Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/090344

Hopf Bifurcations in a Delayed Microscopic Model of Credit Risk Contagion

Carlo Bianca[1,2,*] *and Luca Guerrini*[3]

[1] Sorbonne Universités, UPMC Univ Paris 06, UMR 7600, Laboratoire de Physique Théorique de la Matière Condensée, 4, place Jussieu, case courrier 121, 75252 Paris cedex 05, France
[2] CNRS, UMR 7600 LPTMC, Paris, France
[3] Polytechnic University of Marche, Department of Management, 60121 Ancona, Italy

Received: 10 Aug. 2014, Revised: 10 Oct. 2014, Accepted: 14 Oct. 2014
Published online: 1 May 2015

Abstract: This paper is concerned with the proof of the existence of Hopf bifurcations in a mathematical model recently proposed in [T. Chen, X. Li, and J. He, Abstract and Applied Analysis **2014**, 456764 (2014)] for understanding the complex stochastic dynamics phenomena of credit risk contagion in the financial market. Specifically the model consists in an ordinary differential equation with time-delay. Moreover, by using the normal form theory and center manifold argument, the stability, direction, and period of bifurcating periodic solutions are gained.

Keywords: Steady-state, Time delay, Credit Risk, Stability analysis, Center manifold

1 Introduction

Since the last century, the mathematical modeling of complex phenomena emerging in nature and society has attracted much attention. In particular the general principles that are at the origin of the complexity in a given system are not completely understood and the treatment of a complex system strictly depends on the number of the components of the system and their interactions [1].

Complex phenomena have been modeled in many financial, economic and social systems, and nonlinear dynamics involving Hopf bifurcation, chaos and fractals has been numerically and analitycally identified, see, among others, papers [2,3,4,5,6] and the references cited therein. Moreover in order to take into account that most of the phenomena arising in economics and also in biological systems at a certain time are strictly related to the behavior of the system at a previous time, mathematical models with time delays have been proposed in the pertinent literature, see [7,8,9,10,11,12,13,14,15,16,17]. The main effects of a time delay have been the changing of the stability of the equilibrium (stable equilibrium becomes unstable), fluctuations, and Hopf bifurcation, see [18,19,20].

Recently Chen et al. have proposed in [21] a time-delayed microscopic mathematical model of credit risk contagion in the financial market driven by correlated Gaussian white noises. Moreover the related time-delayed Fokker-Planck model driven by correlated noises has been derived and a sensitivity analysis on the stationary probability distribution function of the dynamical system of credit risk contagion has been performed. According to [21], the deterministic part of the credit risk contagion model with time-delay and correlated noises is described by the following delayed differential equation:

$$\dot{N}(t) = \lambda_1 N(t) + \lambda_2 N_d(t) - \mu \xi \lambda_2^2 [N_d(t)]^2, \qquad (1)$$

where $N(t)$ is the density of the credit activity population subjected to the credit risk in the financial market, $N_d(t) := N(t-\tau)$, being τ the time-delay of credit risk contagion, λ_1 is the contagion rate of credit risk related to direct business relation, λ_2 is the contagion rate of credit risk related to indirect business relation ($\lambda_2 < \lambda_1$), μ is referred to the Nelength scale, ξ is the nonlinear resistance coefficient of the relationship network comprising credit activity participants in the financial market.

* Corresponding author e-mail: bianca@lptmc.jussieu.fr

Letting the time-delay and nonlinear resistance coefficient to vary, the authors of [21] have numerically investigated and analyzed the model (1) showing numerically the existence of Hopf bifurcations and chaotic behaviors of credit risk. Specifically they found that as the infectious scale of credit risk and the wavy frequency of credit risk contagion are increased, the stability of the system of credit risk contagion is reduced, the dynamical system of credit risk contagion gives rise to chaotic phenomena, and the chaotic area increases gradually with the increase in time-delay. The nonlinear resistance only influences the infectious scale and range of credit risk, which is reduced when the nonlinear resistance coefficient increases.

This paper is concerned with the mathematical analysis of the mathematical model (1). Specifically this paper deals with the existence of nontrivial positive steady states and the related asymptotic stability. Moreover, the existence of the Hopf bifurcation numerically shown in [21] is proved and, by using the normal form theory and center manifold argument, the explicit formulas which determine the stability, direction, and period of bifurcating periodic solutions are obtained. It is worth stressing that, differently from [21], our analysis is not restricted to the case $\lambda_1 > \lambda_2$. Therefore the analytical results contained in this paper also generalize the numerical results obtained in [21].

