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# Generators of Certain Function Banach Algebras and Related Questions

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**Abstract:** We study the structure of generators of the Banach algebras $\left(W_p^{(n)}[0,1], \underset{\alpha}{*}\right)$ and $\left(W_p^{(n)}[0,1], \circledast\right)$, where $\underset{\alpha}{*}$ denotes the convolution product $\underset{\alpha}{*}$ defined by $\left(f \underset{\alpha}{*} g\right)(x) := \int_0^x f(x+\alpha-t)g(t)\,dt$, and the so-called Duhamel product $\circledast$. We also give some description of cyclic vectors of usual convolution operators acting in the Sobolev space $W_p^{(n)}[0,1]$ by the formula $K_k f(x) = \int_0^x k(x-t) f(t)\,dy$.

**Keywords:** Banach algebra; Generator of Banach algebra; Convolution operator; Duhamel product; Sobolev space.

## 1 Introduction

Let $W_p^{(n)} := W_p^{(n)}[0,1]$ $(1 \le p < \infty)$ be the Sobolev space of functions $C^{(n-1)}[0,1]$ such that $f^{(n)} \in L_p[0,1]$. The norm in $W_p^{(n)}$ is defined by

$$\|f\|_{W_p^{(n)}} := \|f\|_{C^{(n-1)}} + \left\|f^{(n)}\right\|_{L_p}.$$

It is easy to verify that $W_p^{(n)}$ is a Banach algebra with respect to the classical convolution product

$$(f*g)(x) = \int\limits_0^x f(x-t)g(t)\,dt.$$

We will denote the $n$-th convolution power by $f^{*n} = f * \ldots * f$.

For any $f \in W_p^{(n)}[0,1]$, $f^{*n}(0) = 0$, $n = 1,2,3,\ldots$, so that it is clear that a necessary condition for $f \in W_p^{(n)}[0,1]$ to generate $W_p^{(n)}[0,1]$, that is

$$span\{f, f*f, f*f*f, \ldots\} = W_p^{(n)}[0,1],$$

is that $f(0) \neq 0$. But it is not known if this condition is sufficient.

In this article, we consider the Banach algebra $W_p^{(n)}[0,1]$ and describe its all $\underset{\alpha}{*}$-generators and $\circledast$-generators (see Theorem 2 in Section 3 and Corollary 1 in Section 2). We also study cyclic vectors of some convolution operators (see Theorem 1 in Section 2).

## 2 Cyclic vectors of convolution operators

In $W_p^{(n)}[0,1]$, the Duhamel product is defined (see, for instance [1,4]) by the following formula:

$$\begin{aligned}(f \circledast g)(x) &= \frac{d}{dx}\int_0^x f(x-t)g(t)\,dt \\ &= \int_0^x f'(x-t)g(t)\,dt + f(0)g(x), \qquad (1)\end{aligned}$$

where $f,g \in W_p^{(n)}[0,1]$. One can use results of operational calculus [9] (see also [4]) to show that $W_p^{(n)}[0,1]$ is commutative and associative algebra with respect to the Duhamel product $\circledast$, and it is clear from (1) that an identity function 1 is the unit for the algebra $\left(W_p^{(n)}[0,1], \circledast\right)$. It is also easy to verify that actually $\left(W_p^{(n)}[0,1], \circledast\right)$ is a Banach algebra (see, for instance,

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Karaev [4]). The operator

$$\mathscr{D}_f g := f \circledast g$$

is called the Duhamel operator associated with the function $f \in W_p^{(n)}[0,1]$.

Let us consider the usual convolution operator $\mathscr{K}_k$ on $W_p^{(n)}[0,1]$:

$$(\mathscr{K}_k f)(x) = \int_0^x k(x-t) f(t)\,dt, \qquad (2)$$

where $k \in W_p^{(n)}[0,1]$ is a fixed function. Here we shall examine cyclic vectors of the operator $\mathscr{K}_k$. Recall that $f \in W_p^{(n)}[0,1]$ is a cyclic vector for $\mathscr{K}_k$ if the vectors

$$f, \mathscr{K}_k f, \mathscr{K}_k^2 f, ..., \mathscr{K}_k^n f, ...$$

span the algebra $W_p^{(n)}[0,1]$, that is,

$$span\{\mathscr{K}_k^m f : k \geq 0\} = clos\, lin\{\mathscr{K}_k^m f : m \geq 0\} = W_p^{(n)}[0,1].$$

