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A General Case for the Maximum Norm Analysis of an Overlapping Schwarz Methods of Evolutionary HJB Equation with Nonlinear Source Terms with the Mixed Boundary Conditions

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Abstract: In this paper we provide a maximum norm analysis of an overlapping Schwarz method on non-matching grids for evolutionary HJB equation with nonlinear source terms with the mixed boundary conditions and a general elliptic operator. Moreover, an asymptotic behavior in uniform norm is established.

Keywords: Domain Decomposition, HJB equation, PQVIs,Error Estimate, Asymptotic Behavior

1 Introduction

The main work of this paper is to extend the previous numerical analysis results ([3], [4], [5]) to the following new evolutionary HJB equations with mixed boundary conditions and the general elliptic operator: find $u(x,t)$ such that $u \in L^2(0,T;K(u))$, $u_t \in L^2\left(0,T;L^2(\Omega)\right)$

$$\begin{cases} \dfrac{\partial u^i}{\partial t} + \max\limits_{i=1,\dots,M}\left(A^i u - f^i(u)\right) = 0, \text{ in } \Sigma, \\ \dfrac{\partial u^i}{\partial \eta} = \psi^i \text{ in } \Gamma_0,\ i = 1,\dots,M, \\ u^i = 0 \text{ in } \Gamma/\Gamma_0,\ u^i(x,0) = u_0^i \text{ in } \Omega \end{cases} \tag{1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^d$, $d \geq 1$ and Σ is a set in $\mathbb{R}\times\mathbb{R}^d$ defined as $\Sigma = [0,\ T]\times\Omega$ with $T < +\infty$. A^i are the differential operators defined as follows

$$A^i = -\sum_{j,k=1}^{N} \frac{\partial}{\partial x_j} a^i_{jk}(x) \frac{\partial}{\partial x_k} + \sum_{k=1}^{N} b^i_k(x) \frac{\partial}{\partial x_k} + a^i_0(x) \tag{2}$$

and their bilinear forms are associated with A^i; for $u,v \in H^1_0(\Omega)$

$$a^i(u,v) = \int_\Omega \left(\sum_{j,k=1}^{N} a^i_{jk}(x) \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_k} dx\right) + + \int_\Omega \left(\sum_{j=1}^{N} b^i_k(x)\frac{\partial u}{\partial x_j} v + a^i_0(x)\, uv dx\right), \tag{3}$$

assumed to be noncoercive.

and the smooth functions $a^i_{k,j}(x)$, $b^i_k(x)$, $a^i_0(x) \in \left(L^\infty(\Omega)\cap C^2(\bar{\Omega})\right)^M$, $x \in \bar{\Omega}, 1 \leq k,\ \ j \leq N$ are sufficiently smooth coefficients and satisfy the following conditions

$$a^i_{jk}(x) = a^i_{kj}(x);\ \ a^i_0(x) \geq \beta > 0,\ \beta \text{ is a constant} \tag{4}$$

such that

$$\sum_{j,k=1}^{N} a^i_{jk}(x)\xi_j\xi_k \geq \gamma|\xi|^2;\ \xi \in \mathbb{R}^N,\ \gamma > 0,\ x \in \bar{\Omega} \tag{5}$$

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with the right hand side $f^1(.)$, $f^2(.)$, ..., $f^M(.)$ are M nonlinear and Lipschitz functions with Lipschitz constant c and satisfying the following condition

$$f^i \in \left(L^2(0,T,L^\infty(\Omega)) \cap C^1\left(0,T,H^{-1}(\Omega)\right)\right)^M,$$
$$f^i > 0 \text{ and also it's increasing,} \tag{6}$$
$$c < \beta.$$

We shall also need the following norm

$$\forall\, W = \left(w^1, w^2, \ldots\ldots, w^M\right) \in \prod_{i=1}^{M} L^\infty(\Omega), \tag{7}$$
$$\|W\|_\infty = \max_{1\le i\le M} \left\|w^i\right\|_\infty.$$

Γ_0 is the part of the boundary defined by:

$$\Gamma_0 = \left\{x \in \partial\Omega = \Gamma \text{ such that } \forall \xi > 0,\ x + \xi \notin \bar{\Omega}\right\},$$

where $\dfrac{\partial u^i}{\partial \eta} = \nabla u^i . \overrightarrow{\eta_i}$, such that $\overrightarrow{\eta}_i$ is the normal vector, the symbol $(.,.)_{\Gamma_0}$ stands for the inner product in $L^2(\Gamma_0)$.

$K(u)$ is an implicit convex set defined as follows

$$K\left(u^i\right) = \begin{cases} \left(u^1, u^2 \ldots\ldots u_h^M\right) \in \left(L^2\left(0,T,H_0^1(\Omega)\right)\right)^M, \\ u^i(x) \le l + u^{i+1},\ \dfrac{\partial u^i}{\partial \eta} = \psi^i \text{ in } \Gamma_0, \\ u^i = 0 \text{ in } \Gamma/\Gamma_0,\ u^i(x,0) = u_0^i \text{ in } \Omega. \end{cases} \tag{8}$$

Finally, $\dfrac{\partial u}{\partial \eta} = \nabla u . \overrightarrow{\eta}$, such that $\overrightarrow{\eta}$ is the normal vector. The symbol $(.,.)_\Omega$ stands for the inner product in $L^2(\Omega)$, $(.,.)_{\Gamma_0}$ stands for the inner product in $L^2(\Gamma_0)$.

Domain decomposition ideas have been applied to a wide variety of problems. We did not want to include all these techniques in this work. For an extensive survey of recent advances, we refer to the proceedings of the annual domain decomposition meetings see. http ://www.ddm.org. Domain decomposition algorithms is divided into two classes, those that use overlapping domains, which refer to as Schwarz methods, and those that use non-overlapping domains, which we refer to as substructuring. Any domain decomposition method is based on the assumption that the given computational domain Ω is decomposed into subdomains Ω_i, $i = 1, \ldots, M$, which may or may not overlap. Next, the original problem can be reformulated upon each subdomain Ω_i, yielding a family of subproblems of reduced size that are coupled one to another through the values of the unknowns solution at subdomain interfaces. Fruitful references can be found in [1], [2], [19], [20]. A numerical study of elliptic and parabolic problems by the finite element combined with a finite difference methods ([5], [6], [7],[8],[9], [10], [11], [12], [13], [15]) and by the domain decomposition method combined with finite element method was treated in [3], [4], [14], [16], [17], [19], [20].

