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On Congruences of Principal $GK_2$-Algebras

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Abstract: We investigate some features of principal $GK_2$-algebras ($PGK_2$-algebras). Necessary and sufficient conditions for a principal $GK_2$-algebra to have 2-permutable congruences are obtained. Furthermore, it is established how 2-permutable congruences are characterized using pairs of principal congruences. Also, a generalization of the 2-permutability of the primary congruences of the $GK_2$-algebras concept to the concept of the $n$-permutable congruences is provided. We round off with strong extensions of principal $GK_2$-algebras.

Keywords: MS-algebra, GMS-algebra, $GK_2$-algebra, principal $GK_2$-algebra, congruence pair, 2-permutability of congruences, $n$-permutability of congruences, strong extension

1 Introduction

T.S. Blyth and J.C. Varlet [1] introduced the variety $MS$ of $MS$-algebras. In [2], they determined the subvarieties of $MS$. Many properties of $MS$-algebras, principal $MS$-algebras, principal $p$-algebras and decomposable $MS$-algebras are investigated in [3,4,5,6,7,8]. The variety $GMS$ was defined and characterized by D. Ševčík in [9]. Certain modular generalized $MS$-algebras with distributive skeletons, called $K_2$-algebras, were introduced by A. Badawy [10]. Each $K_2$-algebra was built using quadruples. A. Badawy [11] considered the subclass $GK_2$ of $GK_2$-algebras. He constructed any $PGK_2$-algebra by means of triple. Also, he deduced that each congruence $\alpha$ on a $GK_2$-algebra $L$ can be constructed by a congruence pair $(\alpha_1, \alpha_2)$ in a unique way, where $\alpha_1 \in Con(L^{\circ\circ})$ and $\alpha_2$ is a congruence of lattices on the bounded lattice $D(L)$. Many authors considered the concepts of permutable congruences, strong extensions and related properties (see [12], [13] and [14]).

This paper applies the concepts of 2-permutability of congruences and $n$-permutability of congruences to $PGK_2$-algebras. We characterize such concepts by using congruence pairs $(\alpha_1, \alpha_2)$ of a principal $GK_2$-algebra $L$, where $\alpha_1$ is a congruence on $GK$-algebra $L^{\circ\circ}$ of all closed elements of $L$, and $\alpha_2$ is a lattice congruence on a lattice bounded $D(L)$. Also, we introduce and characterize the notion of strong extensions of $PGK_2$-algebras. We proved that a $GK_2$-algebra $L$ is a strong extension of a subalgebra $L_1$ if and only if $L^{\circ\circ}$ is a strong extension of $L_1^{\circ\circ}$ and $D(L)$ is a strong extension of $D(L_1)$.

2 Preliminaries

This section contains the basic background and results. We refer to [9,11,15,16,17,18] for details. An $MS$-algebra is an algebra $(;\lor, \land, ^*, 0, 1)$ such that $(L;\lor, \land, 0, 1)$ is a bounded distributive lattice and $^*$ is a unary operation satisfying:

1. $r \leq r^{\circ\circ}$,
2. $(r \land s)^* = r^* \lor s^*$,
3. $0^* = 0$.

The subvariety $M$ (De Morgan algebras) of $MS$ is defined by

$$r = r^{\circ\circ} \quad (1)$$

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The subvariety $K$ (Kleene algebras) of $M$ is characterized by:

$$r \land r^0 \leq s \lor s^0$$  \hspace{1cm} (2)

The class $S$ (Stone algebras) of $MS$ is the subvariety which is defined by:

$$r \land r^0 = 0$$  \hspace{1cm} (3)

The subvariety $B$ (Boolean algebras) of $MS$ is defined by the identity

$$r \lor r^0 = 1$$  \hspace{1cm} (4)

A generalized De Morgan algebra (simply $GM$-algebra) $(L; \lor, \land, \circ, 0, 1)$, where $(L; \lor, \land, 0, 1)$ is a bounded lattice with

1. $r = r^{oo}$,
2. $(r \land s)^0 = r^0 \lor s^0$,
3. $1^0 = 0$.

If a $GM$-algebra satisfies:

$$r \land r^0 \leq s \lor s^0$$  \hspace{1cm} (5)

it becomes a generalized Kleene algebra.

If we drop the distributivity condition of $MS$-algebra, we obtain $GMS$-algebra.

Lemma 2.1. [9] For any two elements $r, s$ of a $GMS$-algebra $L$, we have

1. $0^0 = 1$,
2. $r \leq s \implies r^0 \geq s^0$,
3. $r^0 = r^{oo}$,
4. $(r \lor s)^0 = r^0 \land s^0$,
5. $(r \lor s)^{oo} = r^{oo} \lor s^{oo}$,
6. $(r \land s)^{oo} = r^{oo} \land s^{oo}$.