The contents of the present paper are organized as follows: After this introduction, Section 2 is devoted to the existence and stability analysis of steady-state equilibria of the mathematical model (1) including also the proof of the existence of Hopf bifurcations. Section 3 is concerned with the qualitative analysis of the bifurcating periodic solutions.

2 Steady-state equilibria and stability analysis

Setting $\tau = 0$, the mathematical model (1) admits the following unique positive equilibrium

$$N_* = \frac{\lambda_1 + \lambda_2}{\mu\xi\lambda_2^2}.$$

Let $x = N_d - N_*$, then Eq. (1) can be rewritten as

$$\dot{x} = \lambda_1 x - (2\lambda_1 + \lambda_2)x_d - \mu\xi\lambda_2^2 x_d^2, \tag{2}$$

whose linear part reads

$$\dot{x} = \lambda_1 x - (2\lambda_1 + \lambda_2)x_d, \tag{3}$$

The corresponding characteristic equation, obtained by substituting $x = e^{-\eta\tau}$, is

$$\eta - \lambda_1 + (2\lambda_1 + \lambda_2)e^{-\eta\tau} = 0. \tag{4}$$

When $\tau = 0$, we have $\eta = -(\lambda_1 + \lambda_2) < 0$. Hence, the null equilibrium of (2), and so the positive equilibrium N_* of (1), is locally asymptotically stable.

We choose now the time delay $\tau > 0$ as the bifurcation parameter. It is well known that the equilibrium point is locally asymptotically stable if all the roots of the characteristic equation have negative real parts and unstable if at least one root has positive real part. We will examine now the localization of the roots of the transcendental equation (4). First, note that $\eta = 0$ is not a root of (4). For $\omega > 0$, $\eta = i\omega$ is a root of (4) if and only if

$$i\omega - \lambda_1 + (2\lambda_1 + \lambda_2)e^{-i\omega\tau} = 0.$$

Separating the real and imaginary parts we have

$$\omega = (2\lambda_1 + \lambda_2)\sin\omega\tau, \qquad \lambda_1 = (2\lambda_1 + \lambda_2)\cos\omega\tau, \tag{5}$$

which leads to $\omega^2 = (2\lambda_1 + \lambda_2)^2 - \lambda_1^2$.

Lemma 1. *The characteristic equation (4) has a pair of purely imaginary roots $\lambda = \pm i\omega_0$ at $\tau = \tau_j$, where*

$$\omega_0 = \sqrt{3\lambda_1^2 + \lambda_2^2 + 4\lambda_1\lambda_2}, \tag{6}$$

and

$$\tau_j = \frac{1}{\omega_0}\left[\tan^{-1}\left(\frac{\omega_0}{\lambda_1}\right) + 2\pi j\right], \quad j = 0, 1, 2, \ldots \tag{7}$$

Proof. If $\lambda = \pm i\omega_0$ were not simple, then one would have $1 - \lambda_1\tau_j + i\omega_0\tau_j = 0$, leading to a contradiction. Finally, the critical values τ_j are derived from (5).

Let $\eta(\tau) = \nu(\tau) + i\omega(\tau)$ be a root of (4) near $\tau = \tau_j$ satisfying the conditions $\nu(\tau_j) = 0$ and $\omega(\tau_j) = \omega_0$.

Lemma 2. *The root of characteristic equation (4) near τ_j crosses the imaginary axis from the left to the right as τ continuously varies from a number less than τ_j to one greater than τ_j.*

Proof. Differentiating both sides of the characteristic equation (4) with respect to τ, and using (4), we obtain

$$\frac{d\eta}{d\tau} = -\frac{(\eta - \lambda_1)\eta}{1 + (\eta - \lambda_1)\tau}.$$

Then

$$\nu'(\tau_j) = Re\left(\frac{d\eta}{d\tau}\right)_{\tau=\tau_j} = Re\left[-\frac{(i\omega_0 - \lambda_1)i\omega_0}{1 + (i\omega_0 - \lambda_1)\tau_j}\right]$$
$$= \frac{\omega_0^2}{(1 - \lambda_1\tau_j)^2 + \omega_0^2\tau_j^2} > 0, \tag{8}$$

completing the proof.

