac{1}{2}\textstyle\sum_i d_{ii} + 2\cot^2\theta\sum_i(\alpha_i + \beta_i)\right] \\
&\quad - \tfrac{1}{2}\cos\theta\csc^2\theta\,(1+\cos\theta)\textstyle\sum_{i\neq j} d_{j3}\kappa_i \\
&\quad - \tfrac{1}{2}\csc^2\theta\left[2\cos\theta d_{23,1} - \textstyle\sum_i d_{i3,i}\right] \\
&\quad - d_{12}\kappa_1\cot\theta\csc\theta\ [\mathbf{t}_1\mathbf{t}_2\mathbf{n}_1]\,.
\end{aligned}\right\} \tag{44}$$

Theorem 6. *Evolution of the norm of the 2nd fundamental form is given by:*

$$\left.\begin{aligned}
\frac{\partial S^2}{\partial t} &= S^2\textstyle\sum_i d_{ii} + 4\left(S^2 + 3G\right)\cot^2\theta\sum_i(\alpha_i + \beta_i) \\
&\quad - 4H\,\cos\theta\csc^2\theta\,(1+\cos\theta)\textstyle\sum_{i\neq j} d_{j3}\kappa_i \\
&\quad - 4H\,\csc^2\theta\left[2\cos\theta d_{23,1} - \textstyle\sum_i d_{i3,i}\right] \\
&\quad - 2\kappa_1\kappa_2\cot\theta\,(d_{23}\cos\theta - d_{13})\ \textstyle\sum_i [\mathbf{t}_1\mathbf{t}_2\mathbf{n}_i] \\
&\quad - 8d_{12}\,H\ \kappa_1\cot\theta\csc\theta\ [\mathbf{t}_1\mathbf{t}_2\mathbf{n}_1] \\
&\quad - 4H\ \textstyle\sum_i d_{i3,i}\ \ .
\end{aligned}\right\} \tag{45}$$

Theorem 7. *Evolution of the Riemann tensor R_{ijkl} are given by*

$$\left.\begin{array}{l}\frac{\partial R_{2121}}{\partial t}=\frac{\partial R_{1212}}{\partial t}=-\frac{\partial R_{1221}}{\partial t}=-\frac{\partial R_{2112}}{\partial t}\\ \quad = R_{2121}csc\theta\sum_i d_{ii}+2H\,sin^2\theta\sum_i d_{i3,i}\\ \quad -(d_{13}-d_{23})\,\kappa_1\kappa_2\,cos\theta csc^3\theta\sum_i\;[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_i]\end{array}\right\} \tag{46}$$

Theorem 8. *Evolution of the Riemann curvature tensor R^l_{ijk} are given by*

$$\left.\begin{array}{l}\frac{\partial R^2_{121}}{\partial t}=\frac{\partial R^2_{122}}{\partial t}=-\frac{\partial R^1_{221}}{\partial t}=-\frac{\partial R^2_{112}}{\partial t}=-\frac{\partial R^1_{212}}{\partial t}\\ =R^2_{121}d_{12}cos\theta-R^2_{121}\left(d_{13}csc\theta+d_{32}sec\theta sin^2\theta\right)[\mathbf{t}_1\mathbf{n}_1\mathbf{n}_2]\\ +R^2_{121}csc\theta\left(\sum_i\; d_{i3}[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_i]\right)-d_{21,1}\kappa_1 cos\theta csc^2\theta\,(1+cos\theta)\\ +d_{21}cos\theta csc^2\theta\left[\kappa_1^2 csc^2\theta\,(1+cos\theta)+\kappa_1\tau_1+\kappa_1^2-\kappa_{1,1}\right]\\ -2H\;d_{32}cos\theta\,(\kappa_1+\kappa_2\;cos\theta)+(1+\kappa_2)\,cot\theta csc\theta\;\left(\sum_i d_{12,i}\right)\\ -\kappa_1 cos\theta csc^2\theta\left(\sum_{i\neq j}d_{1i,j}\right)-d_{23}\kappa_1\tau_1 cot\theta csc^2\theta\;[\mathbf{t}_1\mathbf{t}_2\mathbf{b}_1]\\ +cot\theta csc^2\theta[d_{23}\kappa_1^2 csc^2\theta\,(1+cos\theta)-d_{32,2}\kappa_1 sec\theta sin^2\theta+\\ -d_{23}\kappa_{1,1}+\kappa_1\left(\sum_{i\neq j}d_{i3,j}\right)][\mathbf{t}_1\mathbf{t}_2\mathbf{n}_1]+d_{12,12}-d_{22,11}\,.\end{array}\right\} \tag{47}$$

The above theorems for the evolution of the curvatures of the translation surfaces describe the change of the shape in time and their solutions have many applications in different areas of science. In geometry, the behavior of evolution equations of the curvatures and their solutions gives us the most geometric information about evolving the surface.