Definition 2.1. [11] A $GK_2$-algebra $L$ is a $GMS$-algebra satisfying:

1. $r \land r^0 = r^{oo} \land r^0 \forall r \in L$,
2. $r \land r^0 \leq s \lor s^0 \forall r, s \in L$.

Let $L$ be a $GK_2$-algebra. An element $r$ of $L$ is called closed if $r^{oo} = r$ and an element $d \in L$ is called dense if $d^0 = 0$. Set $L^{oo}$ to denote the set of all closed elements of $L$ and $D(L)$ for the set of all dense elements of $L$.

Lemma 2.2. [11] Let $L \in GK_2$-algebra. Then

1. $L^{oo}$ is a GK-subalgebra of $L$,
2. $D(L)$ is a filter of $L$.

Example 2.1. (1) Every $MS$-algebra is a $GMS$-algebra.

2) Every $S$-algebra (pseudo-complement lattice satisfying the Stone identity, $r^* \lor r^{**} = 1$, where $r^* = \max \{s : s \land r = 0\}$ is the pseudo-complement of $r$) is a $GMS$-algebra.

3) The following is a $GMS$-algebra $(L_1, \circ)$ satisfying the Stone identity $r^* \lor r^{**} = 1$. We observe that it is not an $S$-algebra; for example, the element $\mu$ has not pseudo-complement.
Also, we have

$L_1^{\circ} = \{0, a, b, c, q, 1\}$ is a modular $GK$-algebra, and $D(L_1) = \{d, x, y, z, 1\}$ is a modular lattice.

**Definition 2.2.**[11] A $GK_2$-algebra $L$ is a $PGK_2$-algebra if:

1. $D(L) = \{d\}$ for some $d \in L$,
2. The generator $d$ is distributive, that is, $(r \wedge s) \vee d = (r \vee d) \wedge (s \vee d)$ for all $r, s \in L$,
3. $r = \bar{r} \wedge (r \vee d)$ for all $r \in L$.

**Example 2.2.** (1) Every $K_2$-algebra is a $GK_2$-algebra.

(2) Every $S$-algebra is a $GK_2$-algebra.

(3) The $GK_2$-algebra $L_1$ of Example 2.4(3) is a $PGK_2$-algebra which is not an $S$-algebra.

(4) The following $GK_2$-algebra represents an $S$-algebra $L_2$, where $L_2^{\circ} = \{0, a, b, 1\}$ is a Boolean subalgebra and $D(L_2) = \{1\}$. It is clear that it is not a principal $S$-algebra as $c^{\circ} \wedge (c \vee 1) \neq c$.

From this example, it is not true that every finite $GK_2$-algebra is principal.
(5) The following GMS-algebra is a PGK₂-algebra.

\[ a = a^2 = x^2 \quad b = c^2 = y^2 \quad c = b^2 = \beta^2 \]

Diagram:

- \( x \)
- \( y \)
- \( z \)
- \( d \)
- \( \alpha \)
- \( \beta \)
- \( \gamma \)

0 = 1° = x° = y° = z° = d°

\[ 1 = 0° \]

**Definition 2.3.**[6] A binary relation \( \alpha \) defined on a lattice \( L \) is said to be a lattice congruence if:

1. \( \alpha \) is an equivalence relation on \( L \).
2. \( (r, s), (u, v) \in \alpha \) implies \( (r \land u, s \land v), (a \lor c, b \lor d) \in \alpha \).

For a congruence relation \( \alpha \) on a lattice \( L \), \( [r]_\alpha \) is given

\[ [r]_\alpha = \{ t \in L : (t, r) \in \alpha \}. \]  

(6)

It can be prove that \( (L / \alpha, \lor, \land) \) forms a lattice, where

\[ L / \alpha = \{ [r]_\alpha : r \in L \} \]

(7)

is the quotient lattice of \( L \) modulo \( \alpha \) and

\[ [r]_\alpha \lor [s]_\alpha = [r \lor s]_\alpha \text{ and } [r]_\alpha \land [s]_\alpha = [r \land s]_\alpha \]

(8)

A lattice congruence \( \alpha \) on a \( \text{GK}_2 \)-algebra \( (L, \circ) \) is called a congruence on \( L \) if \( r \equiv s(\alpha) \) implies \( r^\circ \equiv s^\circ (\alpha) \).

