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Classification of Proper Hyper BCI-Algebras of Order 3

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Abstract: We consider the notion of an isomorphism between two hyper BCI-algebras and obtain all of the hyper BCI-algebras of order 3. It is shown that there exist 14 proper hyper BCI-algebras and 5 proper weak hyper BCI-algebras of order 3 up to isomorphism. Finally, further study on the theory of hyper BCI-algebra and its related hyper structure and applications of our results in information sciences are discussed.

Keywords: hyper-structure, hyper BCI-algebra, weak hyper BCI-algebra, isomorphism

1 Introduction

Logic algebras are the algebraic foundation of reasoning mechanism in many fields such as computer sciences, information sciences, cybernetics and artificial intelligence. BCK-logic and BCI-logic were originated from λ-calculus and combinators in combinational logic. BCK-algebra and BCI-algebra are algebraic representation of BCK-system and BCI-system, respectively. The concepts of BCK/BCI-algebras were firstly formulated in 1966 by K. Iski as a generalization of the concepts of set-theoretic difference and propositional calculus[1]. BCI-algebras are the generalization of BCK-algebras. After that many researches worked in this area and lots of literatures had been produced about the theory of BCK/BCI-algebra. On the theory of BCK/BCI-algebras, please see [2,3].

The hyper-structure theory (called also multi-algebras) was introduced by F. Marty in 1934 at the eighth congress of Scandinavian mathematicians[4]. Algebraic hyper-structure was suitable generalization of classical algebraic structure. In a classical algebraic structure, the composition of two elements is an element, while the composition of two elements is a set of elements in an algebraic hyper-structure. F. Marty introduced the concept of hyper-group in [4], since then other classic hyper-structures had been introduced, such as hyper-rings[5], hyper-fields[5], hyper-lattices[6], hyper BCK-algebras[7], hyper K-algebras[8], hyper BCC-algebras[9,10], hyper BCI-algebras[11] and so on. Now, the theory of algebraic hyper-structure has become a well-established branch in algebraic theory, and there are extensive applications in many branches of mathematics and applied sciences, such as Euclidian and Non Euclidian geometries, graphs and hyper-graphs, binary relations, lattices, fuzzy and rough sets, automata, cryptography, codes, probabilities, information sciences and so on [12,13,14,15,16,17].

In [7], Y.B. Jun et al. introduced the concept of hyper BCK-algebras which is a generalization of BCK-algebras through applying the hyper-structure to BCK-algebras, and they gave the relations between hyper BCK-ideas and weak hyper BCK-ideals in hyper BCK-algebras. In [11], X.L. Xin introduced the concept of hyper BCI-algebras which is a generalization of BCI-algebras, and he proved that every hyper BCK-algebra is a hyper BCI-algebra. In fact, hyper K-algebras, hyper BCC-algebras, hyper BCI-algebras all are the generalization of hyper BCK-algebras. Since then, these algebraic hyper-structures were extensively investigated by many researchers. It should be pointed out that the research of hyper BCI-algebras seems to have been focused on the ideal theory. In [11], X.L. Xin introduced the concepts of hyper BCI-ideals, weak hyper BCI-ideals, strong hyper BCI-ideals and reflexive hyper BCI-ideals in hyper BCI-algebras, and he gave the relations among these hyper BCI-ideals. In [18], F. Nisar et al. introduced the notion of Bi-polar-valued Fuzzy hyper subalgebra (briefly BFHS) based on Bi-polar-valued fuzzy set, and they stated Bi-polar-value fuzzy characteristic hyper subalgebras. In [19], the concepts of the distributive hyper BCI-ideals and fuzzy distributive

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hyper BCI-algebras were introduced, and the related properties were established. In [20], N. Palaniappan et al. investigated the relations between intuitionistic fuzzy distributive hyper BCI-algebras and the distributive hyper BCI-algebras of a hyper BCI-algebra.

