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Non-Additive Entropy Measure and Record Values

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Abstract: Non-additive entropy measures are important for many applications. We study Havrda and Charvat entropy for record values and have shown that this characterizes the underlying distribution function uniquely. Also the non-additive entropy of record values has been derived in case of some specific distributions. Further we propose a generalized residual entropy measure for record value.

Keywords: Generalized entropy, Record value, Probability integral transformation, Residual entropy.

1 Introduction

Suppose $X_1, X_2, \cdots, X_n$ be a sequence of independent and identically distributed (i.i.d.) random variables with a common absolutely continuous distribution function (cdf) F, probability density function (pdf) f, and survival function $\overline{F} = 1 - F$. An observation X_j will be called an *upper record value* if its value exceeds that of all previous observations. Thus X_j is an upper record if $X_j > X_i$ for every $i < j$. An analogous definition can be given for a lower record value. Let $R(j)$ denote the time (index) at which the jth record value is observed. Since the first observation is always a record value, we have

$$R(1) = 1, \cdots, R(J+1) = min\left\{i : X_i > X_{R(j)}\right\},$$

where $R(0)$ is defined to be 0. The sequence of upper record values can thus be defined by $U_j = X_{R(j)}, j = 1, 2, 3, \cdots$ Let $D(j) = R(j+1) - R(j)$ denote the inter-record time between the jth record value and $(j+1)$th record value, and let the jth record value $X_{R(j)}$ be denoted by X_j for simplicity. Then the probability density function of the jth record value X_j is given by

$$g_{X_j}(x) = \frac{\{-\ln(1-F(x))\}^{j-1} f(x)}{\Gamma(j)}, \; x > 0 \tag{1}$$

The survival function is

$$\bar{G}_{X_j}(x) = \sum_{j=0}^{n-1} \frac{\{-\ln(1-F(x))\}^{j} \bar{F}(x)}{\Gamma(j)} = \frac{\Gamma(n\,; -\log \bar{F}(x))}{\Gamma(n)}, \tag{2}$$

where $\Gamma(a;x)$, the incomplete gamma function, is defined as

$$\Gamma(a;x) = \int_x^{\infty} x^{a-1} e^{-x} dx, \; x, \; a > 0,$$

refer to, David and Nagaraja (2003, p.32).

Records can be viewed as order statistics from a sample whose size is determined by the values and the order of occurrence of the observations. In reliability theory, order statistics and record values are used for statistical modeling. The $(n-m+1)$th order statistics in a sample of size n represents the life length of an $'m$ *out of* n' system. Record values are used in shock models and minimal repair systems, refer to Kamps (1994). Record values arise naturally in problems such as industrial stress testing, meteorological analysis, hydrology, sporting and athletic events, and economics, refer to Arnold et al. (1998), Nevzorov (2001) and Ahsanullah (2004). Several authors have studied the characterization of distribution function F based on the properties of order statistics and record values, refer to Nagaraja and Nevzorov (1997), Raqab and Awad (2000), and Balakrishnan and Stepanov (2004).

The idea of information-theoretic entropy was first introduced by Shannon (1948) and later by Weiner (1949) in Cybernetics. Let X be an absolutely continuous random variable which denotes the lifetime of a device or, a system with probability density function $f(x)$. Then the average amount of uncertainty associated with the random

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variable X, as given by Shannon entropy (1948) is

$$H(X) = -\int_0^\infty f(x)\log f(x)dx\,. \quad (3)$$

The measure (3) is additive in nature in the sense that for two independent random variables X and Y

$$H(X*Y) = H(X) + H(Y)\,,$$

where $X*Y$ denotes the joint random variable. In literature many authors have generalized Shannon entropy (3) in different ways. A well-known parametric extension of the Shannon entropy measure was defined by Havrda and Charvat (1967) as

$$H_\alpha(X) = \frac{1}{1-\alpha}\left\{\int_0^\infty f^\alpha(x)dx - 1\right\},\ \alpha \neq 1,\ \alpha > 0\,. \quad (4)$$

Although entropy measure (4) was first introduced by Havrda and Charvat (1967) in the context of cybernetics theory, it was Tsallis (1988) who exploited its non-extensive features and placed it in a physical setting. Hence entropy measure (4) is also known as Tsallis entropy (1988). Clearly as $\alpha \to 1$, (4) reduces to (3). This entropy is similar to Shannon entropy except for its non-additive nature, that is

$$H(X*Y) = H(X) + H(Y) + (1-\alpha)H(X)H(Y).$$

In general, the non-additive measures of entropy find justifications in many biological and chemical phenomena. Some properties and applications of non-additive entropy measure (4) have been studied by Tsallis (1998, 2002) and, Tsallis and Brigatti (2004). Baratpour et al. (2007, 2008) obtained results for the Shannon entropy and Renyi entropy of the order statistics and record values.