For a \( \text{GK}_2 \)-algebra \( L \), \( \text{Con}(L) \) is used to denote the set of all congruence on \( L \) and \( \alpha_{\circ L^\circ}, \alpha_{\circ D(L)} \) are used for \( \alpha \) restricted to \( L^\circ \) and \( D(L) \), respectively. Obviously, \( (\alpha_{\circ L^\circ}, \alpha_{\circ D(L)}) \in \text{Con}(L^\circ) \times \text{Con}(D(L)) \). Also, we use \( \bigtriangleup_L = L \times L \) and \( \bigtriangleup_L = \{ (r, r) : r \in L \} \) for the universal and the identity congruences on \( L \), respectively.

A congruence relation \( \alpha \) on a lattice \( L \) is called principal if there exist \( r, s \in L \) such that \( \alpha \) is the smallest congruence relation for which \( r \equiv s(\alpha) \). Indeed,

\[ \alpha(r, s) = \bigwedge \{ \alpha \in \text{Con}(L) : r \equiv s(\alpha) \} \]

(9)

**Definition 2.4.**[11] Let \( d \) be the smallest dense element of a \( \text{PGK}_2 \)-algebra \( L \). Then a pair \( (\alpha_1, \alpha_2) \in \text{Con}(L^\circ) \times \text{Con}(D(L)) \) is called a congruence pair of \( L \) if \( r \equiv s(\alpha_1) \) implies \( r \lor d \equiv s \lor d(\alpha_2) \).

A characterization of a congruence relation on \( \text{PGK}_2 \)-algebras is given as follows:

**Theorem 2.1.**[11] Let \( d \) be the smallest dense element of a \( \text{PGK}_2 \)-algebra \( L \). Then any \( \alpha \in \text{Con}(L) \) determines a congruence pair \( (\alpha_{\circ L^\circ}, \alpha_{\circ D(L)}) \). Conversely, any congruence pair \( (\alpha_1, \alpha_2) \) uniquely determines an \( \alpha \in \text{Con}(L) \) satisfies \( \alpha_{\circ L^\circ} = \alpha_1 \) and \( \alpha_{\circ D(L)} = \alpha_2 \), by the rules: \( r \equiv s(\alpha) \iff r^\circ \equiv s^\circ(\alpha_1) \) and \( r \lor d \equiv s \lor d(\alpha_2) \).

**Lemma 2.3.**[11] Let \( L \) be a \( \text{PGK}_2 \)-algebra and let \( A(L) \) be the set of all congruence pairs of \( L \). Then:

1. \( (\forall \beta \in \text{Con}(D(L)))(\bigtriangleup_{L^\circ}, \beta) \in A(L) \),
2. \( (\forall \eta \in \text{Con}(L^\circ))(\bigtriangleup_{D(L)}, \eta) \in A(L) \).
3 2-Permutability of $PGK_2$-algebras

We extend the concept of 2-permutability of congruences to $PGK_2$-algebras. Some basic properties are proved, and necessary and sufficient conditions for a principal $GK_2$-algebra to have 2-permutable congruences are provided. Moreover, it is established how to characterise 2-permutable congruences in terms of pairs of main congruences.

**Definition 3.1.** Let $L$ be a $PGK_2$-algebra. Then $\alpha, \delta \in \text{Con}(L)$ are 2-permutable congruences (briefly 2-permutable) if $\alpha \circ \delta = \delta \circ \alpha$, that is, $r \equiv s(\alpha)$ and $s \equiv p(\delta)$ imply the existence of an element $u \in L$ such that $r \equiv u(\delta)$ and $u \equiv p(\alpha)$.

A $PGK_2$-algebra $L$ is called 2-permutable congruences if any pair of congruences permute. Let $L$ be a principal $GK_2$-algebra. Define a relation $\Gamma$ on $L$ as follows:

$$(r, s) \in \Gamma \iff r^{oo} = s^{oo} \iff r^o = s^\delta.$$ 

**Lemma 3.1.** Let $L$ be a $PGK_2$-algebras. Then

1. $\Gamma \in \text{Con}(L)$ with $\text{Ker} \Gamma = \{0\}$ and $\text{Coker} \Gamma = D(L)$,
2. $r^{oo}$ is the maximum element of the $[r]_{\Gamma}$, where $[r]_{\Gamma} = \{s \in L : s^{oo} = r^{oo}\}$,
3. $[r]_{\Gamma} = [r^oo]_{\Gamma}$ for any $r \in L$,
4. $L/\Gamma$ is a $GK$-algebra,
5. $L/\Gamma \cong L^{oo}$.