In [3] we treated the overlapping domain decomposition method combined with a finite element approximation for the elliptic quasi-variational inequalities related with impulse control problem, where it can be provided with a maximum norm analysis of an overlapping Schwarz method on non-matching grids for the elliptic quasi-variational inequalities related to impulse control problem with respect to the mixed boundary conditions for a simple operator Δ. Then, in [4] we extended the last result for the parabolic quasi variational with the previous similar conditions and using the theta time scheme combined with a finite element spatial approximation and proved that the discretization on every subdomain converges in uniform norm. Furthermore a result of asymptotic behavior in uniform norm has been given by the following theorem, for the first case $\theta \ge \frac{1}{2}$

$$\left\|u_h^{\theta,p,2n} - u^\infty\right\|_\infty \le C\left[h^2\left|\log h\right|^3 + \left(\tfrac{1}{1+\beta\theta\Delta t}\right)^p\right] \tag{9}$$

and

$$\left\|u_h^{\theta,p,2n} - u^\infty\right\|_\infty \le C\left[h^2\left|\log h\right|^3 + \left(\tfrac{1}{1+\beta\theta\Delta t}\right)^p\right] \tag{10}$$

and for the second case $0 \le \theta < \dfrac{1}{2}$

$$\begin{aligned}\left\|u_h^{\theta,p,2n+1} - u^\infty\right\|_\infty &\le Ch^2\left|\log h\right|^3 \\ &+ C\left(\frac{2}{2+\theta(1-2\theta)\rho(A)}\right)^p\end{aligned} \tag{11}$$

and

$$\begin{aligned}\left\|u_h^{\theta,p,2n} - u^\infty\right\|_\infty &\le Ch^2\left|\log h\right|^3 \\ &+ C\left(\frac{2}{2+\theta(1-2\theta)\rho(A)}\right)^p,\end{aligned} \tag{12}$$

where C is a constant independent of h, k and $u_h^\theta(T,x)$, the discrete solution calculated at the moment $T = p\Delta t$ and u^∞, the asymptotic continuous solution. and $\rho(A)$ is the spectral radios of operator A.

Moreover, in [5], we concerned with the system of parabolic quasi-variational inequalities (PQVIs) related to HJB equation with non linear source terms, our goal is to show that evolutionary HJB equations can be properly approximated by a semi- implicit time scheme combined with a finite element spatial method which turns out to be quasi-optimally accurate in uniform norm. as we have carried out before for HJB equation with linear source terms. We approximate the HJB equation by a weakly coupled system of parabolic quasi-variational inequalities and introduce discrete iterative scheme which based in Bensoussan-Lions' algorithm. At the same time, we proved its geometric convergence. Then, we established an L^∞-asymptotic behavior similar to that in, [14], [21] which investigated the stationary and the evolutionary of the free boundary problem and Hamilton-Jacobi-Bellman equations with two cases: linear and nonlinear source terms, and we gave the following estimate

$$\left\|U_h^p - U^\infty\right\|_\infty = \max_{1\leq\, i\, \leq M} \left\|u_h^{i,p} - u^{i,\infty}\right\|_\infty$$

$$\leq C^* \left[h^2 \left|\log h\right|^3 + \left(\tfrac{1+kc}{1+k\beta}\right)^p\right],$$

with C^* a constant independent of both h and k, where $U_h^{,p} = \left(u_h^1, ..., u_h^p\right)$, the discrete solution calculated at the moment-end $T = p\Delta t$ for an index of the time discretization $k = 1, ..., p$, and U^∞, the asymptotic continuous solution with respect the right hand side condition.

We consider a domain which the union of two overlapping sub-domains, where each sub-domain has its own generated triangulation. The grid points on the sub-domain boundaries need not much the grid points from the other sub-domain. Under a discrete maximum principle [9], we show that the discretization on each sub-domain converges quasi-optimally in the L^∞-norm . For that purpose, further to the above arguments, our main tool is a discrete L^∞-stability property with respect the obstacle, the right-hand side and the mixed boundary conditions.

The outline of the paper is as follows. In Section 2, we lay down some notations and assumptions needed through out the paper and state both the continuous and discrete parabolic quasi variational inequalities. In section 3, we state the continuous alternating Schwarz sequence for parabolic quasi-variational inequalities and define their respective the theta scheme combined with a finite element counterparts in the context of overlapping grids. Then, we prove the L^∞-stability analysis of the θ-scheme for PVIs, and finally in Section 4, we associate the discrete PQVIs problem with a fixed point mapping and we use that in proving the existence of a unique discrete solution, In section 5 the geometrical convergence is established using the new iterative discrete algorithm stands in theta scheme. Then, an L^∞-asymptotic behavior estimate for each sub-domain is derived in uniform norm.

2 The Schwarz method for the parabolic Quasi-variational inequalities.

We begin by down some definitions and classical results related to Quasi-variational inequalities.

2.1 The continuous parabolic quasi-variational inequalities

The problem (1) can be approximated by the following system of the continuous parabolic inequalities: find $\left(u^1, u^2.....u_h^M\right) \in \left(L^2\left(0, T, H_0^1(\Omega)\right)\right)^M$ solution to

$$\begin{cases} \dfrac{\partial u^i}{\partial t} + A^i u^i \leq f^{\,i}\left(u^i\right) \text{ in } \Sigma, \\ u^i \leq l + u^{i+1},\ u^{M+1} = u^1, \\ \left(\dfrac{\partial u^i}{\partial t} + A^i u^i - f^i\left(u^i\right)\right)\left(u^i - \left(l + u^{i+1}\right)\right) = 0, \\ u^i(0,x) = u_0^i \text{ in } \Omega,\ i = 1, ..., M, \\ \dfrac{\partial u^i}{\partial \eta} = \psi^i \text{ in } \Gamma_0 \text{ and } u^i = 0 \text{ in } \Gamma/\Gamma_0, \end{cases}, \quad (13)$$

which is similar to that in [15] which investigated the stationary Hamilton-Jacobi-Bellman equations.