2 Preliminaries

Let H be a non-empty set and $\circ$ a function form $H \times H$ to $P(H)\backslash\{\o\}$, where $P(H)$ denotes the power set of H. For two subsets A and B of H, denote by $A \circ B$ the set $\{a \circ b | a \in H, b \in H\}$. Then we call $(H, \circ)$ a hyper groupoid and $\circ$ a hyperoperation. We also define $x \ll y$ by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ means that for all $a \in A$ there is $b \in H$ such that $a \ll b$. We shall use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$, or $\{x\} \circ \{y\}$. In such case, we call "$\ll$" the hyperorder of H.

Definition 2.1 [7] By a hyper BCK-algebra we mean a hyper groupoid $(H, \circ)$ that contains a constant 0 and satisfies the following axioms for all $x, y, z \in H$

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x \circ H \ll \{x\}$,
(HK4) $x \ll y$ and $y \ll x$ imply $x = y$.

Definition 2.2 [11] By a hyper BCI-algebra we mean a hyper groupoid $(H, \circ)$ that contains a constant 0 and satisfies the following axioms for all $x, y, z \in H$

(HI1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(HI2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HI3) $x \ll x$,
(HI4) $x \ll y$ and $y \ll x$ imply $x = y$,
(HI5) $0 \circ (0 \circ x) \ll x$.

Definition 2.3 [9] By a hyper BCC-algebra we mean a nonempty subset H endowed with a hyper operation "$\circ$" and a constant 0 satisfying the following axioms for all $x, y, z \in H$

(HC1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(HC2) $x \ll x$,
(HC3) $x \circ y \ll x$,
(HC4) $x \ll y$ and $y \ll x$ imply $x = y$.

Proposition 2.4 [11] Let $(H, \circ)$ be a hyper BCK-algebra, then $(H, \circ)$ is also a hyper BCI-algebra. The converse is not true.

Proposition 2.5 [9] Every hyper BCC-algebra $(H, \circ)$ satisfying the equality $(x \circ y) \circ z = (x \circ z) \circ y, \forall x, y, z \in H$ is a hyper BCK-algebra.

Definition 2.6 [21] In a hyper BCI-algebra $(H, \circ)$, the set $S_K = \{x \in H : x \circ H \ll x\}$ is defined as hyper BCK-part of H. If $H \neq S_K$, then H is known as a proper hyper BCI-algebra.

Definition 2.7 [21] Let H be a hyper BCI-algebra. Then $x, y \in H$ are said to comparable if $x \ll y$ or $y \ll x$. Otherwise x, y are said to be incomparable, and denoted as $x \parallel y$.

3 Weak hyper BCI-algebra

Definition 3.1[9] By a hyper BCI-algebra we mean a hyper groupoid $(H, \circ)$ that contains a constant 0 and satisfies the following axioms for all $x, y, z \in H$

(HI1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(HI2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HI3) $x \ll x$,
(HI4) $x \ll y$ and $y \ll x$ imply $x = y$,
(HI5*) $0 \circ (0 \circ x) \ll x, x \neq 0$.

In fact, this definition is different from the original concept of hyper BCI-algebra by X.L. Xin in [11]. For the convenience of discussion, hyper BCI-algebras that F. Nisar et al. mentioned in [21] is named after weak hyper BCI-algebras.

According to Definition 2.2 and Definition 3.1, we directly obtain the following proposition.

Proposition 3.2 Every hyper BCI-algebra is a weak hyper BCI-algebra, but the converse is not true.

Definition 3.3 (1) For a hyper BCI-algebra, if it is not a hyper BCK-algebra, we call it a proper hyper BCI-algebra. (2) For a weak hyper BCI-algebra, if it is not a hyper BCI-algebra, we call it a proper weak hyper BCI-algebra.

Example 3.4 Let $H = \{0, a, b\}$ and hyper operations "$\circ$" be defined as in Table 1. Then $(H, \circ)$ is a proper weak hyper BCI-algebra.