**Proof.** (1) It is straightforward to show that $\Gamma$ is an equivalent relation on $L$. Let $(r, s), (u, v) \in \Gamma$. Then $r^{oo} = s^{oo}$ and $u^{oo} = v^{oo}$. Now we have

$$(r \land u)^{oo} = r^{oo} \land u^{oo} = s^{oo} \land v^{oo} = (s \land v)^{oo}.$$ 

Then $(r \land u, s \land v) \in \Gamma$. Also, we have

$$(r \lor u)^{oo} = r^{oo} \lor u^{oo} = s^{oo} \lor v^{oo} = (s \lor v)^{oo}.$$ 

Then $(r \lor u, s \lor v) \in \Gamma$. Now, let $(r, s) \in \Gamma$. Then we have

$$(r, s) \in \Gamma \implies r^{oo} = s^{oo}$$

$$\implies (r^o, s^\delta) \in \Gamma.$$ 

Then $\Gamma \in \text{Con}(L)$. We observe that

$$\text{Ker} \Gamma = \{r \in L : (r, 0) \in \Gamma\} = \{r \in L : r^{oo} = 0^{oo} = 0\} = \{r \in L : r^o = 1\} = \{0\}.$$ 

Moreover,

$$\text{Coker} \Gamma = \{r \in L : (r, 1) \in \Gamma\} = \{r \in L : r^{oo} = 1^{oo} = 1\} = \{r \in L : r^o = 0\} = D(L).$$
Since \( \omega = \omega \), then \( \omega \in [\alpha] \). Let \( s \in [\alpha] \). Then \( s \leq \omega = \omega \). Hence, \( \omega \) is the greatest element of \([\alpha] \).

(3) Since \( \omega = \omega \), then \( \omega \) implies \( \omega \) and \( \forall r \in L \).

(4) We have \( (L/G; \lor, \land, [0]G), [1]G \) is a bounded lattice with bounds \([0]G\) and \([1]G\), where \([r]G \land [s]G = [r \land s]G\) and \([r]G \lor [s]G = [r \lor s]G\). Define \( \triangleleft \) on \( L/G \) by \( ([r]G)^\triangleleft = [r^0]G \). Now, we have the following equalities

\[
(0]G)^\square = [1]G, \quad ([1]G)^\square = [0]G,
\]

\[
([r]G)^\square = [r^0]G = [s]G,
\]

\[
([r]G \land [s]G)^\square = ([r \land s]G)^\square = [r^0 \land s^0]G = [r^0 \lor s^0]G = ([r]G)^\square \lor ([s]G)^\square.
\]

Then \( L/G \) is a GM-algebra. Since \( r \land r^0 \leq s \lor s^0 \), then \( [r \land r^0]G \leq [s \lor s^0]G \). Hence,

\[
\]

Thus, \( L/G \) is a GK-algebra.

(5) Define \( f : L^\ominus \longrightarrow L/G \) by

\[
f(r) = [r]G \quad \forall r \in L^\ominus
\]

It is clear that \( f \) is well-defined. Let \( f(r) = f(s) \). Then \( [r]G = [s]G \) implies \( r \equiv s \). Then \( r = s^\ominus = s \) as \( r,s \in L^\ominus \). Then \( f \) is one-to-one. Let \( [s]G \in L/G \) for some \( s \in L \). Then \( [s]G = [s^\ominus]G \) and so \( f(s^\ominus) = [s^\ominus]G = [s]G \). Then \( f \) is onto Also, we need to show that \( f \) is a homomorphism. Clearly, \( f(r \lor s) = f(r) \lor f(s) \) and \( f(r \land s) = f(r) \land f(s) \). Also,

\[
f(r^0) = [r^0]G = [r^0 \ominus]G = ([r]G)^\square = ([f(r)]^\square = ([f(r)]^\square = ([f(r)]^\square.
\]

Clearly \( f(0^\ominus) = [0]G \) and \( f(1) = [1]G \). Hence, \( L^\ominus \cong L/G \).

**Lemma 3.2.** Let \( L \) be a PGK\(_2\)-algebras. Then:

1. \( \Gamma \) permutes with any \( \alpha \in \text{Con}(L) \).
2. \( \triangle_L \) permutes with any \( \alpha \in \text{Con}(L) \).
3. \( \triangledown_L \) permutes with any \( \alpha \in \text{Con}(L) \).

**Proof.** (1) Let \( \alpha \in \text{Con}(L) \). Then we need to show that \( \alpha \circ \Gamma = \Gamma \circ \alpha \). Let \( r \equiv s(\alpha \circ \Gamma) \). Then \( r \equiv p(\alpha) \) and \( p \equiv s(\Gamma) \) for some \( r \in L \). So, \( r \equiv p(\alpha) \) and \( p^0 = s^\ominus \). Now

\[
r \equiv p(\alpha) \implies p^0 \equiv p^0(\alpha), s \lor d \equiv s \lor d(\alpha)
\]

\[
\implies p^0 \equiv s^\ominus(\alpha), s \lor d \equiv s \lor d(\alpha) \text{ as } p^0 = s^\ominus
\]

\[
\implies p^0 \land (s \lor d) \equiv s^\ominus \land (s \lor d)(\alpha) = s(\alpha) \text{ as } s = s^\ominus \land (s \lor d)
\]

Since \( p^0 \land (s \lor d) = r^0 \), then \( p^0 \land (s \lor d) = [r^0]G \). Since \( r \equiv r^0 \land (s \lor d)(\Gamma) \) and \( r^0 \land (s \lor d)(\alpha) \equiv s(\alpha) \), then \( r \equiv s(\Gamma \circ \alpha) \).