So, after a simple mathematical development and by using the Riez presentation, the problem (13)can be transformed into the following continuous parabolic quasi-variational inequalities: find $\left(u^1, u^2.....u_h^M\right) \in \left(L^2\left(0, T, H_0^1(\Omega)\right)\right)^M$ solution of

$$\begin{cases} \left(\dfrac{\partial u^i}{\partial t}, v^i - u^i\right)_\Omega + a^i\left(u^i, v^i - u^i\right) \\ \geq \left(f^i\left(u^i\right), v^i - u^i\right) - \left(\psi, v - u^i\right)_{\Gamma_0}, \\ u^i \leq l + u^{i+1},\ v^i \leq l + u^{i+1}, \\ u^i(0,x) = u_0^i \text{ in } \Omega,\ i = 1, ..., M, \\ \dfrac{\partial u^i}{\partial \eta} = \psi^i \text{ in } \Gamma_0 \text{ and } u^i = 0 \text{ in } \Gamma/\Gamma_0, \end{cases}, \quad (14)$$

where $a^i(.,.)$ is the bilinear form associated with operator A^i defined in (3).

and

$$\left(f^i\left(u^i\right), v\right)_\Omega = \int_\Omega f^i\left(u^i\right).vdx$$

with

$$\left(\varphi^i, v\right)_{\Gamma_0} = \int_{\Gamma_0} \varphi.vd\sigma.$$

2.2 The discrete system of parabolic quasi-variational inequalities

Let Ω be decomposed into triangles and τ_h denote the set of all those elements $h > 0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform. We consider the usual basis of affine functions φ_l, $l=\{1,...,m(h)\}$ defined by $\varphi_l(M_s)=\delta_{ls}$ where M_s is a vertex of the considered triangulation. We introduce the following discrete spaces V^h of finite element

$$V^h=\begin{cases} v\in\left(L^2\left(0,T,H_0^1(\Omega)\right)\cap C\left(0,T,H_0^1(\bar{\Omega})\right)\right)^M, \\ \text{such that} \\ v\,|_K\in P_1,\; K\in\tau_h, \\ u(.,0)=u_0 \text{ in } \Omega, \\ \dfrac{\partial u^i}{\partial\eta}=\psi^i \text{ in } \Gamma_0 \text{ and } u^i=0 \text{ in } \Gamma/\Gamma_0, \end{cases} \tag{15}$$

where r_h is the usual interpolation operator defined by

$$\begin{aligned} &v\in L^2\left(0,T,H_0^1(\Omega)\right)\cap C\left(0,T,H_0^1(\bar{\Omega})\right), \\ &r_h v=\sum_{i=1}^{m(h)} v(M_i)\,\varphi_i(x) \end{aligned} \tag{16}$$

and P_1 denotes the space of polynomials with degree at most 1.

In the sequel of the paper, we shall make use of the discrete maximum principle assumption (dmp). In other words, we shall assume that the matrices $\left(A^i\right)_{ps}=a(\varphi_p,\varphi_s)$, $1\le i\le M$ are M-matrices (cf. [9]).

We discretize in space the problem (14), i.e. that we approach the space H_0^1 by a space discretization of finite dimensional $V_h\subset H_0^1$. Then we discretize the previous semi discrete spatial problem with respect to time by using the semi–implicit scheme. Therefore, we search a sequence of elements $u^{i,k}\in\left(H_0^1(\Omega)\right)^M$ which approaches $u^i(t_k)$, $t_k=k\Delta t$, with initial data $u^{i,0}=u_0^i$.

Thus, we have for $k=1,...,p$,

$$\begin{cases} \left(\dfrac{u_h^{i,k}-u_h^{i,k-1}}{\Delta t},v_h^i-u_h^{i,k}\right)_\Omega+a^i\left(u_h^{i,k},v_h^i-u_h^{i,k}\right)\ge \\ \ge\left(f^i\left(u_h^{i,k}\right),v_h^i-u_h^{i,k}\right)_\Omega-\left(\psi^i,v_h^i-u_h^{i,k}\right)_{\Gamma_0}, \\ u_h^{i,k}\le r_h\left(l+u_h^{i+1,k}\right), \\ v_h^i\le r_h\left(l+u_h^{i+1,k}\right), \\ u^{i,0}(x)=u_0^i \text{ in } \Omega,\; i=1,...,M, \\ \dfrac{\partial u^{i,k}}{\partial\eta}=\psi^{i,k} \text{ in } \Gamma_0 \text{ and } u^i=0 \text{ in } \Gamma/\Gamma_0, \end{cases} \tag{17}$$

which implies

$$\begin{cases} \left(\dfrac{u_h^{i,k}}{\Delta t},v_h^i-u_h^{i,k}\right)_\Omega+a^i\left(u_h^{i,k},v_h^i-u_h^{i,k}\right)\ge \\ \ge\left(f^{i,k}\left(u_h^k\right)+\dfrac{u_h^{i,k-1}}{\Delta t},v_h^i-u_h^{i,k}\right)_\Omega-\left(\psi^i,v_h^i-u_h^{i,k}\right)_{\Gamma_0}, \\ u_h^{i,k}\le r_h\left(l+u_h^{i+1,k}\right),\; u_h^{M+1}=u_h^1,\; l>0, \\ u_h^{i,k}(0)=u_{0h}^{i,k} \text{ in } \Omega,\; i=1,...,M, \\ \dfrac{\partial u^{i,k}}{\partial\eta}=\psi^{i,k} \text{ in } \Gamma_0, \\ u^i=0 \text{ in } \Gamma/\Gamma_0, \end{cases} \tag{18}$$

Then, the problem(18) can be reformulated into the following coercive discrete system of elliptic quasi-variational inequalities (EQVIs)

$$\begin{cases} b^i\left(u_h^{i,k},v_h^i-u_h^{i,k}\right)\ge \\ \left(f\left(u_h^{i,k-1}\right)+\lambda u_h^{i,k-1},v_h^i-u_h^{i,k}\right)_\Omega- \\ -\left(\psi^i,v_h^i-u_h^{i,k}\right)_{\Gamma_0},\quad u_h^{i,k}\in\left(V^h\right)^M \\ u_h^{i,k}\le r_h\left(l+u_h^{i+1,k}\right),\; u_h^{M+1}=u_h^1,\; l>0, \\ u_h^{i,k}(0)=u_{0h}^{i,k} \text{ in } \Omega,\; i=1,...,M \\ \dfrac{\partial u^{i,k}}{\partial\eta}=\varphi^{i,k} \text{ in } \Gamma_0 \text{ and } u^i=0 \text{ in } \Gamma/\Gamma_0, \end{cases} \tag{19}$$

such that

$$\begin{cases} b\left(u_h^{i,k},v_h^i-u_h^{i,k}\right)=\lambda\left(u_h^{i,k},v_h^i-u_h^{i,k}\right)+ \\ +a^i\left(u_h^{i,k},v_h^i-u_h^{i,k}\right),\; u_h^{i,k}\in\left(V^h\right)^M, \\ \lambda=\dfrac{1}{\Delta t}=\dfrac{1}{k}=\dfrac{T}{n},\; k=1,...,n. \end{cases} \tag{20}$$