Clearly, when $k \in Cyc(\mathscr{K}_k)$ (the set of all cyclic vectors of the operator $\mathscr{K}_k$), this is just a problem of description of $*$-generators of the Banach algebra $\left(W_p^{(n)}[0,1], *\right)$. Recall that about description of generators of the Banach algebras of smooth functions is initiated by Ginsberg and Newman in [2].

The following key lemma can be proved by the same methods as in [4,5,6,7,8,10], and therefore omitted.

**Lemma 1.** *Let $f \in W_p^{(n)}[0,1]$. Then $f$ is $\circledast$-invertible if and only if $f(0) \neq 0$.*

An immediate corollary of Lemma 1, is the following, which characterizes $\circledast$-generators of the Banach algebra $\left(W_p^{(n)}[0,1], \circledast\right)$.

**Corollary 1.** *The function $f \in W_p^{(n)}[0,1]$ generates the Banach algebra $\left(W_p^{(n)}[0,1], \circledast\right)$ if and only if $f(0) \neq 0$.*

**Theorem 1.** *Let $k \in W_p^{(n)}[0,1]$, $f \in W_p^{(n)}[0,1]$ be two functions and $\mathscr{K}_k$ be a corresponding convolution operator defined by $(2)$. Let us denote $F := \int_0^x k(t)\,dt$. Suppose that $\{F^{\circledast m}\}_{m=0}^{\infty}$ (for $m = 0$ we put 1) is a complete system in $W_p^{(n)}[0,1]$. Then $f \in Cyc(\mathscr{K}_k)$ if and only if $f(0) \neq 0$.*

*Proof.* We use the similar arguments in [5]. Clearly, $F'(x) = k(x)$. Therefore, for every $g \in W_p^{(n)}[0,1]$ we have

$$(\mathscr{K}_k g)(x) = \int_0^x k(x-t) g(t)\,dt = \frac{d}{dx}\int_0^x F(x-t) g(t)\,dt$$
$$= (F \circledast g)(x).$$

By induction we obtain that

$$\mathscr{K}_k^m f = (F \circledast ... \circledast F) \circledast f = F^{\circledast m} \circledast f = \mathscr{D}_f F^{\circledast m}$$

for $m = 0,1,2,...$, from which we have

$$span\{\mathscr{K}_k^m f : m \geq 0\} = span\{\mathscr{D}_f F^{\circledast m} : m \geq 0\}$$
$$= \overline{\mathscr{D}_f span\{F^{\circledast m} : m \geq 0\}}.$$

Now, since $\{F^{\circledast m} : m \geq 0\}$ is a complete system in $W_p^{(n)}[0,1]$, by applying Lemma 1, it is easy to show that $f \in Cyc(\mathscr{K}_k)$ if and only if $f(0) \neq 0$ (because it is immediate from Lemma 1 that the Duhamel operator $\mathscr{D}_f$ is invertible in $W_p^{(n)}$ if and only if $f(0) \neq 0$), which proves the theorem.

## 3 $\underset{\alpha}{*}$-generators of $W_p^{(n)}[0,1]$

Here we will consider the following convolutional product $\underset{\alpha}{*}$, which is defined by the formula

$$\left(f \underset{\alpha}{*} g\right)(x) := \int_0^x f(x+\alpha-t) g(t)\,dt$$

for any two functions $f, g \in W_p^{(n)}[0,1]$, where $\alpha \in [0,1)$ is a fixed number. It is not difficult to prove that $W_p^{(n)}[0,1]$ is a commutative Banach algebra with respect to the convolutional product $\underset{\alpha}{*}$ (we omit it). We will denote the corresponding $\underset{\alpha}{*}$-convolution operator by the symbol $K_{f,\alpha}$:

$$K_{f,\alpha} g(x) := \left(f \underset{\alpha}{*} g\right)(x).$$

Our following result gives some characterization of $\underset{\alpha}{*}$-generators of the radical Banach algebra $\left(W_p^{(n)}[0,1], \underset{\alpha}{*}\right)$, which is the main result of Section 3.