(2) Let \( r \equiv s(\alpha \circ \triangle_L) \). Then \( r \equiv p(\alpha) \), \( p \equiv s(\triangle_L) \) for some \( r \in L \). Hence \( r \equiv s(\alpha) \) as \( p = s \). Then, \( r \equiv r(\triangle_L) \) and \( r \equiv s(\alpha) \).

Thus, we deduced that \( r \equiv s(\triangle_L \circ \alpha) \). Therefore, \( \triangle_L \) permutes with any element of \( \text{Con}(L) \).

(3) Let \( r \equiv s(\alpha \circ \triangledown_L) \). Then \( r \equiv p(\alpha) \), \( p \equiv s(\triangledown_L) \) for some \( r \in L \). Then we have \( r \equiv s(\triangledown_L) \) and \( s \equiv s(\alpha) \). Thus, \( r \equiv s(\triangledown_L \circ \alpha) \). Therefore, \( \triangledown_L \) permutes with any element of \( \text{Con}(L) \).
Now, we provide a characterization of 2-permutable congruences.

**Theorem 3.1.** Let $d$ be the smallest dense element of a PGK$_2$-algebra $L$. Then $L$ has 2-permutable congruences if and only if:

1. $L^0$ has 2-permutable congruences,
2. $D(L)$ has 2-permutable congruences.

**Proof.** Suppose that $\alpha, \delta$ are 2-permutable on $L$. First, we prove that $\alpha_d = \delta_d$ are 2-permutable on $L^0$. Consider that $r, s, p \in L^0$ be such that $r \equiv s(\alpha_d)$ and $s \equiv p(\delta_d)$. Then $r \equiv s(\alpha)$ and $s \equiv p(\delta)$. Since $\alpha, \delta$ are 2-permutable, we have $r \equiv q(\delta), q \equiv p(\alpha)$ for some $q \in L$. Now,

$$r \equiv q(\delta), q \equiv p(\alpha) \implies r^0 \equiv q^0(\delta), q^0 \equiv p^0(\alpha) \implies r \equiv u(\delta), u \equiv p(\alpha) \text{ as } r, p \in L^0 \implies r \equiv q^0(\delta), q^0 \equiv p^0(\alpha) \text{ as } q^0 \in L^0.$$

Therefore $\alpha_d = \delta_d$ are 2-permutable on $L^0$ and (1) is proved. Secondly, we show that 2-permutability of $\alpha, \delta$ implies 2-permutability of $\alpha_d$ and $\delta_d$. Let $r, s, p \in D(L)$ such that $r \equiv s(\alpha_d)$ and $s \equiv p(\delta_d)$. Then $r \equiv s(\alpha)$, $s \equiv p(\delta)$. Since $\alpha, \delta$ are 2-permutable, then $r \equiv u(\delta)$ and $u \equiv p(\alpha)$ for some $u \in L$. Now,

$$r \equiv u(\delta), u \equiv p(\alpha) \implies u \equiv u(\delta), u \equiv p(\alpha) \text{ as } u, p \geq d \implies r \equiv u \equiv u(\delta), u \equiv p(\alpha) \text{ where } u \equiv d \in D(L).$$

Hence $r \equiv u \equiv d(\delta_d)$ and $u \equiv p(\alpha_d)$. Therefore $\alpha_d$ and $\delta_d$ are 2-permutable congruences on $D(L)$. For the converse direction, let $\alpha, \delta \in \text{Con}(L)$ such that $\alpha_d = \delta_d$ and $\text{Con}(L)$ are 2-permutable on $L^0$ and $D(L)$, respectively. Consider the elements $r, s, p \in L$ with $r \equiv s(\alpha)$ and $s \equiv p(\delta)$. We have, by Theorem 2.9, that $r^0 \equiv s^0(\alpha_d)$ and $s^0 \equiv p^0(\delta_d)$. Since $\alpha_d, \delta_d$ are 2-permutable congruences on $L^0$, then $r^0 \equiv u^0(\delta_d)$ and $u \equiv p^0(\alpha_d)$ with $u \in L^0$ implies that $r^0 \equiv u(\delta)$ and $u \equiv p^0(\alpha)$. On the other hand, also by Theorem 2.9, we get $r \equiv u \equiv v(\delta_d)$ and $v \equiv p \equiv d(\delta_d)$. Since $\alpha_d, \delta_d$ are 2-permutable congruences on $D(L)$, then $r \equiv u \equiv v(\delta_d)$ and $v \equiv p \equiv d(\delta)$. It follows that

$$r \equiv r^0 \land (r \lor d) \land p = p^0 \land (p \lor d).$$

(11)