2.3 Approximation of the HJB equation by a system of discrete PQVIs

As we have defined before, A^i denote the finite elements matrices defined by

$$\left(A^i\right)_{ls}=a^i(\varphi_l,\;\varphi_s)\quad 1\le i\le M,\;1\le l,\;s\le m(h)$$

and let B^i denote the finite elements matrices defined by

$$\left(B^i\right)_{ls} = b^i\left(\varphi_l,\ \varphi_s\right) \quad 1 \le i \le M,\ 1 \le l,\ s \le m(h) \quad (21)$$

respectively, where

$$b^i\left(\varphi_l,\ \varphi_s\right) = a^i\left(\varphi_l,\ \varphi_s\right) + \lambda\left(\varphi_l,\ \varphi_s\right).$$

Now, in the light of the above definitions, notations, and assumptions and according to the above discretization by the semi–implicit scheme, we are in position to define the discrete HJB equation. This latter consists of solving the following semi discrete problem: find $u^{i,k} \in \left(H_0^1(\Omega)\right)^M$

$$\max_{1\le i\le M}\left(B^i u^k - F^{i,k}\left(u^{i,k}\right)\right) = 0. \quad (22)$$

Additionally, according to the above discretization by the finite element approximation applied to (14), and it can be easily reformulated (22) as: for $u_h^{i,k} \in V_h$

$$\max_{1\le i\le M}\left(B^i u_h^k - F^{i,k}(u_h^{i,k})\right) = 0, \quad (23)$$

where

$$F_l^{i,k}(u_h^k) = \left(f^{i,k}\left(u_h^{i,k}\right) + \lambda u_h^{k-1},\ \varphi_l\right)_\Omega - \left(\psi^i, v_h^i - u_h^{i,k}\right)_{\Gamma_0}$$

and

$$\lambda = \frac{T}{n},\ k = 1,\ldots,n.$$

Thanks to [6], [11], [12], [13], [15], the problem (23) can be approximated by the following system of discrete elliptic quasi-variational inequalities (EQVIs): find $\left(u_h^{1,k}, u_h^{2,k}.....u_h^{M,k}\right) \in \left(V_h^i\right)^M$ solution to

$$\begin{cases} b^i\left(u_h^{i,k}, v_h^i - u_h^{i,k}\right) \ge \\ \left(f^{i,k}\left(u_h^{i,k}\right) + \lambda u_h^{i,k-1}, v_h^i - u_h^{i,k}\right)_\Omega \\ -\left(\psi^i, v_h^i - u_h^{i,k}\right)_{\Gamma_0}, \\ u_h^{i,k} \le r_h\left(l + u_h^{i+1,k}\right), \\ v_h^i \le r_h\left(l + u_h^{i+1,k}\right), \\ u_h^{M+1,k} = u_h^{1,k},\ i = 1,\ldots,M, \\ \dfrac{\partial u^{i,k}}{\partial \eta} = \psi^i \text{ in } \Gamma_0 \text{ and } u_h^i = 0 \text{ in } \Gamma/\Gamma_0. \end{cases}.$$

Let $(M\xi,\varphi)$, $\left(M\tilde{\xi},\tilde{\varphi}\right)$ be a pair of data, and $\xi = \sigma(M\xi,\varphi)$, $\tilde{\xi} = \sigma\left(M\tilde{\xi},\tilde{\varphi}\right)$ be the corresponding solutions to the following parabolic quasi-variational inequalities (PQVI):

$$b^i\left(\xi^i, v - \xi^i\right) \ge \left(f^{i,}\left(\xi^i\right) + \lambda w, v^i - \xi^i\right)_\Omega + \\ + \left(\varphi^i, v^i - \xi^i\right)_{\Gamma_0}$$

and

$$\begin{cases} b\left(\tilde{\xi}, v - \tilde{\xi}\right) \ge \left(f^{\theta,k}, v - \tilde{\xi}\right)_\Omega + \\ + \left(\tilde{\varphi}, (v - \xi)\right)_{\Gamma_0}, \forall v \in H^1(\Omega) \end{cases}.$$

Lemma 1.*(cf.[3])Under the previous hypotheses and the notation..*

$$\text{If } \varphi \ge \tilde{\varphi}. \text{Then } \sigma(M\xi,\varphi) \ge \sigma\left(M\tilde{\xi},\tilde{\varphi}\right).$$

Proposition 1.*(cf.[3])Under the previous hypotheses, we have the following inequality*

$$\|u - \tilde{u}\|_{L^\infty(\Omega_i)} \le \|Mu - M\tilde{u}\|_{L^\infty(\Omega_i)} \\ + \|\varphi - \tilde{\varphi}\|_{L^\infty(\partial\Omega_i \cap \Omega_j)}, \quad (24)$$

$$\text{such that } i \ne j,\ i,j = 1,2,$$

where $Mu = l + u$.

3 The discrete Schwarz sequences.

The discrete maximum principle assumption (dmp) ??: We assume the matrix whose coefficients $a(\varphi_i,\varphi_j)$ are $M-$matrix. For convenience in all the sequels, C will be a generic constant independent on h.

As we have defined before Ω be a bounded open domain in $\mathbb{R}^2$ and we assume that Ω is smooth and connected.

Then we decompose Ω in two sub-domains Ω_1, Ω_2 such that

$$\Omega = \Omega_1 \cup \Omega_2 \quad (25)$$

and u satisfies the local regularity condition

$$u\,|_{\Omega_i} \in L^2(0,T,W^{2,p}(\Omega_i)) \quad (26)$$

and we denote by $\Gamma = \partial\Omega$, $\Gamma_1 = \partial\Omega_1$, $\Gamma_2 = \partial\Omega_2$, $\gamma_1 = \partial\Omega_1 \cap \Omega_2$, $\gamma_2 = \partial\Omega_2 \cap \Omega_1$, $\Omega_{1,2} = \Omega_1 \cap \Omega_2$.

For $d = 1,\ 2$, let τ^{hd} be a standard regular and quasi-uniform finite element triangulation in Ω_d; $h_i\,(h_1 = h_2 = h)$, being the meshsize. We assume that the two triangulations are mutually independent on $\Omega_{1,2}$ in the sense that a triangle belonging to one triangulation does not necessarily belong to the other.

Let V^{h_d} be the space of continuous piecewise linear functions on τ^{h_d} which vanish on $\Omega_d \cap \partial\Omega_j,\ p \neq j,\ p,\ j = 1,\ 2$. For $w \in C\left(\partial\bar{\Omega}_d\right)$ we define

$$V_w^{h_d} = \begin{cases} v_h \in V^{h_d} :\ v_h = \pi_{h_d}(w) \ \text{ on } \Omega_d \cap \partial\Omega_j, \\ v_h(.,0) = v_{h0} \text{ in } \Omega, \\ \dfrac{\partial v_h}{\partial \eta} = \psi \text{ in } \Gamma_0\ , \\ v_h = 0 \text{ in } \Gamma/\Gamma_0;\ \ d \neq j,\ i,\ j = 1,\ 2, \end{cases}, \quad (27)$$

where π_{h_d} denotes the interpolation operator on $\partial\Omega_d$.