In this communication we study results for the record values based on the non-additive entropy measure (4). The paper is organized as follows. A general expression for the entropy measure (4) of a record value distribution is derived in Section 2. The entropies of record values associated with the uniform, exponential, weibull, pareto, finite range and gamma are presented in Section 3. In an attempt to establish a coherence between the entropies of the parent and the corresponding record value distributions, Section 4 is devoted to a characterization result. Generalized residual entropy of record values has been studied in Section 5.

2 Generalized Entropy for Different Univariate distribution

In this section, we derive generalized entropy measure (4) for some specific univariate continuous distributions.

2.1 Exponentiated Exponential Distribution

A random variable X is said to have the exponentiated exponential distribution denoted by $X \sim EE(\gamma,\ \theta)$, if its probability density function (pdf) and cumulative distribution function (cdf) are given by

$$f(x) = \gamma\theta\exp(-\theta x)\{1-\exp(-\theta x)\}^{\gamma-1},\ x > 0 \quad (5)$$

and

$$F(x) = \{1-\exp(-\theta x)\}^\gamma,\ \gamma > 0,\ \theta > 0, \quad (6)$$

respectively. In particular for $\gamma = 1$, (5) is the exponential distribution.

The exponentiated exponential distribution introduced by Gupta and Kundu (1999) has some interesting physical interpretations. Consider a parallel system consisting of γ components, the system works, only when at least one of the γ-components works. If the lifetime distributions of the components are independent identically distributed (i.i.d.) exponential random variables, then the lifetime distribution of the system is defined as (6).

Entropy of a random variable X is a measure of its uncertainty. Let $X \sim EE(\gamma,\ \theta)$; we derive explicit form for the generalized entropy measure (4) for X. For the pdf given by (5),

$$\int_0^\infty f^\alpha(x)dx = (\gamma\theta)^\alpha \int_0^\infty \exp(-\theta\alpha x)\{1-\exp(-\theta x)\}^{\alpha\gamma-\alpha}dx. \quad (7)$$

On substituting $y = \exp(-\theta x)$, (7) reduces to

$$\gamma^\alpha\theta^{\alpha-1}\int_0^1 y^{\alpha-1}\{1-y\}^{\alpha\gamma-\alpha}dy = \gamma^\alpha\theta^{\alpha-1}B(\alpha;\alpha\gamma-\alpha+1).$$

So (4) takes the form

$$H_\alpha(X) = \frac{1}{1-\alpha}\left\{\gamma^\alpha\theta^{\alpha-1}B(\alpha;\alpha\gamma-\alpha+1)-1\right\}. \quad (8)$$

If $\gamma = 1$, then (8) reduces to $H_\alpha(X) = \frac{1}{1-\alpha}\left\{\frac{\theta^{\alpha-1}}{\alpha}-1\right\}$, the entropy measure (4) for exponential distribution.

Table 1 gives the non-additive Havrda and Charvat entropy measure (1967) for some specific probability distributions.

3 Generalized Entropy of Record Value Obtained for Specific Distributions

We will use the probability integral transformation of the random variable $U = F(x)$, where the distribution of U is the standard uniform distribution. The probability integral transformation provides the following useful