Since $L$ is a PGK$_2$-algebra, then we have $r = r^0 \land (r \lor d)$ and $p = p^0 \land (p \lor d)$. Then we have

$$r^0 \equiv u(\delta), r \lor d \equiv v(\delta) \implies r = r^0 \land (r \lor d) \equiv u \lor v(\delta),$$

(12)

and

$$u \equiv p^0(\alpha), v \equiv p \lor d(\alpha) \implies a \lor v \equiv p^0 \land (p \lor d)(\alpha) = p \equiv p,$$

(13)

Consequently, we deduce that $r \equiv u \lor v(\delta)$ and $u \lor v \equiv p(\alpha)$. Therefore $\alpha, \delta$ are 2-permutable congruences.

**Theorem 3.2.** A PGK$_2$-algebra $L$ has 2-permutable congruences if and only if every pair of principal congruences on $L$ permutes.

**Proof.** The first statement is obvious. Assume that any pair of principal congruences on $L$ permute. Let $\alpha, \delta \in \text{Con}(L)$. Consider $r, s, p \in L$ with $r \equiv s(\alpha)$ and $s \equiv p(\delta)$. It is clear that $\alpha \cup \delta \subseteq \alpha$ and $\delta \cup (s, p) \subseteq \delta$. Hence, $r \equiv p(\alpha \cup \delta)$. Since $\alpha \cup \delta$ is 2-permutable, then $r \equiv u(\delta(s, p))$ and $u \equiv p(\alpha(s, p))$ for some $u \in L$. Consequently, $r \equiv u(\delta)$ and $u \equiv p(\alpha)$ and hence $r \equiv p(\alpha \cup \delta)$.

4. $n$-Permutability of PGK$_2$-algebras

The results of this section extend the 2-permutability of congruences of PGK$_2$-algebras to $n$-permutable congruences. Two congruences $\alpha, \delta$ are $n$-permutable if

$$\alpha \circ \delta \circ \alpha \circ \ldots \ldots \ldots \ldots (n - \text{ times}) = \delta \circ \alpha \circ \delta \circ \ldots \ldots \ldots \ldots (n - \text{ times}),$$

(14)

where $n = 1, 2, \ldots, n - 1$.

**Definition 4.1.** A principal GK$_2$-algebra $L$ has $n$-permutable congruences, if every two congruences in $L$ are $n$-permutable.
Lemma 4.1. Let \( d \) be the smallest dense element of a PGK\(_2\)-algebra \( L \). Let \( \theta, \psi \) be congruences on \( L \). Then

\[
\begin{align*}
(1) \quad \{ \alpha \circ \delta \circ \alpha \circ \ldots \} \big|_{L^{\infty}} &= \alpha_{L^{\infty}} \circ \delta_{L^{\infty}} \circ \alpha_{L^{\infty}} \circ \ldots \quad (n - \text{times}), \\
(2) \quad \{ \alpha \circ \delta \circ \alpha \circ \ldots \} \big|_{D(L)} &= \alpha_{D(L)} \circ \delta_{D(L)} \circ \alpha_{D(L)} \circ \ldots \quad (n - \text{times}).
\end{align*}
\]

Proof. (1) To show the equality (15)

Now, let \( r, s \in L^{\infty} \) with \( r \equiv s(\alpha \circ \delta \circ \ldots) \). Then \( r \equiv s(\alpha \circ \delta \circ \ldots) \). Thus there exist elements \( t_1, t_2, \ldots, t_{n-1} \in L \) be such that \( r \equiv t_1(\alpha), t_1 \equiv t_2(\delta), \ldots, t_{n-1} \equiv s(\nu) \), where

\[
\nu = \begin{cases} 
\alpha & \text{if } n \text{ is odd} \\
\delta & \text{if } n \text{ is even}
\end{cases}
\]

We have, \( s(\nu) = t_1(\alpha), t_1 \equiv t_2(\delta), \ldots, t_{n-1} \equiv s(\nu) \). Then, \( r \equiv s(\delta_{L^{\infty}} \circ \alpha_{L^{\infty}} \circ \ldots) \) because of \( t_n \wedge d \in L^{\infty} \) for \( n = 1, 2, \ldots, n - 1 \). The reverse inclusion is obvious. Hence,

\[
\{ \alpha \circ \delta \circ \ldots \} \big|_{L^{\infty}} = \{ \alpha_{L^{\infty}} \circ \delta_{L^{\infty}} \circ \ldots \}.
\]