We consider the model obstacle problem: Find $u_h^{i,k} \in V_h$such that

$$b^i\left(u_h^{i,k}, v_h - u_h^{i,k}\right) \geq \left(f^{\ i,k}\left(u_h^{i,k}\right) + \mu u_h^{k-1}, v_h - u_h^{i,k}\right)_{\Omega} + \left(\psi_h^i, v_h - u_h^{i,k}\right)_{\Gamma_0},\ v_h,\ u_h^{i,k} \in V_h \quad (28)$$

We define the discrete counterparts of the discrete Schwarz sequences defined in (28), respectively by $u_h^{i,k,2n+1},\ v_h \in V^h_{\left(u_h^{i,k,2n}\right)}$, such that

$$\begin{cases} b\left(u_h^{i,k,2n+1}, v_h - u_h^{2n+1}\right) - \\ \left(f^{\ i,k}\left(u_h^{i,k}\right) + \mu u_h^{i,k-1,2n-1}, \left(v_h - u_h^{i,k,2n+1}\right)\right)_{\Omega_1} \\ -\left(\psi_h^i, v - u_h^{i,k,2n+1}\right)_{\Gamma_0} \geq 0, \\ u_h^{i,k,2n+1} = u_h^{i,k,2n} \text{ on } \partial\Omega_1, \\ v_h = u_h^{i,k,2n} \text{ on } \partial\Omega_1, \\ u_h^{i,k,2n+1} \leq r_h\left(l + u_h^{i+1,k,2n-1}\right), \\ \dfrac{\partial u^{i,k,2n+1}}{\partial \eta} = \psi^i \text{ in } \Gamma_0 \text{ and } u_h^{i,2n+1} = 0 \text{ in } \Gamma/\Gamma_0, \end{cases} \quad (29)$$

and $u_h^{i,k,2n}, v^h \in V^h_{\left(u_h^{i,k,2n-1}\right)}$ such that

$$\begin{cases} b\left(u_h^{i,k,2n}, v_h - u_h^{2n}\right) \\ -\left(f^{\ \theta,k} + \mu u_h^{i,k-1,2n-2}, \left(v_h - u_h^{i,k,2n}\right)\right)_{\Omega_2} \\ -\left(\psi_h^i, v - u_h^{i,k,2n}\right)_{\Gamma_0} \geq 0, \\ u_h^{i,k,2n} = u_h^{i,k,2n-1} \text{ on } \partial\Omega_2,\ v_h = u_h^{i,k,2n-1} \text{ on } \partial\Omega_2, \\ u_h^{i,k,2n} \leq r_h\left(l + u_h^{i+1,k,2n-2}\right), \\ \dfrac{\partial u^{i,k,2n}}{\partial \eta} = \psi^{i,2n} \text{ in } \Gamma_0, \\ u_h^{i,2n} = 0 \text{ in } \Gamma/\Gamma_0. \end{cases} \quad (30)$$

3.1 Existence and uniqueness for discrete PQVIs.

Next using the preceding assumptions, we shall prove the existence of a unique solution for problem (23) by means of the Banach's fixed point theorem.

3.1.1 A fixed point mapping associated with discrete problem

We defined : $\mathbf{H}^+ = \prod_{i=1}^{M} L_+^\infty(\Omega)$, where $L_+^\infty(\Omega)$ denotes the positive cone of $L^\infty(\Omega)$. Now we define the following mapping

$$\begin{aligned} T_h : \mathbf{H}^+ &\longrightarrow \left(L^\infty(\Omega)\right)^M \\ W &\longrightarrow TW = \xi_h^{i,k} = \left(\xi_h^{1,k}, \xi_h^{2,k}, ..., \xi_h^{M,k}\right) \\ &= \partial_h\left(F^{i,k}\left(w^i\right), l + w^{i+1}\right), \end{aligned} \quad (31)$$

such that $\xi_h^{i,k}$, $\forall i = 1,...,M$ is the solution of the following problem

$$\begin{cases} b^i\left(\xi_h^{i,k}, v_h^i - \zeta_h^{i,k}\right) \geq \left(f^{i,k}\left(\xi_h^{i,k}\right) + \lambda w^i, v_h^i - \xi_h^{i,k}\right)_\Omega \\ -\left(\psi_h^i, v_h^i - \xi_h^{i,k}\right)_{\Gamma_0}, \; v_h^i \in V_h, \\ \xi_h^{i,k} \leq r_h\left(l + w^{i+1}\right), \\ v_h^i \leq r_h\left(l + w^{i+1}\right), \; i = 1,2,...,M, \\ \xi_h^{M+1,k} = \xi_h^{1,k}. \; k = 1,...,p, \\ \dfrac{\partial u^{i,k}}{\partial \eta} = \psi^i \text{ in } \Gamma_0 \text{ and } u_h^i = 0 \text{ in } \Gamma/\Gamma_0, \end{cases} . \tag{32}$$

4 An iterative discrete algorithm

We choose u_h^0 as the solution of the following discrete equation

$$b(u_h^{i,0}, v_h) = \left(g^{i,0}, v_h\right), \; v_h \in V^h, \tag{33}$$

where g^0 is a regular function give.

Now we give the following discrete algorithm

$$\begin{aligned} &U_h^{k,2n+1} = T_h u_h^{i,k-1,2n+1}, k = 1,..,p, \\ &U_h^{k,2n+1} \in V^h_{\left(u_h^{i,k,2n}\right)}, \end{aligned} \tag{34}$$

and

$$\begin{aligned} &U_h^{k,2n} = T_h U_h^{k-1,2n}, k = 1,..,p, \\ &U_h^{k,2n} \in V^h_{\left(u_h^{i,k,2n-1}\right)}, \end{aligned} \tag{35}$$

where $U_h^{k,2n+1} = \left(u_h^{1,k,2n+1},...,u_h^{M,k,2n+1}\right)$ and $U_h^{k,2n} = \left(u_h^{1,k,2n},...,u_h^{M,k,2n}\right)$ are the solutions of the problems (34) (resp (35))

*Remark.*We denote by

$$\mathbf{Q} = \left\{W \in \mathbf{H}^+, \text{ such that } 0 \leq W \leq U^0\right\}, \tag{36}$$

where $U^0 = U_0 = \left(u_0^1,...,u_0^M\right)$.