5 Application

In this section, we give some interesting results about how the evolving curves will look like after a period of time and the effect of the evolution on the translation surfaces. This is given through some illustrated figures. For this purpose we consider two examples to illustrate our investigation:

(1) Consider the translation surfaces for a circular helix curves $\alpha(s_1,t)$ and $\beta(s_2,t)$ as an example in E^3, so that their points can be represented in the form

$$\mathbf{X}\;(s_i,t)=\sum_i\left(a_i(t)\cos\,[\frac{s_i}{A}],a_i(t)\sin\,[\frac{s_i}{A}],2b_i(t)\,\frac{s_i}{A}\right)$$
$$,\;\; A=\sqrt{a_i^2+b_i^2} \tag{48}$$

(2) Consider the translation surfaces for a circular helix curve $\alpha(s_1,t)$ and the circle curve $\beta(s_2,t)$, where

$$\beta(s_2,t)=(a_2(t)\cos\,\theta,a_2(t)\sin\,\theta),\;\;\theta=\frac{s_2}{a_2},\;a_2\neq 0 \tag{49}$$

Evolution of these surfaces are plotted for different values of the time t. In examples (1) and (2) which show that these surfaces are collapsing in length and expanding in width as in the following cases:

(i) Evolution of the two circular helix curves α and β as in Fig.1 (b,c,d).
(ii) Evolution of the circular helix α and fixed circular helix β as in Fig.2 (b,c,d).
(iii) Evolution of the circular helix α and the circle curve β as in Fig.3 (b,c,d).
(iv) Evolution of the circular helix α and fixed the circle curve β as in Fig.4 (b,c,d).

From the above examples we note that the effect of the evolution of the curves α and β on the translation surfaces is very weak under $a_i(t)=e^t$, and $b_i(t)=e^{-t}$ for $t\in(0,0.5]$, and hence we can know these surfaces after the evolution. In other words the translation surfaces are still having some geometric properties which they had before the evolution, as in Figs.$(1\ (a),2\ (a),3\ (a),4\ (a))$, i.e., the geometric properties in this period are hereditary properties. Also, it is noticed that the effect of the evolution of the curves α and β on the translation surfaces is very strong under $a_i(t)=e^t$, and $b_i(t)=e^{-t}$ for $t\succ 0.5$ and hence we can not know any information on the translation surfaces after the evolution, i.e., the geometric properties in this period are not hereditary properties.

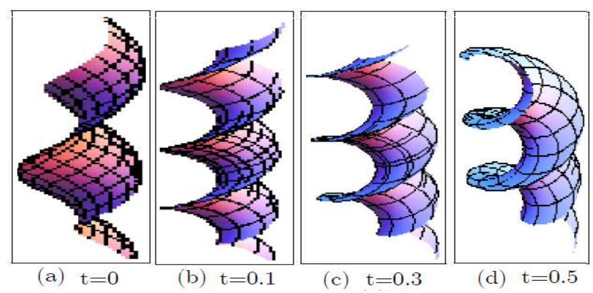

Fig. 1: Evolution of the two circular helix α , β.

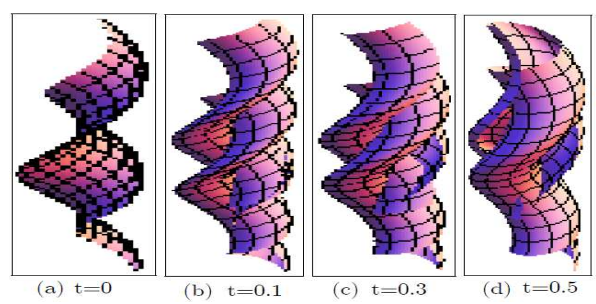

Fig. 2: Evolution of the circular helix α and fixed circular helix β.

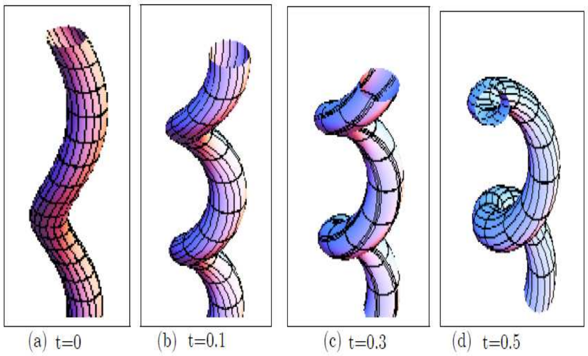

Fig. 3: Evolution of the circular helix α and the circle curve β.

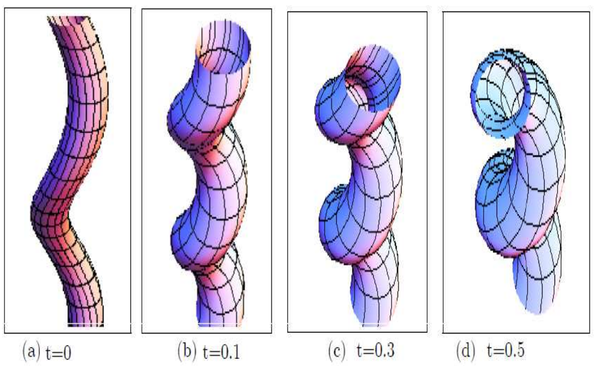

Fig. 4: Evolution of the circular helix α and fixed the circle curve β.

6 Conclusion

The study of curves finds applications in various branches of physics. As illustrative examples of a space curve we mention the propagation of light in a twisted optical fiber, the time evolution of the (normalized) classical spin vector field of a one-dimensional Heisenberg chain and the vector field of the O(3) nonlinear sigma model in field theory. This latter model is known to be related to one-dimensional antiferromagnets. In recent years, there has been much interest in the notion of anholonomy in physics. The geometric phases and the corresponding gauge fields have attracted attention in a wide spectrum of problems in classical and quantum systems [26]

As an application of the translation surfaces evolution it is interesting to consider a system, such as fluid and spins, on a moving translation surface. It is also interesting to see what kind of integrable systems is included in this motion. In one dimensional case, it is possible to explain the relation between the motion of the generating curves and integrable system in terms of the inverse scattering method. Situation is, however, much more involved in the case of the motion of translation surfaces.

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