(2) Let \( r, s \in D(L) \) be such that \( r \equiv s(\theta \circ \psi \circ \ldots) \). Then there exist \( t_1, t_2, \ldots, t_{n-1} \in L \) be such that \( r \equiv t_1(\alpha), t_1 \equiv t_2(\delta), \ldots, t_{n-1} \equiv b(\nu) \). Then, \( r \equiv r \vee d \equiv t_1 \vee d(\alpha), \ldots, t_{n-1} \vee d \equiv b \vee d \equiv s(\nu) \). Therefore,

\[
\{ \alpha \circ \delta \circ \ldots \} \big|_{D(L)} = \{ \alpha_{D(L)} \circ \delta_{D(L)} \circ \ldots \} \quad (n - \text{time})
\]

Theorem 4.1. Let \( d \) be the smallest dense element of a PGK\(_2\)-algebra \( L \). Then \( L \) has \( n \)-permutable congruences if and only if \( L^{\infty} \) and \( D(L) \) are \( n \)-permutable congruences.

Proof. (\( \Rightarrow \)) By using Lemma 4.2(1) we have

\[
\begin{align*}
\alpha_{L^{\infty}} \circ \delta_{L^{\infty}} \circ \ldots &= \{ \alpha \circ \delta \circ \ldots \} \big|_{L^{\infty}} \\
&= \delta \circ \alpha \circ \ldots \big|_{L^{\infty}} \\
&= \delta_{L^{\infty}} \circ \alpha_{L^{\infty}} \circ \ldots
\end{align*}
\]

Again by using Lemma 4.2(2) we have

\[
\begin{align*}
\alpha_{D(L)} \circ \delta_{D(L)} \circ \ldots &= \{ \alpha \circ \delta \circ \ldots \} \big|_{D(L)} \\
&= \delta \circ \alpha \circ \ldots \big|_{D(L)} \\
&= \delta_{D(L)} \circ \alpha_{D(L)} \circ \ldots
\end{align*}
\]

(\( \Leftarrow \)) Let \( r \equiv s(\alpha \circ \beta \circ \ldots) \). Then \( r^{\infty} \equiv s^{\infty}(\{ \alpha \circ \beta \circ \ldots \} \big|_{L^{\infty}}) \) and \( r \vee d \equiv s \vee d(\{ \alpha \circ \beta \circ \ldots \} \big|_{D(L)}) \) by Theorem 2.9. Applying Lemma 4.2 we have

\[
\begin{align*}
\text{Since } \alpha_{L^{\infty}} \circ \delta_{L^{\infty}} \circ \ldots &= \delta_{L^{\infty}} \circ \alpha_{L^{\infty}} \circ \ldots \quad (n - \text{times}) \quad \text{and } \alpha_{D(L)} \circ \delta_{D(L)} \circ \ldots \equiv \delta_{D(L)} \circ \alpha_{D(L)} \circ \ldots \quad (n - \text{times}), \\
\text{then we get } \\
r^{\infty} \equiv s^{\infty}(\{ \alpha \circ \beta \circ \ldots \} \big|_{L^{\infty}}) \quad \text{and } r \vee d \equiv s \vee d(\{ \alpha \circ \beta \circ \ldots \} \big|_{D(L)}).
\end{align*}
\]

Now, by using Definition 2.5(3) and Theorem 2.9, we get

\[
\begin{align*}
\text{Therefore, } r \equiv s(\delta \circ \alpha \circ \ldots). \text{ Thus, we deduce that } \delta \text{ and } \alpha \text{ are } n \text{ permutable.}
\end{align*}
\]
5 Strong extensions of \(PGK_2\)-algebras

The concept of strong extensions of \(PGK_2\)-algebras is investigated in this section. An algebra \(L\) satisfies the congruence extension property (CEP); if for every subalgebra \(L_1\) of \(L\) and every \(\alpha\) of \(L_1\), \(\alpha\) extends to a congruence of \(L\). (see [19])

**Definition 5.1.** Let \(M_1\) and \(N\) be a \(PGK_2\)-algebra. Then we call the algebra \(K\) a strong extension of the algebra \(K_1\) if \(K_1\) is a subalgebra of \(K\) and for any \(\alpha_1 \in \text{Con}(K_1)\), there exists a unique congruence relation \(\alpha \in \text{Con}(K)\) such that \(\alpha_{K_1} = \alpha_1\).