Table 1

$\circ$	0	a	b
0	$\{0, a\}$	$\{0, a\}$	$\{b\}$
a	$\{a\}$	$\{0, a\}$	$\{b\}$
b	$\{b\}$	$\{b\}$	$\{0, a\}$

Proof: Obviously, (HI3), (HI4) hold. Because $0 \circ (0 \circ 0) = \{0, a\} \not\ll 0$, $0 \circ (0 \circ a) = \{0, a\} \ll a$, $0 \circ (0 \circ b) = \{b\} \ll b$, So (HI5*) holds, but (HI5) does not hold. In the following, we will prove that (HI1) (HI2) also hold.

For all $x, y, z \in H$, we have the four situations: (i) $x \neq b, y \neq b$; (ii) $x = b, y = b$; (iii) $x = b, y \neq b$; (iv) $x \neq b, y = b$. Firstly, for $x \neq b, y \neq b$, we have $(x \circ z) \circ (y \circ z) \subseteq \{0, a\}$. According $\{0, a\} \ll a, a \in x \circ y$, we have $(x \circ z) \circ (y \circ z) \ll x \circ y$. Secondly, for $x = b, y = b$, we have $(x \circ z) \circ (y \circ z) = (b \circ z) \circ (b \circ z) = \{0, a\} = b \circ b$. Finally, for $(x = b, y \neq b)$ or $(x \neq b, y = b)$, we have $(x \circ z) \circ (y \circ z) = \{b\} = x \circ y$. Thus, (HI1) holds.

Obviously, if $y = z, (x \circ y) \circ z = (x \circ z) \circ y$ hold. For $y \neq z$, we have the following situations: (i) $(x = b, y \neq b, z \neq b), (x \neq b, y = b, z \neq b)$ or $(x \neq b, y \neq b, z = b)$; (ii) $(x = y = b, z \neq b)$ or $(x = z = b, y \neq b)$; (iii) $x \neq b, y \neq b, z \neq b$. Firstly, for $(x = b, y \neq b, z \neq b), (x \neq b, y = b, z \neq b)$ or $(x \neq b, y \neq b, z = b)$, we have $(x \circ y) \circ z = \{b\} = (x \circ z) \circ y$. Secondly, for $(x = y = b, z \neq b)$ or $(x = z = b, y \neq b)$, we have $(x \circ y) \circ z = \{0, a\} = (x \circ z) \circ y$. Finally, for $x \neq b, y \neq b, z \neq b$, we have we have $(x \circ y) \circ z = \{0, a\} = (x \circ z) \circ y$. Thus, (H2) holds.

To sum up, $(H, \circ)$ is not a hyper BCI-algebra, but it is a weak hyper BCI-algebra.

Example 3.5 Let $H=\{0,a,b\}$ and hyper operations "$\circ$" be defined as in Table 2. Then $(H,\circ)$ is a (weak) hyper BCI-algebra.

Table 2

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a,b\}$	$\{0,a,b\}$	$\{0,b\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

Proof: Obviously, (HI3), (HI4) hold. $\forall x\in H, 0\circ(0\circ x)=\{0\}\ll x$, So (HI5) holds. In the following, we will prove that (HI1), (HI2) also hold.

For all $x,y,z\in H$, we have the following situation: (i) $x=0$; (ii) $x=a$; (iii) $x=b,y\neq b$; (iv) $x=b,y=b$. Firstly, if $x=0$, we have $(x\circ z)\circ(y\circ z)=(0\circ z)\circ(y\circ z)=0\circ(y\circ z)=\{0\}, x\circ y=0\circ y=\{0\}$. So, $(x\circ z)\circ(y\circ z)\ll x\circ y$. Secondly, if $(x=a)$ or $(x=b,y\neq b)$, we have $b\in x\circ y$, $H\ll b$. So, $(x\circ z)\circ(y\circ z)\ll x\circ y$. Finally, if $x=b,y=b$, we get $(x\circ z)\circ(y\circ z)=(b\circ z)\circ(b\circ z)=\{0\}=b\circ b=x\circ y$, so $(x\circ z)\circ(y\circ z)\ll x\circ y$. Thus, (HI1) holds.