Proposition 1. *If $\tau \in [0, \tau_0)$, all roots of (4) have negative real parts. If $\tau = \tau_0$, all roots of (4) except $\lambda = \pm i\omega_0$ have negative real parts. If $\tau \in (\tau_j, \tau_{j+1})$, for $j = 0, 1, 2, \ldots$, then (4) has $2(j+1)$ roots with positive real parts.*

*Proof.*The conclusions are straightforward from the above results.

Bearing all above in mind, the following theorem holds true.

Theorem 1.*Let ω_0 and τ_j be defined as in (6) and (7), respectively. Then the equilibrium N_* of (1) is locally asymptotically stable for $\tau \in [0,\tau_0)$, and unstable for $\tau > \tau_0$. Furthermore, Eq. (1) undergoes a Hopf bifurcation at the positive equilibrium N_* when $\tau = \tau_j$, $j = 0,1,2,\ldots$*

3 Qualitative analysis of the bifurcating periodic solutions

This section deals with analytical results on the direction, stability and period of the bifurcating periodic solutions in Eq. (2). The analysis is based on the normal form theory and the center manifold theorem proposed by Hassard et al. in [22].

Without loss of generality, we will investigate the critical value $\tau = \tau_0$ at which (2) undergoes a Hopf bifurcation from the null equilibrium. For notational convenience we set $\tau = \tau_0 + \mu$, $\mu \in \mathbb{R}$. Then $\mu = 0$ is the Hopf bifurcation value for (2). Set

$$L_\mu(\varphi) = \lambda_1\varphi(0) - (2\lambda_1+\lambda_2)\,\varphi(-\tau), \quad \varphi \in C([-\tau_0,0],\mathbb{R},$$

and

$$f(\mu,\varphi) = -\mu\xi\lambda_2^2\varphi(-\tau)^2.$$

By the Riesz representation theorem, there exists a bounded variation function $\eta(\theta,\mu)$, $\theta \in [-\tau_0,0]$, such that

$$L_\mu\varphi = \int_{-\tau_0}^{0} d\eta(\theta,\mu)\varphi(\theta), \qquad \varphi \in C([-\tau_0,0],\mathbb{R}).$$

In fact, we can choose

$$\eta(\theta,\mu) = \lambda_1\delta(\theta) - (2\lambda_1+\lambda_2)\,\delta(\theta+\tau),$$

where δ is the Dirac delta function. For $\varphi \in C([-\tau_0,0],\mathbb{R})$, we define

$$A(\mu)(\varphi) := \begin{cases} \dfrac{d\varphi(\theta)}{d\theta}, & \theta \in [-\tau_0,0), \\ \displaystyle\int_{-\tau_0}^{0} d\eta(r,\mu)\varphi(r), & \theta = 0, \end{cases}$$

and

$$R(\mu)(\varphi) := \begin{cases} 0, & \theta \in [-\tau_0,0), \\ f(\mu,\varphi), & \theta = 0. \end{cases}$$

Then we can rewrite Eq. (2) as

$$\dot{x}_t = A(\mu)x_t + R(\mu)x_t, \tag{9}$$

where $x_t = x(t+\theta)$, for $\theta \in [-\tau_0,0]$. For $\psi \in C([0,\tau_0],\mathbb{R})$, the adjoint operator A^* of $A = A(0)$ is defined as follows:

$$A^*(\mu)\psi(r) = \begin{cases} -\dfrac{d\psi(r)}{dr}, & r \in (0,\tau_0], \\ \displaystyle\int_{-\tau_0}^{0} d\eta(r,\mu)\psi(-r), & r = 0. \end{cases}$$

For $\varphi,\psi \in C([-\tau_0,0],\mathbb{R})$, we consider the following inner product:

$$< \psi,\varphi >:=$$
$$\bar{\psi}(0)\varphi(0) - \int_{\theta=-\tau_0}^{0}\int_{r=0}^{\theta} \bar{\psi}(r-\theta)d\eta(\theta,0)\varphi(r)dr.$$

In order to determine the normal form of the operator A, we need to calculate the eigenvector q (resp. q^*) of A (resp. A^*) belonging to the eigenvalue $i\omega_0$ (resp. $-i\omega_0$). By direct calculation, it can be verified that $q(\theta) = e^{i\omega_0\theta}$ and $q^*(s) = De^{i\omega_0 s}$, where

$$D = \frac{1}{1-(2\lambda_1+\lambda_2)\,e^{i\omega_0\tau_0}}.$$

Furthermore, $< q^*,q >= 1$ and $< q^*,\bar{q} >= 0$.