**Theorem 2.** *Let $f \in W_p^{(n)}[0,1]$ and $f(\alpha) \neq 0$. Then $f$ is a $\underset{\alpha}{*}$-generator of the algebra $\left(W_p^{(n)}[0,1], \underset{\alpha}{*}\right)$ if and only if*

$$span\{1, F, \mathscr{K}_{f,\alpha} F, \mathscr{K}_{f,\alpha}^2 F, ...\} = W_p^{(n)}[0,1],$$

*where $F(x) = \int_\alpha^x f(t)\,dt$.*

*Proof.* Note that it is not difficult to see that the method of the Karaev's paper [4] allow us to prove that the Sobolev space $W_p^{(n)}[0,1]$ is also Banach algebra with respect to the product $\underset{\alpha}{\circledast}$, which is defined by

$$f \underset{\alpha}{\circledast} g = \frac{d}{dx}\int_\alpha^x f(x+\alpha-t) g(t)\,dt.$$

Therefore, "the $\alpha$-Duhamel operator" $\mathscr{D}_{f,\alpha} g := \frac{d}{dx}\int_\alpha^x f(x+\alpha-t) g(t)\,dt$ is a bounded operator in $\left(W_p^{(n)}[0,1], \underset{\alpha}{\circledast}\right)$, and $\|\mathscr{D}_{f,\alpha}\| = \|f\|_{W_p^{(n)}}$. Since

$F'(x) = f(x)$, we have (see the proof of Theorem 1) $\mathscr{K}_{f,\alpha} = \mathscr{D}_{F,\alpha}$, that is $\mathscr{K}_{f,\alpha}g = \mathscr{D}_{F,\alpha}g$ for all $g \in W_p^{(n)}[0,1]$. In particular,

$$\begin{aligned}(\mathscr{K}_{f,\alpha}f)(x) = (\mathscr{D}_{F,\alpha}f)(x) &= \frac{d}{dx}\int_\alpha^x f(x+\alpha-t)F(t)\,dt \\ &= \int_\alpha^x f'(x+\alpha-t)F(t)\,dt + f(\alpha)F(x) \\ &= (\mathscr{D}_{f,\alpha}F)(x),\end{aligned}$$

where $\mathscr{D}_{f,\alpha}$ is an invertible operator in $W_p^{(n)}[0,1]$, because it can be also shown by the similar arguments of the paper by Gürdal and Şöhret [3] that element $f \in \left(W_p^{(n)}, \underset{\alpha}{\circledast}\right)$ is invertible if and only if $f(\alpha) \neq 0$. Thus,

$$f = \mathscr{D}_{f,\alpha}1 \tag{$3_1$}$$

and

$$f \underset{\alpha}{*} f = \mathscr{D}_{f,\alpha}F. \tag{$3_2$}$$

Further, we have:

$$\begin{aligned}f \underset{\alpha}{*} f \underset{\alpha}{*} f = \mathscr{K}_{f,\alpha}^2 f &= \mathscr{K}_{f,\alpha}(\mathscr{K}_{f,\alpha}f) = \mathscr{K}_{f,\alpha}(\mathscr{D}_{f,\alpha}F) \\ &= \mathscr{K}_{f,\alpha}(\mathscr{K}_{f',\alpha} + f(\alpha)I)F \\ &= (\mathscr{K}_{f,\alpha}\mathscr{K}_{f',\alpha} + f(\alpha)\mathscr{K}_{f,\alpha})F \\ &= (\mathscr{K}_{f',\alpha} + f(\alpha)I)(\mathscr{K}_{f,\alpha}F) \\ &= \mathscr{D}_{f,\alpha}(\mathscr{K}_{f,\alpha}F),\end{aligned}$$

and thus

$$f \underset{\alpha}{*} f \underset{\alpha}{*} f = \mathscr{D}_{f,\alpha}(\mathscr{K}_{f,\alpha}F); \tag{$3_3$}$$