Obviously, if $y=z$, $(x\circ y)\circ z=(x\circ z)\circ y$ hold. For $x\neq y$, we have the following situations: (i) $x=0$; (ii) $(x=b, z=b, y\neq b)$ or $(x=b, y=b, z\neq b)$; (iii) $x=b, y\neq b, z\neq b$; (iv) $x=a, y\neq b, z\neq b$; (v) $(x=a, z=b, y\neq b)$ or $(x=a, y=b, z\neq b)$. Firstly, if $x=0$, we have $(x\circ y)\circ z=(0\circ y)\circ z=\{0\}$, $(x\circ z)\circ y=(0\circ z)\circ y=\{0\}$. So, $(x\circ y)\circ z=(x\circ z)\circ y$. Secondly, if $(x=b, z=b, y\neq b)$ or $(x=b, y=b, z\neq b)$, we have $(b\circ y)\circ b=\{0\}$, $(b\circ b)\circ y=\{0\}$. So, $(x\circ y)\circ z=(x\circ z)\circ y$. Thirdly, if $x=b, y\neq b, z\neq b$, i.e. $(x=b, y=a, z=0)$ or $(x=b, y=0, z=a)$, we have $(b\circ a)\circ 0=\{b\}$, $(b\circ 0)\circ a=\{b\}$. So, $(x\circ y)\circ z=(x\circ z)\circ y$. Fourthly, if $x=a, y\neq b, z\neq b$, i.e. $(x=a, y=a, z=0)$ or $(x=a, y=0, z=a)$, we have $(a\circ a)\circ 0=\{0,a,b\}$, $(a\circ 0)\circ a=\{0,a,b\}$. So, $(x\circ y)\circ z=(x\circ z)\circ y$. Finally, if $(x=a, z=b, y\neq b)$ or $(x=a, y=b, z\neq b)$, we have $(a\circ y)\circ b=\{0,b\}=(a\circ b)\circ y$, $(a\circ b)\circ z=\{0,b\}=(a\circ z)\circ b$. So, $(x\circ y)\circ z=(x\circ z)\circ y$. Thus, (HI2) holds.

To sum up, $(H,\circ)$ is a hyper BCI-algebra and is also weak.

F. Nisar et al. in [9] investigated relative properties of weak hyper BCI-algebras and proved that: If $(H,\circ)$ be a hyper BCI-algebra, then $(H,\circ)$ satisfies the following properties: (1) $x\circ 0=\{x\}, \forall x\in H, x\neq 0$; (2) $x\circ y\ll z$ implies $x\circ z\ll y, \forall x,y,z\in H$.

Example 3.5 shows that the above properties are not correct, because (1) $a\circ 0=\{a,b\}\neq\{a\}$, (2) $a\circ 0=\{a,b\}\ll\{b\}$, but $a\circ b=\{0,b\}\not\ll\{0\}$. Nisar et al. in [9] applied the wrong property (i.e. $x\circ 0=\{x\}$, $\forall\, x\in H, x\neq 0$) to prove the conclusion that the number of proper hyper BCI-algebras of order 3 upto isomorphism is 8. Thus, the conclusion is not correct.

4 Classification of proper hyper BCI-algebras of order 3

In this section, we will obtain all non-isomorphic proper hyper BCI-algebras of order 3 by programming calculation using Matlab software.

According to Definition 2.1, Definition 2.2 and Definition 2.6, the following definition of proper hyper BCI-algebra can be directly obtained.

Definition 4.1 By a proper hyper BCI-algebra we mean a hyper groupoid $(H,\circ)$ that contains a constant 0 satisfying the following axioms for all $x,y,z\in H$

(HI1) $(x\circ z)\circ(y\circ z)\ll x\circ y$,

(HI2) $(x\circ y)\circ z=(x\circ z)\circ y$,

(HI3) $x\ll x$,

(HI4) $x\ll y$ and $y\ll x$ imply $x=y$,

(HI5) $0\circ(0\circ x)\ll x$,

(HK3*) $\exists x\in H, x\circ H\not\ll\{x\}$.