For x_t solution of Eq. (9) at $\mu = 0$, we define

$$z(t) =< q^*,x_t >$$

and

$$W(t,\theta) = x_t(\theta) - 2Re\{z(t)q(\theta)\}.$$

On the center manifold, we have

$$\begin{aligned} W(t,\theta) &= W(z(t),\bar{z}(t),\theta) \\ &= W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \end{aligned} \tag{10}$$

where z and $\bar{z}$ are local coordinates for the center manifold in the direction of q^* and $\bar{q}^*$, respectively. Then (9) on the center manifold is described by

$$\dot{z}(t) = i\omega_0 z(t) + \bar{q}^*(0)\,f_0(z,\bar{z}), \tag{11}$$

where

$$f_0(z,\bar{z}) = f(0,W(z(t),\bar{z}(t),0) + 2Re\{z(t)q(0)\}).$$

Eq. (11) can be written in the abbreviated form as

$$\dot{z}(t) = i\omega_0 z + g(z,\bar{z}),$$

with

$$\begin{aligned} g(z,\bar{z}) &= \bar{q}^*(0)\,f_0(z,\bar{z}) \\ &= g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \end{aligned} \tag{12}$$

Therefore, we have

$$\dot{W} = \dot{x}_t - \dot{z}q - \dot{\overline{zq}} =$$

$$\begin{cases} AW - 2Re\{\bar{q}^*(0) f_0 q(\theta)\}, \ \theta \in [-\tau_0, 0), \\ AW - 2Re\{\bar{q}^*(0) f_0 q(0)\} + f_0, \qquad \theta = 0, \end{cases} \tag{13}$$

namely

$$\dot{W} = AW + H(z, \bar{z}, \theta),$$

where

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots.$$

From $x_t(\theta) = W(z,\bar{z},\theta) + ze^{i\omega_0\theta} + \bar{z}e^{-i\omega_0\theta}$, we get

$$\begin{aligned} x_t(0) &= W(z,\bar{z},0) + z + \bar{z}, \\ x_t(-\tau_0) &= W(z,\bar{z},-\tau_0) + ze^{-i\omega_0\tau_0} + \bar{z}e^{i\omega_0\tau_0}, \end{aligned}$$

and

$$\begin{aligned} f_0(z,\bar{z}) &= -\mu\xi\lambda_2^2\left[W(z,\bar{z},-\tau_0) + ze^{-i\omega_0\tau_0} + \bar{z}e^{i\omega_0\tau_0}\right]^2 \\ &= -\mu\xi\lambda_2^2 e^{-2i\omega_0\tau_0}z^2 - 2\mu\xi\lambda_2^2 z\bar{z} - \mu\xi\lambda_2^2 e^{2i\omega_0\tau_0}\bar{z}^2 \\ &\quad + [-2\mu\xi\lambda_2^2 e^{-i\omega_0\tau_0}W_{11}(-\tau_0) \\ &\quad - \mu\xi\lambda_2^2 e^{i\omega_0\tau_0}W_{20}(-\tau_0)]z^2\bar{z} \\ &\quad + [-2\mu\xi\lambda_2^2 e^{i\omega_0\tau_0}W_{11}(-\tau_0) \\ &\quad - \mu\xi\lambda_2^2 e^{-i\omega_0\tau_0}W_{20}(-\tau_0)]z\bar{z}^2 + \cdots. \end{aligned}$$

Hence, from (12), comparing coefficients, it follows that

$$g_{20} = -2\bar{D}\mu\xi\lambda_2^2 e^{-2i\omega_0\tau_0}, \tag{14}$$

$$g_{11} = -2\bar{D}\mu\xi\lambda_2^2, \tag{15}$$

$$g_{02} = -2\bar{D}\mu\xi\lambda_2^2 e^{2i\omega_0\tau_0}, \tag{16}$$

$$\begin{aligned} g_{21} &= 2\bar{D}[-2\mu\xi\lambda_2^2 e^{-i\omega_0\tau_0}W_{11}(-\tau_0) \\ &\quad -\mu\xi\lambda_2^2 e^{i\omega_0\tau_0}W_{20}(-\tau_0)] \end{aligned} \tag{17}$$