Since $f^{i,k}\ (.) \geq 0$, and $u_h^{i,0} = u_{h0}^i \geq 0$, combining comparison results in variational inequalities with a simple induction, it follows that $u^{i,k} \geq 0$, i.e., $U^k \geq 0$, $\forall k = 1,...,p$ and $TW \geq 0$.
Furthermore, by (35)and (36)we have

$$U^{1,2n} = TU^{0,2n} \leq U^{0,2n}.$$

Similar to that in previous works [4], [5], [6], the mapping T is a monotone increasing for the stationary HJB equation with non linear source term. Then it can be easily verified that

$$U^{2,2n} = TU^{1,2n} \leq TU^{0,2n}$$

$$= U^{1,2n} \leq U^{0,2n},$$

thus, inductively

$$U^{k+1,2n} = TU^{k,2n} \leq U^{k,2n}$$

$$\leq ... \leq U^{0,2n}, \; \forall k = 1,...,p$$

and also it can be seen the sequence $\left(u^k\right)_k$ stays in $\mathbf{Q}$.

Let

$$F^{i,k}\left(v^i\right) = f^{i,k}\left(u^i\right) + \lambda v^i,$$

$$G^{i,k}(w) = f^{i,k}\left(u^i\right) + \lambda w^i \in \left(L^\infty(\Omega)\right)^M$$

be the corresponding right-hand sides to the PQVIs.

Proposition 2.*The mapping T_h is Lipchitz on $\mathbf{H}^+$i.e.,*

$$\|T_hV - T_hW\|_\infty \leq \|V - W\|_\infty, V, W \in \mathbf{H}^+.$$

*Proof.*We clearly have

$$\|T_hV - T_hW\|_\infty = \max_{1\leq i\leq M}\left\|(T_hV)^i - (T_hW)^i\right\|_\infty =$$

$$\max_{1\leq i\leq M}\left\|\partial_h\left(F^{i,k}, Mv_h^{i,k-1}\right) - \partial_h\left(G^{i,k}, Mw_h^{i,k-1}\right)\right\|_\infty,$$

where $(T_hW)^i$ and $(T_hV)^i$ denote the i^{th} components of the vectors W and V, respectively.
Setting

$$\phi^{i,k} = \max\begin{pmatrix} \left\|r_h\left(l + v^{i+1}\right) - r_h\left(l + w^{i+1}\right)\right\|_\infty, \\ \left\|F^{i,k}\left(v_h^i\right) - G^{i,k}\left(w_h^i\right)\right\|_\infty \end{pmatrix}.$$

We have

$$r_h\left(l + v^{i+1}\right) \leq r_h\left(l + w^{i+1}\right) + \left\|r_hv_h^{i+1} - r_hw^{i+1}\right\|_\infty$$

$$\leq r_h\left(l + w^{i+1}\right) + \phi^{i,\theta,k}.$$

Moreover, we have

$$F^{i,k}\left(v^i\right) \leq G^{i,k}\left(w^i\right) + \left\|F^{i,k}\left(v^i\right) - G^{i;k}\left(w^i\right)\right\|_\infty$$

Under the assumption (6), we have

$$F^{i,k}\left(v^i\right) \le G^{i,k}\left(w^i\right) + \frac{a_0}{\beta+\lambda}\left(c+\lambda\right)\left\|v^i + w^i\right\|_\infty$$

$$\le G^{i,k}\left(w^i\right) + a_0\phi^{i,k} + \lambda,$$

it follows that

$$\begin{cases} \partial_h\left(F^{i,k}\left(u^i\right), l + w^{i+1}\right) \\ \le \partial_h\left(G^{i,k}\left(w^i\right) + a_0\left(x\right)\phi^{i,k} + \lambda, l + w^{i+1}\right) \\ \le \partial_h\left(G^i\left(w^i\right), l + w^{i+1}\right) + \phi^{i,k}, \end{cases}$$

Therefore

$$T_hV \le T_hW + \phi^{i,k}.$$

Similarly, interchanging the roles of $(v)^i$ and $(w)^i$ we also get

$$T_hW \le T_hV + \phi^{i,k}.$$

Thus

$$\left\|T_hW - T_h\tilde{W}\right\|_\infty = \max_{1\le\ i\ \le M}\left\|(T_hW)^i - \left(T_h\tilde{W}\right)^i\right\|_\infty =$$

$$\max_{1\le\ i\ \le M}\left\|\partial_h\left(F^{i,k}, l + v^{i+1}\right) - \partial_h\left(G^{i,k}, l + w^{i+1}\right)\right\|_\infty.$$

Then, we can easily deduce

$$\begin{cases} \left\|T_hV - T_hW\right\|_\infty \le \\ \le \max\begin{pmatrix} \left\|r_h\left(l + v_h^{i+1}\right) - r_h\left(l + w_h^{i+1}\right)\right\|_\infty, \\ \left\|F^{i,k} - G^{i,k}\right\|_\infty \end{pmatrix} \\ \le \max\begin{pmatrix} \left\|r_h\left(l + v_h^{i+1}\right) - r_h\left(l + w_h^{i+1}\right)\right\|_\infty, \\ \left(\frac{1+kc}{1+k\beta}\right)\left\|v_h^{i,k} - w_h^{i,k}\right\|_\infty \end{pmatrix} \\ \le \left\|V - W\right\|_\infty \end{cases}$$

Remark. If we only use the right hand side properties (see [5]), we get the the mapping T_h is contraction with the rate of contraction $\dfrac{1+kc}{1+k\beta}$. Therefore, $T_{h\lambda}$ admits a unique fixed point which coincides with the solution of EQVIs (28).

Proposition 3. *Under the previous hypotheses and notations, we have the following estimate of convergent*

$$\left\|U_h^k - U_h^\infty\right\|_\infty \le \left(\tfrac{1+kc}{1+k\beta}\right)^k\left\|U_h^\infty - U_{h_0}\right\|_\infty. \tag{37}$$

C is a constant independent of h and k.