Based on above definition, we design an algorithm for checking whether $(X,\circ)$ is a proper hyper BCI-algebra in the following Algorithm 1.

Algorithm 1: Checking whether (X,o) is a proper hyper BCI-algebra.

```
Input (X:set, ∘:hyper operation)
Output("X is a proper hyper BCI-algebra or not")
Begin
  If X = φ then
      go to (1.);
  EndIf
  If 0 ∉ X then
      go to (1.);
  EndIf
  Stop:=false;
  i:=1;
  s:=0;
  While i ≤ |X| and not(Stop) do
      If (0 ∉ x_i ∘ x_i) then
        Stop:=true;
      EndIf
      Y = 0 ∘ (0 ∘ x_i);
      l := 1;
      While l ≤ |Y| and not(Stop) do
        If(0 ∉ y_l ∘ x_i) then
           Stop:=true;
        EndIf
      EndWhile
      Z = x_i ∘ X;
      m := 1;
      While m ≤ |Z| and not(Stop) do
        If 0 ∉ z_m ∘ x_i then
           s=s+1;
        EndIf
      EndWhile
      j := 1;
      While j ≤ |X| and not(Stop) do
        If (0 ∈ x_i ∘ x_j) and (0 ∈ x_j ∘ x_i) and (x_i ≠ x_j) then
           Stop:=true;
        EndIf
        k := 1;
        While k ≤ |X| and not(Stop) do
           If (x_i ∘ x_j) ∘ x_k ≠ (x_i ∘ x_k) ∘ x_j then
              Stop:=true;
```

```
            EndIf
            W = (x_i ∘ x_k) ∘ (x_j ∘ x_k);
            t := 1;
            While t ≤ |W| and not(Stop) do
               If 0 ∉ w_l ∘ (x_i ∘ x_j) then
                  Stop:=true;
               EndIf
            EndWhile
         EndWhile
      EndWhile
   EndWhile
   If s < 1 Then
      Stop:=true;
   EndIf
   If Stop then
      (1.) Output("X is not a proper hyper BCC-algebra.")
   Else
      Output("X is a proper hyper BCC-algebra.")
   EndIf
End
```

Definition4.2 Hyper BCI-algebras $(A,\circ)$ and $(B,*)$ are said to be isomorphic if

$$\exists f : A \to B, \forall x,y \in A, f(x\circ y) = f(x) * f(y).$$

Example 4.3 Let $H = \{0,a,b\}$ and hyper operations "$\circ$" and "$*$" are defined as in Table 3 and Table 4, respectively. Then $(H,\circ)$ and $(H,*)$ are isomorphic.

Proof: Let f is a one-to-one mapping, and $f(0) = f(0)$, $f(a) = f(b)$, $f(b) = a$. According to the operations of "$\circ$" and "$*$", we get $f(0\circ 0) = f(0) = 0 = 0*0 = f(0)*f(0)$, $f(0\circ a) = f(0) = 0 = 0*b = f(0)*f(a)$, $f(0\circ b) = f(b) = a = 0*a = f(0)*f(b)$, $f(a\circ 0) = f(a) = b = b*0 = f(a)*f(0)$, $f(a\circ a) = f(\{0,a\}) = \{0,b\} = b*b = f(a)*f(a)$, $f(a\circ b) = f(b) = a = b*a = f(a)*f(b)$, $f(b\circ 0) = f(b) = a = a*0 = f(b)*f(0)$, $f(b\circ a) = f(b) = a = a*b = f(b)*f(a)$, $f(b\circ b) = f(0) = 0 = a*a = f(b)*f(b)$. Thus, Hyper BCI-algebras $(H,\circ)$ and $(H,*)$ are isomorphic.