We need to compute $W_{11}(-\tau_0)$ and $W_{20}(-\tau_0)$ that appear in g_{21}. From

$$\begin{aligned} H(z,\bar{z},\theta) &= -2Re\{q^*(0) f_0 q(\theta)\} \\ &= -g(z,\bar{z})q(\theta) - \bar{g}(z,\bar{z})q(\theta) \\ &= -\left(g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots\right)q(\theta) \\ &\quad -\left(\bar{g}_{20}\frac{\bar{z}^2}{2} + \bar{g}_{11}z\bar{z} + g_{02}\frac{z^2}{2} + \cdots\right)\bar{q}(\theta), \end{aligned} \tag{18}$$

we obtain

$$\begin{aligned} H_{20}(\theta) &= -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \\ H_{11}(\theta) &= -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \end{aligned}$$

On the center manifold, $\dot{W} = W_z\dot{z} + W_{\bar{z}}\dot{\bar{z}}$. Hence, we get the equations

$$\begin{cases} (A - 2i\omega_0)W_{20}(\theta) = -H_{20}(\theta), \\ AW_{11}(\theta) \qquad = -H_{11}(\theta), \\ (A + 2i\omega_0)W_{02}(\theta) = -H_{02}(\theta). \end{cases} \tag{19}$$

From (13), we have

$$\dot{W}_{20}(\theta) = 2i\omega_0 W_{20}(\theta) - g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta).$$

Solving for $W_{20}(\theta)$, we find

$$W_{20}(\theta) = \frac{g_{20}}{i\omega_0}q(\theta) + \frac{\bar{g}_{02}}{3i\omega_0}\bar{q}(\theta) + E_1 e^{2i\omega_0\theta}, \tag{20}$$

and similarly

$$W_{11}(\theta) = -\frac{g_{11}}{i\omega_0}q(\theta) + \frac{\bar{g}_{11}}{i\omega_0}\bar{q}(\theta) + E_2,$$

where E_1, E_2 are both real constants that can be determined by setting $\theta = 0$ in $H(z,\bar{z},\theta)$. In fact,

$$\begin{aligned} H_{20}(0) &= -g_{20}q(0) - \bar{g}_{20}\bar{q}(0) + 2b_2 e^{-2i\omega_0\tau_0}, \\ H_{11}(0) &= -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + 2b_2. \end{aligned}$$

Eqs. (19) yield

$$\lambda_1 W_{20}(0) - (2\lambda_1 + \lambda_2)W_{20}(-\tau_0) =$$

$$= 2i\omega_0\tau_0 W_{20}(0) - H_{20}(0) \tag{21}$$

$$\lambda_1 W_{11}(0) - (2\lambda_1 + \lambda_2)W_{11}(-\tau_0) = -H_{11}(0).$$

Substituting (20) in (21), we can calculate E_1. Similarly, for E_2.

Based on the above analysis, one has that each g_{ij} in (14) is determined. Thus, we can compute the following quantities:

$$C_1(0) = \frac{i}{2\omega_0}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \tag{22}$$

$$\mu_2 = -\frac{Re\{C_1(0)\}}{Re\{\lambda'(\tau_0)\}},$$

$$\beta_2 = 2Re\{C_1(0)\},$$

$$T_2 = -\frac{Im\{C_1(0)\} + \mu_2 Im\{\lambda'(\tau_0)\}}{\omega_0}.$$

It is well known that μ_2 determines the direction of the Hopf bifurcation: if $\mu_2 > 0$ (resp. $\mu_2 < 0$), then the Hopf bifurcation is supercritical (resp. subcritical) and the bifurcating periodic solutions exist for $\tau > \tau_0$ (resp. $\tau < \tau_0$); β_2 determines the stability of bifurcating periodic solutions: the bifurcating periodic solutions on the center manifold are stable (resp. unstable) if $\beta_2 < 0$ (resp. $\beta_2 > 0$); T_2 determines the period of the bifurcating periodic solutions: the period increases (resp. decreases) if $T_2 > 0$ (resp. $T_2 < 0$). Since (9) gives $Re\{\lambda'(\tau_0)\} > 0$, we thus have the main result of the present paper.