Table 1: yyyyyy

Distribution	Density function $f(x)$	Entropy $H_\alpha(X)$
Uniform	$\frac{1}{(b-a)}$; $x \in [a,b]$	$\frac{1}{(1-\alpha)}\left\{(b-a)^{1-\alpha}-1\right\}$
Exponential	$\theta e^{-\theta x}$; $x>0,\ \theta>0$	$\frac{1}{(1-\alpha)}\left\{\frac{\theta^{\alpha-1}}{\alpha}-1\right\}$
Pareto	$\frac{\theta\beta^\theta}{x^{(\theta+1)}}$; $x\geq\beta>0,\ \theta>0$	$\frac{1}{(1-\alpha)}\left[\frac{\theta^\alpha}{[\alpha\theta+\alpha-1]\beta^{\alpha-1}}-1\right]$
Finite Range	$\frac{a}{b}(1-\frac{x}{b})^{a-1}$;$0\leq x\leq b,\ a>1$	$\frac{1}{(1-\alpha)}\left\{\frac{a^\alpha b^{1-\alpha}}{(1+(a-1)\alpha)}-1\right\}$
Beta	$\frac{x^{a-1}(1-x)^{b-1}}{\beta(a,b)}$; $0\leq x\leq 1,\ a,\ b>0$	$\frac{1}{(1-\alpha)}\left[\frac{\beta(\alpha(a-1)+1,\alpha(b-1)+1)}{\beta^\alpha(a,b)}-1\right]$
Levy	$(\frac{\sigma}{2\pi})^{1/2}\frac{1}{(x-\mu)^{3/2}}e^{-\frac{\sigma}{2(x-\mu)}}$; $x>\mu,\ \mu>0$	$\frac{1}{(1-\alpha)}\left\{(\frac{\sigma}{2})^{\frac{3(1-\alpha)}{2}}(\frac{\alpha^{1-\frac{3\alpha}{2}}}{\pi^{1/2}})\Gamma(\frac{3\alpha}{2}-1)-1\right\}$
Folded Cramer	$\frac{\theta}{(1+\theta x)^2}$; $x\geq 0,\ \theta>0$	$\frac{1}{(1-\alpha)}\left\{\frac{\theta^{\alpha-1}}{(2\alpha-1)}-1\right\}$

representation of the entropy measure (4) for the random variable X

$$H_\alpha(X)=\frac{1}{1-\alpha}\left\{\int_0^1 f^{(\alpha-1)}(F^{-1}(u)du-1\right\}. \qquad (9)$$

Next we prove the following result.

Lemma 31 *The entropy measure (4) of the j^{th} record value X_j can be expressed as*

$$H_\alpha(X_j)=\frac{\Gamma[(j-1)\alpha+1]}{\{\Gamma(j)\}^\alpha(1-\alpha)}E[f^{\alpha-1}\{F^{-1}(1-e^{-v})\}]-\frac{1}{(1-\alpha)}, \qquad (10)$$

where $v\sim\Gamma\{(j-1)\alpha+1\}$ and E is the expectation.

Proof Generalized entropy (4) of the j^{th} record value is defined as

$$H_\alpha(X_j)=\frac{1}{1-\alpha}\left\{\int_0^\infty\{g_{X_j}(x)\}^\alpha dx-1\right\},\ \alpha\neq1,\ \alpha>0\,.$$

Using (1), this can be rewritten as

$$H_\alpha(X_j)=\frac{1}{(1-\alpha)\{\Gamma(j)\}^\alpha}\{\int_0^\infty\{-\ln\bar{F}(x)\}^{(j-1)\alpha} f^\alpha(x)dx-\{\Gamma(j)\}^\alpha\}$$

Substituting $-\ln\bar{F}(x)=u$, and hence, $x=F^{-1}(1-e^{-u})$, we have

$$H_\alpha(X_j)=\frac{1}{(1-\alpha)\{\Gamma(j)\}^\alpha}\{\int_0^\infty u^{(j-1)\alpha}e^{-u}\left[f^{\alpha-1}\{F^{-1}(1-e^{-u})\}\right]du-\{\Gamma(j)\}^\alpha\} \qquad (11)$$

It can be rewritten as

$$H_\alpha(X_j)=\frac{1}{(1-\alpha)}\{\frac{\Gamma[(j-1)\alpha+1]}{\{\Gamma(j)\}^\alpha}E[f^{\alpha-1}\{F^{-1}(1-e^{-v})\}]-1\}.$$

So, the result follows.