Table 3

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{b\}$
a	$\{a\}$	$\{0,a\}$	$\{b\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

Table 4

$*$	0	a	b
0	$\{0\}$	$\{a\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{a\}$
b	$\{b\}$	$\{a\}$	$\{0,b\}$

According Definition 4.2, we design an algorithm to judge whether $(X,\circ)$ and $(X,*)$ are isomorphic as follows:

Algorithm 2: checking whether (X,o) and (X,*) are isomorphic

```
Input(∘,∗: two hyper operations of hyper BCC-algebra,
X:set; f:one-to-one mapping)
Output("f is a isomorphism from X toX or not.")
Begin
   Stop:=flase;
   i := 1;
   While i ≤ |X| and not(Stop) do
      j := 1;
      While j ≤ |X| and not(Stop) do
         If {f(t)|t ∈ x_i ∘ x_j} ≠ f(x_i) ∗ f(x_j) then
            Stop:=true;
         EndIf
      EndWhile
   EndWhile
   If Stop then
      Output("(X,∘) is not isomorphic with (X,∗).")
   Else
      Output("(X,∘) is isomorphic with (X,∗).")
   EndIf
End
```

Proposition 4.4 Let $H = \{0,a,b\}$. Then there are 15 hyperorder sets $(H,\ll)$ upto isomorphism as follows:

(1) $a \ll b$, $0 \parallel a$, $0 \parallel b$; (2) $0 \ll b$, $a \ll b$, $0 \parallel a$;
(3) $0 \ll a$, $a \ll b$, $0 \parallel b$; (4) $0 \ll a$, $0 \ll b$, $a \parallel b$;
(5) $b \ll 0$, $0 \parallel a$, $a \parallel b$; (6) $b \ll 0$, $a \ll b$, $0 \parallel a$;
(7) $b \ll 0$, $0 \ll a$, $a \parallel b$; (8) $b \ll 0$, $0 \ll a$, $a \ll b$;
(9) $b \ll 0$, $b \ll a$, $0 \parallel a$; (10) $b \ll 0$, $b \ll a$, $0 \ll a$;
(11) $a \ll 0$, $b \ll 0$, $a \parallel b$; (12) $a \ll 0$, $b \ll 0$, $a \ll b$;
(13) $0 \parallel a$, $0 \parallel b$, $a \parallel b$; (14) $0 \ll a$, $0 \parallel a$, $a \parallel b$;
(15) $0 \ll a$, $0 \ll b$, $a \ll b$.

For every above situation, we calculate all non-isomorphic proper hyper BCI-algebras by the algorithems in Algorithm 1 and Algorithm 2.

Proposition 4.5 Let $H = \{0,a,b\}$. If its underlying hyperorder is one of the following situations:

(1) $a \ll b$, $0 \parallel a$, $0 \parallel b$; (2) $0 \ll b$, $a \ll b$, $0 \parallel a$;
(3) $0 \ll a$, $a \ll b$, $0 \parallel b$; (4) $0 \ll a$, $0 \ll b$, $a \parallel b$;
(5) $b \ll 0$, $0 \parallel a$, $a \parallel b$; (6) $b \ll 0$, $a \ll b$, $0 \parallel a$;
(7) $b \ll 0$, $0 \ll a$, $a \parallel b$; (8) $b \ll 0$, $0 \ll a$, $a \ll b$;
(9) $b \ll 0$, $b \ll a$, $0 \parallel a$; (10) $b \ll 0$, $b \ll a$, $0 \ll a$;
(11) $a \ll 0$, $b \ll 0$, $a \parallel b$; (12) $a \ll 0$, $b \ll 0$, $a \ll b$.

Then, there is no proper hyper BCI-algebras in H.

Proposition 4.6 Let $H = \{0,a,b\}$ with $0 \parallel a, 0 \parallel b, a \parallel b$. Then there exists only one proper hyper BCI-algebra upto isomorphism as in Table 5.

Table 5

$\circ$	0	a	b
0	$\{0\}$	$\{b\}$	$\{a\}$
a	$\{a\}$	$\{0\}$	$\{b\}$
b	$\{b\}$	$\{a\}$	$\{0\}$

Proposition 4.7 Let $H = \{0,a,b\}$ with $0 \ll a, 0 \parallel a, a \parallel b$. Then there exist two proper hyper BCI-algebras upto isomorphism as in Table 3 and Table 6.