Theorem 2. *Let N_* be the unique positive equilibrium of the delayed mathematical model (1) and $Re\{C_1(0)\}$ be the real part of (22). Then the direction of the Hopf bifurcation of (1) at the equilibrium N_* when $\tau = \tau_0$ is supercritical (resp. subcritical) and the bifurcating periodic solutions on the center manifold are stable (resp. unstable) if $Re\{C_1(0)\} < 0$ (resp. $Re\{C_1(0)\} > 0$).*

Acknowledgement

The first author acknowledges the financial support by ANR T-KiNeT Project.

References

[1] Y. Bar-Yam, Dynamics of Complex Systems, Studies in Nonlinearity, Westview Press, 2003.
[2] A.C.L. Chian, E.L. Rempel, and C. Rogers, Chaos, Solitons and Fractals **29**, 1194-1218 (2006).
[3] D.A. Hesieh, The Journal of Finance **46**, 1839-1877 (1991).
[4] Q. Gao and J. Ma, Nonlinear Dynamics **58**, 209-216 (2009).
[5] C. Bianca, L. Guerrini, Acta Applicandae Mathematicae **128**, 39-48 (2013).
[6] C Xu, Y Wu, L Lu, On Permanence and Asymptotically Periodic Solution of A Delayed Three-level Food Chain Model with Beddington-DeAngelis Functional Response, International Journal of Applied Mathematics, **44**, (2014).
[7] P.J. Cunningham and W.J. Wangersky, Time Lag in Population Models, Yale, 1958.
[8] T. Erneux, Applied Delay Differential Equations, Springer, New York, NY, USA, 2009.
[9] C.T.H. Baker, G. A. Bocharov, and C. A.H. Paul, Journal of Theoretical Medicine **2**, 117-128 (1997).
[10] J. Blair, M.C. Mackey, and J.M. Mahay, Mathematical Biosciences **128**, 317-346 (1995).
[11] Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, vol. 191 of Mathematics in Science and Engineering, Academic Press, Boston, Mass, USA, 1993.
[12] H. M. Byrne, Mathematical Biosciences **144**, 83-117 (1997).
[13] M. J. Piotrowska, Mathematical and Computer Modelling **47**, 597-603 (2008).
[14] Q Liu, Y Lin, J Cao, Global Hopf Bifurcation on Two-Delays Leslie-Gower Predator-Prey System with a Prey Refuge, Computational and Mathematical Methods in Medicine, **2014**, (2014).
[15] C. Bianca, L. Guerrini, The Scientific World Journal **2014**, 207806 (2014).
[16] C. Bianca, M. Ferrara, L. Guerrini, Abstract and Applied Analysis **2013**, 736058 (2013).
[17] C. Bianca, M. Ferrara, L. Guerrini, Abstract and Applied Analysis **2013**, 901014 (2013).
[18] Y. Ma, Nonlinear Analysis: Real World Application **13**, 370-375 (2012).
[19] Y. Song, J.Wei, and M. Han, International Journal of Bifurcation and Chaos in Applied Sciences and Engineering **14**, 3909-3919 (2004).
[20] J. Wei, Nonlinearity **20**, 2483-2498 (2007).
[21] T. Chen, X. Li, and J. He, Abstract and Applied Analysis **2014**, 456764 (2014).
[22] B. D. Hassard, N. D. Kazarinoff, and Y. H. Wan, Theory and Applications of Hopf Bifurcation, Cambridge University Press, 1981.

C. Bianca received the PhD degree in Mathematics for Engineering Science at Politecnico of Turin. His research interests are in the areas of applied mathematics and mathematical physics including the mathematical methods and models for complex systems, mathematical billiards, chaos, anomalous transport in microporous media and numerical methods for kinetic equations. He has published research articles in reputed international journals of mathematical and engineering sciences. He is referee and editor of mathematical journals.

Luca Guerrini received the PhD degree in Pure Mathematics at University of California, Los Angeles. He is Associate Professor of Mathematical Economics at Polytechnic University of Marche, Italy. His research interests are in the areas of pure and applied mathematics as well as mathematical economics. He has published extensively in internationally refereed journals. He is referee and editor of mathematical journals