Proof. We have

$$u_h^{i,\infty} = T_hu_h^{i,\infty},$$

$$\left\|u_h^{i,1} - u_h^{i,\infty}\right\|_\infty = \left\|T_hu_h^{i,0} - T_hu_h^{i,\infty}\right\|_\infty$$

$$\le \left(\tfrac{1+kc}{1+k\beta}\right)\left\|u_h^{i,0} - u_h^{i,\infty}\right\|_\infty$$

and also we have

$$\left\|u_h^{i,k+1} - u_h^{i,\infty}\right\|_\infty = \left\|T_hu_h^{i,k-1} - T_hu_h^{i,\infty}\right\|_\infty$$

$$\le \left(\tfrac{1+kc}{1+k\beta}\right)\left\|u_h^{i,k} - u_h^{i,\infty}\right\|_\infty.$$

Then

$$\max_{1\le\ i\le\ M}\left\|u_h^{i,k+1} - u_h^{i,\infty}\right\|_\infty \le$$

$$\le \left(\tfrac{1+kc}{1+k\beta}\right)^{k+1}\max_{1\le\ i\le\ M}\left\|u_h^{i,0} - u_h^{i,\infty}\right\|_\infty,$$

thus

$$\left\|U_h^k - U_h^\infty\right\|_\infty \le \left(\frac{1+kc}{1+k\beta}\right)^k\left\|U_h^\infty - U_{h_0}\right\|_\infty.$$

5 L^∞-Asymptotic Behavior

5.1 Convergence proof via the maximum principle for a system of elliptic quasi variational inequalities with non linear source terms: non coercive case

We introduce the sets

$$T^{\ i,2n} = \begin{cases} u^{i,2n} \in V^h_{u^{i,2n-1}} : u_t^{2n} + A^iu^{2n} \le f^{\ i}\left(u^{i,2n}\right)\ , \\ u^{i,2n} = u^{i,2n-1} \text{ on } \partial\Omega_2, \\ u^{i,2n} \le l + u^{i,2n-2} \text{ on } \Omega_2, \\ u^{i,2n} = 0 \text{ in } \Gamma/\Gamma_0, \\ u^{i,2n} = \psi^i \text{ in } \Gamma_0 \end{cases}$$

and

$$T^{\ i,2n+1} = \begin{cases} u^{i,2n+1} \in V^h_{u^{2n}} : \\ u_t^{2n+1} + A^iu^{2n+1} \le f^{\ i}\left(u^{i,2n+1}\right), \\ u^{i,2n+1} = u^{i,2n} \text{ on } \partial\Omega_1, \\ u^{i,2n+1} \le l + u^{i,2n-1} \text{ on } \Omega_1, \\ u^{i,2n+1} = 0 \text{ in } \Gamma/\Gamma_0, \\ u^{i,2n} = \psi^i \text{ in } \Gamma_0. \end{cases}$$

Lemma 2.[20]*If A is the $M-$matrice and $u_h^{i,2n}$ (resp. $u_h^{i,2n+1}$) is the solution (29)(resp. (30)). Then $u_h^{i,2n}$ (resp.$u_h^{i,2n+1}$) is the minimal of $T^{i,2n}$ (resp.$T^{i,2n+1}$).*

Theorem 1.*Let u_h^i be a solution of (18). Then the iterative sequence $\left\{u_h^{i,2n}\right\}$(resp. $\left\{u_h^{i,2n+1}\right\}$) is monotone; that is, $u_h^{i,2n} \in T^{i,2n}$ (resp.$u_h^{i,2n+1} \in T^{i,2n+1}$) and $u_h^i \leq u_h^{i,2n+2} \leq u_h^{i,2n} \leq .... \leq u_h^{i,0}$.*

*Proof.*We take $u_h^0 = u_h \mid \Omega_2$ such that $A^i u_h^0 = f$,.We know that if $u_h^{i,0} \leq l + u_h^{i+1,0}$ then $\left(u_t^0 + A^i u_h^0 - f^i\left(u^i\right)\right) |_{\Omega_2} \leq 0$. Therefore, using the Riez presentation, it can be deduced that

$$b^i\left(u_h^{i,0},\, v_h - u_h^{i,0}\right)_{\Omega_2} - \left(f\left(u^i\right),\, \left(v_h - u_h^{i,0}\right)\right)_{\Omega_2} - \left(\psi^i,\, \left(v - u_h^{i,0}\right)\right)_{\Gamma_0} \geq 0.$$

Thus

$$u_h^{i,0} \in T^{i,0}.$$

From Lemma (2) we know that $u_h^{i,2}$ is the minimal element of $T^{i,0}$. So

$$u_h^{i,2} \leq r_h\left(l + u_h^{i,0}\right),$$

we yields that

$$u_h^{i,2} \leq u_h^{i,0}.$$

By induction, for index n we obtain

$$u_h^{i,2n} \leq u_h^{i,2n-2} \leq ... \leq u_h^{i,2} \leq u_h^{i,0} = u_h^i.$$

We know that if

$$u_h^{i,3} \leq r_h M u_h^{i,1},$$

then

$$\left(u_t^3 + A^i u_h^{i,3} - f\left(u^i\right)\right) |_{\Omega_1} \leq 0$$

that is

$$\begin{cases} b^i\left(u_h^{i,3},\, v_h - u_h^{i,3}\right)_{\Omega_1} - \left(f\left(u^i\right),\, \left(v_h - u_h^{i,3}\right)\right)_{\Omega_1} \\ -\left(\psi^i,\, \left(v - u_h^{i,3}\right)\right)_{\Gamma_0} \geq 0. \end{cases}.$$

Therefore $u_h^{i,3} \in T^{i,3}$. Also from Lemma (2), we know that $u_h^{i,3}$ is the minimal element of $T^{i,3}$.We yields that $u_h^{i,3} \leq u_h^{i,1}$.

By induction, for index n we obtain

$$u_h^{i,2n+1} \leq u_h^{i,2n-1} \leq ... \leq u_h^{i,1}.$$

Lemma 3.*If $A = (a_{ij})_{i,j=\{1....N\}}$ is the M-matrix. Then there exists two constants $k_1,\ k_2$*

$$k_1 = \sup\left\{w_h(x),\ x \in \gamma_2\right\} \in (0,1)$$

and

$$k_1 = \sup\left\{w_h(x),\ x \in \gamma_1\right\} \in (0,1),$$

such that

$$\sup_{\gamma_1}\left|u_h - u_h^{2n+1}\right| \leq k_1 \sup_{\gamma_1}\left|u_h - u_h^{2n}\right| \tag{38}$$

and

$$\sup_{\gamma_2}\left|u_h - u_h^{2n+1}\right| \leq k_2 \sup_{\gamma_2}\left|u_h - u_h^{2n}\right|. \tag{39}$$

*Remark.*The demonstration of Lemma (3) is an adaptation of the one in [20]) given for the problem of variational inequality.

*Remark.*The Lemma (3) remains true for the coercive case.