Table 6

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{b\}$
a	$\{a\}$	$\{0\}$	$\{b\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

Proposition 4.8 Let $H = \{0,a,b\}$ with $0 \ll a, 0 \ll b, a \ll b$. Then there exist eleven proper hyper BCI-algebras upto isomorphism as in Table 2 and Table 7 $\sim$ Table 16.

Table 7

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a,b\}$	$\{0,b\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

Table 8

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a,b\}$	$\{0,a,b\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

Table 9

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a,b\}$	$\{0,b\}$	$\{0,b\}$
b	$\{b\}$	$\{b\}$	$\{0,b\}$

Table 10

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a,b\}$	$\{0,a,b\}$	$\{0,b\}$
b	$\{b\}$	$\{b\}$	$\{0,b\}$

Table 11

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a,b\}$	$\{0,a,b\}$	$\{0,a,b\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

Table 12

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a,b\}$	$\{0,b\}$	$\{0,a,b\}$
b	$\{b\}$	$\{b\}$	$\{0,b\}$

Table 13

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a,b\}$	$\{0,a,b\}$	$\{0,a,b\}$
b	$\{b\}$	$\{b\}$	$\{0,b\}$

Table 14

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a,b\}$	$\{0,a,b\}$	$\{0,a,b\}$
b	$\{b\}$	$\{b\}$	$\{0,a,b\}$

Table 15

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a,b\}$	$\{0,a,b\}$	$\{0,a,b\}$
b	$\{b\}$	$\{a,b\}$	$\{0,b\}$

Table 16

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a,b\}$	$\{0,a,b\}$	$\{0,a,b\}$
b	$\{b\}$	$\{a,b\}$	$\{0,a,b\}$

Proposition 4.9 Let $H = \{0,a,b\}$. Then there exist five proper weak hyper BCI-algebras upto isomorphism. Table 1 and Table 17 $\sim$ Table 20 give these Cayley tables.

Table 17

$\circ$	0	a	b
0	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{a\}$	$\{0,a\}$

Table 18

$\circ$	0	a	b
0	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0,a\}$

Table 19

$\circ$	0	a	b
0	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0,a,b\}$

Table 20

$\circ$	0	a	b
0	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{a,b\}$	$\{0,a,b\}$

According to Proposition 4.4 $\sim$ Proposition 4.9, we have the following conclusion.

Proposition 4.10 There exist 15 proper hyper BCI-algebras and 4 proper weak hyper BCI-algebras of order 3 up to isomorphism.

5 Applications

In this paper, we obtained a full classification of proper hyper BCI-algebras of Order 3 and compared our results with those obtained previously in [21]. Our results are important supplements for the theory of hyper BCI-algebra and related hyper structure. In addition, they are also useful for the development of classical and non-classical propositional calculi [12,23] in artificial intelligence and information sciences. There are some systems which contain the only implication functor among the logical functors, and these examples are the system of positive implicational calculus, weak positive implicational calculus by A. Church, and BCI, BCK-systems by C.A. Meredith [12]. The interest for these algebras is justified by the fact that the objects utilized in artificial and intelligence information sciences are the "weak representations" of these algebras. So, its weak representations of an interval algebra are the objects of interest in the information sciences and artificial intelligence. Our results can be applied to study “interval calculi” used in artificial intelligence and information sciences for representing temporal knowledge.

6 Conclusion

The study of the numeration problem up to isomorphism in various kinds of algebraic system is one of ultimate goals of algebra research. And the numeration problem of lower order algebra system is the basis and starting point of the corresponding algebraic system research. In this paper, we investigate the numeration problem on proper hyper BCI-algebras of order 3. First of all, we investigated that there are 15 hyper-order structure of order 3 up to isomorphism. Then, for every hyper-order structure, we calculated all non-isormorphic proper hyper BCI-algebras by Matlab program calculation. Finally, we calculated that there exist 14 proper hyper BCI-algebras of order 3 up to isomorphism. These Cayley tables will play an important role in the development of the theory of hyper BCI Calgebra. We hope that this work would serve as a foundation for further study of the theory of hyper BCI-algebra and its related hyper structure. The results can be perhaps applied in information sciences.