3.1 Uniform Distribution

If a random variable X is uniformly distributed over $(a,\ b)$, $a<b$, then its density and distribution functions are given respectively by

$$f(x)=\frac{1}{b-a}\ \text{and}\ F(x)=\frac{x-a}{b-a},\ a<x<b.$$

We have

$$H_\alpha(X_j)=\frac{1}{(1-\alpha)\{\Gamma(j)\}^\alpha}\{\int_0^\infty u^{(j-1)\alpha}e^{-u}f^{\alpha-1}(x)du-\{\Gamma(j)\}^\alpha\}.$$

Thus generalized entropy (4) of the j^{th} record value for uniform distribution is given as

$$H_\alpha(X_j)=\frac{1}{(1-\alpha)\{\Gamma(j)\}^\alpha}\left\{\frac{\int_0^\infty u^{(j-1)\alpha}e^{-u}du}{(b-a)^{\alpha-1}}-\{\Gamma(j)\}^\alpha\right\},$$

which gives

$$H_\alpha(X_j)=\frac{1}{(1-\alpha)\{\Gamma(j)\}^\alpha}\left\{\frac{\Gamma\{(j-1)\alpha+1\}}{(b-a)^{\alpha-1}}-\{\Gamma(j)\}^\alpha\right\}.$$

The generalized entropy of the first record, that is, X_1, is

$$H_\alpha(X_1)=\frac{(b-a)^{1-\alpha}}{1-\alpha}-\frac{1}{1-\alpha}. \qquad (12)$$

This is the entropy of the parent distribution for uniform variate as indicated in Table 2.1. The entropy of the non-trivial record, that is, X_2, is given as

$$H_\alpha(X_2)=\frac{1}{(1-\alpha)}\left\{\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha-1}}-1\right\}.$$

3.2 Exponential Distribution

Let X be a random variable having the exponential distribution with pdf $f(x)=\theta\{\exp-(\theta x)\}$. Substituting

$-\ln\bar{F}(x) = u$, hence, $\frac{u}{\theta} = F^{-1}(1-e^{-u})$ we have

$$f^{\alpha-1}\{F^{-1}(1-e^{-u})\} = \left[f\left(\frac{u}{\theta}\right)\right]^{\alpha-1} = \theta^{\alpha-1}e^{-u(\alpha-1)}. \tag{13}$$

Thus generalized entropy (4) of the j^{th} record value for exponential distribution is given as

$$H_\alpha(X_j) = \frac{1}{(1-\alpha)\{\Gamma(j)\}^\alpha}\{\frac{\int_0^\infty u^{(j-1)\alpha}e^{-\alpha u}du}{\theta^{1-\alpha}} - \{\Gamma(j)\}^\alpha\},$$

which gives

$$H_\alpha(X_j) = \frac{1}{(1-\alpha)\{\Gamma(j)\}^\alpha}\{\frac{\Gamma\{(j-1)\alpha+1\}\theta^{\alpha-1}}{\alpha^{(j-1)\alpha+1}} - \{\Gamma(j)\}^\alpha\}. \tag{14}$$

For $j=1$, that is, the entropy of the first record, we have

$$H_\alpha(X_1) = \frac{\theta^{\alpha-1}}{\alpha(1-\alpha)} - \frac{1}{1-\alpha}.$$

This is the entropy of the parent distribution. The entropy of the non-trivial record, that is, X_2, is given as

$$H_\alpha(X_2) = \frac{1}{(1-\alpha)}\left\{\frac{\Gamma(\alpha+1)\theta^{\alpha-1}}{\alpha^{\alpha+1}} - 1\right\}.$$

3.3 Pareto Distribution

Let X be a random variable having Pareto distribution with pdf

$$f(x) = \frac{\theta\beta^\theta}{x^{\theta+1}},\ x \geq \beta > 0,\ \theta > 0. \tag{15}$$

Substituting $-\ln\bar{F}(x) = u$, we observe that $x = F^{-1}(1-e^{-u}) = \beta e^{\frac{u}{\theta}}$ and for computing $H_\alpha(X_j)$, we have

$$f^{\alpha-1}\{F^{-1}(1-e^{-u})\} = \left(\frac{\theta}{\beta}\right)^{\alpha-1} e^{-u(\alpha-1)(1+\frac{1}{\theta})}.$$

Thus generalized entropy (4) of the j^{th} record value for Pareto distribution is given as

$$H_\alpha(X_j) = \frac{1}{(1-\alpha)\{\Gamma(j)\}^\alpha}\{\left(\frac{\theta}{\beta}\right)^{\alpha-1}\int_0^\infty u^{(j-1)\alpha} e^{-(\alpha+\frac{\alpha-1}{\theta})u}du - \{\Gamma(j)\}^\alpha\},$$

which gives,

$$H_\alpha(X_j) = \frac{1}{(1-\alpha)\{\Gamma(j)\}^\alpha}\{\frac{\Gamma\{(j-1)\alpha+1\}\theta^{j\alpha}}{\beta^{\alpha-1}[\alpha\theta+\alpha-1]^{(j-1)\alpha+1}} - \{\Gamma(j)\}^\alpha\}. \tag{16}$$