**Theorem 5.1.** Let \(K_1\) be a subalgebra of a \(PGK_2\)-algebra \(K\). Then \(K\) is a strong extension of \(K_1\) if and only if

1. \(D(K)\) is a strong extension of \(D(K_1)\),

2. \(K^{\infty}\) is a strong extension of \(K_1^{\infty}\).

**Proof.** Let \(K\) be a strong extension of \(K_1\). Let \(\eta_2 \in \text{Con}(D(K_1))\). Assume that \(\tilde{\eta}_2, \tilde{\eta}_2 \in \text{Con}(D(K))\) such that \(\tilde{\eta}_2 \circ \eta_2 = \eta_2\). Then, by Lemma 2.10(1), we have

\[
(\Delta_{K^{\infty}}, \tilde{\eta}_2) \in A(K) \text{ and } (\Delta_{K_1^{\infty}}, \eta_2) \in A(K_1). 
\]

According to Theorem 2.9, we have \(\tilde{\eta}, \eta \in \text{Con}(K)\) and \(\eta \in \text{Con}(K_1)\) corresponding to \((\Delta_{K^{\infty}}, \tilde{\eta}_2), (\Delta_{K_1^{\infty}}, \eta_2)\) and \(\eta = (\Delta_{K_1^{\infty}}, \eta_1)\), respectively. We see that \(\tilde{\eta}_1 = \tilde{\eta}_1 = \eta\). We have \(\tilde{\eta} = \eta\). Hence, \(\tilde{\eta} = \tilde{\eta}_2\) proving (1). On the other hand, we need to show that \(K^{\infty}\) is a strong extension of \(K_1^{\infty}\). Let \(\eta_1 \in \text{Con}(K_1^{\infty})\) and \(\eta\) extend to a congruence of \(K^{\infty}\). Let \(\eta_1 \in \text{Con}(K^{\infty})\) with \(\eta_1 \circ \eta_1 = \eta_1 \circ \eta_1 = \eta_1\). Then, by Lemma 2.10(2), we have

\[
(\eta_1, \eta_2)(\eta_2, \eta_2) \in A(K) \text{ and } (\eta_1, \eta_2)(\eta_2, \eta_2) \in A(K_1). 
\]

Again, by Theorem 2.9, we have \(\tilde{\eta}, \eta \in \text{Con}(K)\) and \(\eta \in \text{Con}(K_1)\) corresponding to \((\eta_1, \eta_2)(\eta_2, \eta_2), (\eta_1, \eta_2)(\eta_2, \eta_2)\) and \(\eta = (\eta_1, \eta_2)(\eta_2, \eta_2)\), respectively. We see that \(\tilde{\eta}_1 = \tilde{\eta}_1 = \eta_1\). Since \(K\) is a strong extension of \(K_1\), then \(\eta_1 = \eta_1\). Therefore \(\tilde{\eta}_1 = \tilde{\eta}_1\), proving (2). Conversely, suppose that conditions (1) and (2) hold and let \(\eta \in \text{Con}(K_1)\). Let \(\tilde{\eta}, \eta\) be extensions of \(\eta\) in \(\text{Con}(K)\). By Theorem 2.9, the congruences \(\tilde{\eta}, \eta\) and \(\eta\) can be represented by the congruence pairs \((\tilde{\eta}_1, \tilde{\eta}_2)(\tilde{\eta}_2, \tilde{\eta}_2)\) and \((\eta_1, \eta_2)\), respectively. Where

\[
\tilde{\eta}_1 \circ \tilde{\eta}_1 = \tilde{\eta}_1 \circ \tilde{\eta}_1 = \eta_1 \text{ and } \tilde{\eta}_2 \circ \tilde{\eta}_2 = \tilde{\eta}_2 \circ \tilde{\eta}_2 = \eta_2. 
\]

By (1) and (2) we get

\[
\tilde{\eta}_1 = \tilde{\eta}_1 \text{ and } \tilde{\eta}_2 = \tilde{\eta}_2. 
\]

Therefore, \(\tilde{\eta} = \eta\).

**Corollary 5.1.** Let \(K_1\) and \(K\) be \(PGK_2\)-algebras. If \(K_1\) is a strong extension of \(K\), then \(\text{Con}(K_1) \cong \text{Con}(K)\).

6 Conclusion

The following three key concepts in algebraic structures: 2-Permutability, \(n\)-Permutability, and strong extensions were examined for the \(PGK_2\)-algebras via congruence pairs. This paper's work could be further developed to study many aspects of \(GK_2\)-algebras and related structures. For instance, it can be applied to triple construction of \(GK_2\)-algebras, perfect extensions of \(PGK_2\)-algebras, and substructures of \(PGK_2\)-algebras.

References