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References

[1] K. Iski, An algebras related with a propositional calculus[J]. Proc. Japan Acad. **42**, 26-29 (1966).
[2] J. Meng, Y.B. Jun, BCK-algebras[M]. Kyung Moon Sa Co., Seoul, Korea, (1994).
[3] Y.S. Huang, BCI-algebra[M]. Science Press, Beijing, China, (2006).
[4] F. Marty, Sur une generalization de la notion de group[J]. 8th Congress Math. Scandinaves, Stockholm, 45-49 (1934).
[5] M. Krasner, A class of hyperrings and hyperfields[J]. International Journal of Mathematica and Mathematical Sciences, **6**, 207-312 (1983).
[6] X.Z. Guo, X.L. Xin, On hyperlattice[J]. Pure and Applied Mathematics, **20**, 40-43 (2004).
[7] Y. B. Jun, M.M. Zahedi, X.L. Xin and R.A. Borzooei, On hyper BCK-algebras[J]. Italian Journal of Pure and Applied Mathematics, **8**, 127-136 (2000).
[8] R.A. Borzooei, A. hasankhani, M.M. Zahedi and Y.B. Jun, On hyper-K-algebras[J]. Math. Japonica, **52**, 113-121 (2000).
[9] Y.B. Jun, E.G. Boh and H. Harizavi, Hyper BCC-algebras[J]. Honam Mathematical J. **28**, 57-67 (2006).
[10] R.A. Borzooei, W. A. Dudek and N. Koohestani, On Hyper BCC-algebras[J]. International Journal of Mathematics and Mathematical Sciences, 1-18 (2006).
[11] X.L. Xin, Hyper BCI-algebras[J]. Discussiones Mathematicae General Algebra and Applications, **26**, 5-19 (2006).
[12] P. Corsini, V. Leoreanu, Applications of hyperstructure theory[M]. Kluwer Academic Publishers, Dordrecht, Netherlands, (2003).
[13] B. Davvaz, V. Leoreanu-Fotea, Hyperring theory and applications[M]. International Academic Press, USA, (2007).
[14] X.H. Yuan, H.X. Li and K.B. Sun, Granular computing theory based on hypergroups[J]. Fuzzy Systems and Mathematics, **25**, 133-142 (2011).
[15] A. D. Khalafi, B. Davvaz, Algebraic hyper-structures associated to convex analysis and applications[J]. Filomat, **26**, 55-65 (2012).
[16] B. Davvaz, A.D. Nezhad, and A. Benvidi, Chemical hyperalgebra: dismutation reactions[J]. MATCH Commun. Math. Comput. Chem., **67**, 55-63 (2012).
[17] B. Davvaz, A. Dehghan and M.M. Heidari, Inheritance examples of algebraic hyperstructures[J]. Information Sciences, **224**, 180-187 (2013).
[18] F. Nisar, R.S. Tariq and S.A. Bhatti, Bi-polar-valued fuzzy hyper subalgebras of a hyper BCI-algebra[J]. World Applied Sciences Journal, **9**, 25-33 (2010).
[19] F. Nisar et al., Fuzzy ideals in hyper BCI-algebras[J]. World Applied Sciences Journal, **16**, 1771-1777 (2012).
[20] N. Palaniappan et al., Intuitionistic fuzzy ideals in hyper BCI-algebras[J]. International Journal of Computational Science and Mathematics, **4**, 271-285 (2012).
[21] F. Nisar, R. S. Tariq and S.A. Bhatti, On proper hyper BCI-algebras of order 3[J]. European Journal of Scientific Research, **37**, 319-336 (2009).
[22] A.L. Zhang, X.L. Xin, On quotient hyper-algebras of hyper BCI-algebras[J]. Pure and Applied Mathematics, **24**, 774-778 (2008).
[23] X. H. Zhang, Strong NMV-algebras, commutative basic algebras and naBL-algebras[J], Mathematica Slovaca, **63**, 661-678 (2013).

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