The main convergence result is given by the following theorem:

Theorem 2.[3]*The sequences $\left(u_h^{i,2n+1}\right)$; $\left(u_h^{i,2n}\right)$, $n \geq 0$ produced by the Schwarz alternating method converge geometrically to the solution u of the stationary obstacle problem. More precisely, there exist $k_1,\ k_2 \in (0,1)$ which depend only respectively of (Ω_1, γ_2) and (Ω_2, γ_1) such that all $n \geq 0$.*

$$\sup_{\bar{\Omega}_1}\left|u_h^i - u_h^{i,2n+1}\right| \leq k_1^n k_2^n \sup_{\gamma_1}\left|u_h^i - u_h^{i,0}\right| \tag{40}$$

and

$$\sup_{\bar{\Omega}_2}\left|u_h^i - u_h^{i,2n}\right| \leq k_1^n k_2^{n-1} \sup_{\gamma_2}\left|u_h^i - u_h^{i,0}\right|. \tag{41}$$

Theorem 3.[16]*Under the previous assumptions, and the maximum principle assumption, there exists a constant C independent of h such that*

$$\left\|u^{i,\infty} - u_h^{i,\infty}\right\|_\infty \leq Ch^2 \left|\log h\right|^3,$$

where u^∞ is an asymptotic continuous solution.

5.2 Error estimate for the EQVIs.

Theorem 4.*Let u be a solution of the stationary problem of (14). Then there exists a constant C independent of both h and n such that*

$$\left\|u^i - u_h^{i,2n+1}\right\|_{L^\infty(\bar{\Omega}_1)} \leq Ch^2 \left|\log h\right|^3 \tag{42}$$

and

$$\left\|u^i - u_h^{i,2n}\right\|_{L^\infty(\bar{\Omega}_2)} \leq Ch^2 \left|\log h\right|^3. \tag{43}$$

*Proof.*Setting here $k = k_1 = k_2$, and using Theorem (3) and Theorem (4), we have

$$\left\| u^i - u_h^{i,2n+1} \right\|_{L^\infty(\bar{\Omega}_1)} \leq \| u^i - u_h^i \|_{L^\infty(\bar{\Omega}_1)} + \left\| u_h^i - u_h^{i,2n+1} \right\|_{L^\infty(\bar{\Omega}_1)}$$

$$\leq \| u^i - u_h^i \|_{L^\infty(\bar{\Omega}_1)} + k^{2n} \left\| u_h^i - u_h^{i,0} \right\|_{L^\infty(\gamma_1)}$$

$$\leq Ch^2 |\log h|^2 + k^{2n} \left\| u_h^i - u_h^{i,0} \right\|_{L^\infty(\gamma_1)}$$

$$\leq Ch^2 |\log h|^2$$

$$+k^{2n} \left(\| u^i - u_h^i \|_{L^\infty(\gamma_1)} + \left\| u^i - u_h^{i,0} \right\|_{L^\infty(\gamma_1)} \right)$$

Thus we can deduce

$$\left\| u^i - u_h^{i,2n+1} \right\|_{L^\infty(\bar{\Omega}_1)} \leq Ch^2 |\log h|^2 + Ch^2 k^{2n} |\log h|^2$$

and also setting

$$k^{2n} \leq |\log h|,$$

we get

$$\left\| u^i - u_h^{i,2n+1} \right\|_{L^\infty(\bar{\Omega}_1)} \leq Ch^2 |\log h|^3.$$

Thus, we can deduce

$$\| U_h^{2n+1} - U^\infty \|_\infty = \max_{1 \leq i \leq M} \left\| u_h^{i,2n+1} - u^{i,\infty} \right\|_\infty \leq Ch^2 |\log h|^3.$$

The proof of the (43) case is similar.

5.3 Asymptotic behavior for the PQVIs

This section is devoted to the proof of main result of the present paper, where we prove the theorem of the asymptotic behavior in L^∞-norm for Hamilton-Jacobi-Billman.

Now we evaluate the variation in $(L^\infty(\Omega))^M$-norm between $U_h(T,x)$, the discrete solution calculated at the moment $T = n\Delta t$ and $u^{i,\infty}$, the following asymptotic continuous solution

$$\begin{cases} b^i\left(u^{i,\infty}, v^i - U^\infty\right) \geq \\ \geq \left(f^i\left(u^{i,\infty}\right) + \lambda u^{i,\infty}, v^i - u^{i,\infty}\right), \\ u^{i,\infty} \leq l + u^{i+1,\infty}, \\ v^i \leq l + u^{i+1,\infty},\ i = 1, \ldots, M, \end{cases}, \qquad (44)$$

where $f^i\left(u^{i,\infty}\right)$ is a bounded on $(L^\infty(\Omega))^M$ and $v^i \in H_0^1(\Omega)$.

Theorem 5.*Under the previous hypotheses and notations, we have*

$$\left\| U_h^{2n} - U^\infty \right\|_\infty \leq C^* \left[h^2 |\log h|^3 + \left(\tfrac{1+kc}{1+k\beta} \right)^p \right] \qquad (45)$$

$$\left\| U_h^{2n+1,p} - U^\infty \right\|_\infty \leq C^* \left[h^2 |\log h|^3 + \left(\tfrac{1+kc}{1+k\beta} \right)^p \right] \qquad (46)$$

where $C^* = \max(1, C)$

*Proof.*Using Theorem (4) and Proposition (3), we get

$$\left\| u_h^{i,2n,p} - u^\infty \right\|_\infty \leq C^* \left[h^2 |\log h|^3 + \left(\tfrac{1+kc}{1+k\beta} \right)^p \right]$$

and also it can be easily found

$$\left\| U_h^{2n,p} - U^\infty \right\|_\infty = \max_{1 \leq i \leq M} \left\| u_h^{i,2n,p} - u^{i,\infty} \right\|_\infty \leq C^* \left[h^2 |\log h|^3 + \left(\tfrac{1+kc}{1+k\beta} \right)^p \right],$$

which completes the proof.
The proof of (46) case is similar.

*Remark.*It can be seen that in the previous estimates (45)and (46) $\left(\frac{1+kc}{1+k\beta} \right)^p$ tends to 0 when p approaches to infinity. Therefore, [4], [5], thanks to Theorem (5), the convergence order for the both cases: the coercive and noncoercive problems are:

$$\| U^{2n,\infty} - U_h^\infty \|_{L^\infty(\Omega)} \leq Ch^2 |\log h|^3$$

6 Conclusion

In this paper, we have introduced a new approach of an overlapping Schwarz method on non-matching grids for parabolic quasi-variational inequalities related to impulse control problem with respect to the mixed boundary conditions and with a general case for the elliptic operator, where we have established the asymptotic behavior in uniform norm similar to that in the previous published paper [3] regarding the overlapping Schwarz method for the stationary free boundary problems,. The type of estimation, which we have obtained here, is important for the calculus of quasi-stationary state for the simulation of petroleum or gaseous deposit.

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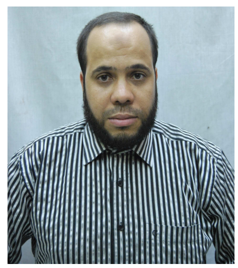

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