$$\begin{aligned}f \underset{\alpha}{*} f \underset{\alpha}{*} f \underset{\alpha}{*} f = \mathscr{K}_{f,\alpha}^3 f &= \mathscr{K}_{f,\alpha}(\mathscr{K}_{f,\alpha}^2 f) \\ &= \mathscr{K}_{f,\alpha}\mathscr{D}_{f,\alpha}(\mathscr{K}_{f,\alpha}F) \\ &= \mathscr{D}_{f,\alpha}\mathscr{K}_{f,\alpha}(\mathscr{K}_{f,\alpha}F) \\ &= \mathscr{D}_{f,\alpha}(\mathscr{K}_{f,\alpha}^2 F),\end{aligned}$$

and thus

$$f \underset{\alpha}{*} f \underset{\alpha}{*} f \underset{\alpha}{*} f = \mathscr{D}_{f,\alpha}(\mathscr{K}_{f,\alpha}^2 F). \tag{$3_4$}$$

By induction we deduce that

$$\mathscr{K}_{f,\alpha}^m f = \mathscr{D}_{f,\alpha}\left(\mathscr{K}_{f,\alpha}^{m-1}F\right) \quad (\forall m \geq 1). \tag{$3_m+1$}$$

Now, from formulas $(3_{m+1})$, $m \geq 0$, we have:

$$\begin{aligned}&span\left\{f, f \underset{\alpha}{*} f, f \underset{\alpha}{*} f \underset{\alpha}{*} f, \ldots\right\} \\ &= span\left\{\mathscr{D}_{f,\alpha}1, \mathscr{D}_{f,\alpha}F, \mathscr{D}_{f,\alpha}(\mathscr{K}_{f,\alpha}F), \mathscr{D}_{f,\alpha}(\mathscr{K}_{f,\alpha}^2 F), \ldots\right\} \\ &= clos\mathscr{D}_{f,\alpha}span\left\{1, F, \mathscr{K}_{f,\alpha}F, \mathscr{K}_{f,\alpha}^2 F, \ldots\right\}.\end{aligned}$$

From this, by considering that the condition $f(\alpha) \neq 0$ means invertibility of the corresponding Duhamel operator $\mathscr{D}_{f,\alpha}$, we deduce that

$$span\left\{f, f \underset{\alpha}{*} f, f \underset{\alpha}{*} f \underset{\alpha}{*} f, \ldots\right\} = W_p^{(n)}[0,1]$$

if and only if

$$span\left\{1, F, \mathscr{K}_{f,\alpha}F, \mathscr{K}_{f,\alpha}^2 F, \ldots\right\} = W_p^{(n)}[0,1],$$

which proves Theorem 2.

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## References


[1] Dimovski I., Convolutional calculus, Kluwer Academic Publishers, Dordrecht, (1990).

[2] Ginsberg J.I. and Newman D.J., Generators of certain radical algebras, J. Approx. Theory, **3** (1970), 229-235.

[3] Gürdal M. and Şöhret F., On some operator equations in the space of analytic functions and related questions, Proc. Est. Acad. Sci., **62** (2013), 81-87.

[4] Karaev M.T., Invariant subspaces, cyclic vectors, commutant and extended eigenvectors of some convolution operators, Methods of Funct. Anal. and Topology, **11** (2005), 48-59.

[5] Karaev M.T., Gürdal M. and Saltan S., Some applications of Banach algebra techniques, Math. Nachrichten, **284** (2011), 1678-1689.

[6] Karaev M.T. and Tuna H., Description of maximal ideal space of some Banach algebra with multiplication as Duhamel product, Complex Variables: Theory and Applications, **49** (2004), 449-457.

[7] Karaev M.T. and Tuna H., On some applications of Duhamel product, Linear and Multilinear Algebra, **54** (2006), 301-311.

[8] Karaev M.T. and Saltan S., A Banach algebra structure for the Wiener algebra $W(\mathbb{D})$ of the *disc*, Complex Variables: Theory and Applications, **50** (2005), 299-305.

[9] Krabbe G., Operational Calculus, New York, Springer-Verlag, (1970).

[10] Wigley N.M., The Duhamel product of analytic functions, Duke Math. J., **41** (1974), 211-217.

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