Therefore,

$$H_\alpha(X_1) = \frac{1}{1-\alpha}\left\{\frac{\theta^\alpha}{[\alpha\theta+\alpha-1]\beta^{\alpha-1}} - 1\right\},$$

the entropy for the parent distribution. The entropy for non trivial record X_2 is given as

$$H_\alpha(X_2) = \frac{1}{1-\alpha}\left\{\frac{\Gamma(\alpha+1)\theta^{2\alpha}}{\beta^{\alpha-1}[\alpha\theta+\alpha-1]^{\alpha+1}} - 1\right\}. \tag{17}$$

3.4 Finite Range Distribution

The pdf of the finite range distribution is given by

$$f(x) = \frac{a}{b}\left(1-\frac{x}{b}\right)^{a-1},\quad a > 1,\ 0 \leq x \leq b. \tag{18}$$

The survival function is

$$\bar{F}(x) = 1 - F(x) = \left(1-\frac{x}{b}\right)^a.$$

Substituting $-\ln\bar{F}(x) = u$, we observe that $x = F^{-1}(1-e^{-u}) = b(1-e^{-\frac{u}{a}})$ and for computing $H_\alpha(X_j)$, we have

$$f^{\alpha-1}\{F^{-1}(1-e^{-u})\} = \left(\frac{a}{b}\right)^{\alpha-1} e^{-u(\alpha-1)(1-\frac{1}{a})} \tag{19}$$

Lemma 3.1 gives

$$H_\alpha(X_j) = \frac{1}{(1-\alpha)\{\Gamma(j)\}^\alpha}\{\frac{\Gamma\{(j-1)\alpha+1\}a^{j\alpha}}{b^{\alpha-1}[a\alpha-\alpha+1]^{(j-1)\alpha+1}} - \{\Gamma(j)\}^\alpha\}. \tag{20}$$

For $j=1$, the entropy for parent distribution is

$$H_\alpha(X_1) = \frac{1}{1-\alpha}\left\{\frac{\theta^\alpha}{[\alpha\theta+\alpha-1]\beta^{\alpha-1}} - 1\right\}$$

The entropy for non trivial record X_2 is given as

$$H_\alpha(X_2) = \frac{1}{1-\alpha}\left\{\frac{\Gamma(\alpha+1)a^{2\alpha}}{b^{\alpha-1}[a\alpha-\alpha+1]^{\alpha+1}} - 1\right\}. \tag{21}$$

3.5 Weibull Distribution

A non-negative random variable X is Weibull distributed, if its pdf is

$$f(x) = \lambda\beta x^{\beta-1}\exp\left\{-\lambda x^\beta\right\},\ \lambda,\ \beta > 0,\ x > 0$$

where λ and β are scale and shape parameters respectively. The survival function is

$$\bar{F}(x) = 1 - F(x) = e^{-\lambda x^\beta}.$$

Substituting $-\ln\bar{F}(x) = u$, we observe that $x = F^{-1}(1-e^{-u}) = \{\frac{u}{\lambda}\}^{\frac{1}{\beta}}$ and for computing $H_\alpha(X_j)$, we have

$$f^{\alpha-1}\{F^{-1}(1-e^{-u})\} = \left(\beta\lambda^{\frac{1}{\beta}}\right)^{\alpha-1}\{u\}^{\frac{(\alpha-1)(\beta-1)}{\beta}} e^{-u(\alpha-1)}$$

Lemma 3.1 gives

$$H_\alpha(X_j) = \frac{1}{(1-\alpha)\{\Gamma(j)\}^\alpha}\{\frac{\left(\beta\lambda^{\frac{1}{\beta}}\right)^{\alpha-1}\Gamma\{j\alpha-\frac{\alpha-1}{\beta}\}}{\{\alpha\}^{j\alpha-\frac{\alpha-1}{\beta}}} - \{\Gamma(j)\}^\alpha\}. \tag{22}$$

For $j=1$ the entropy for parent distribution is

$$H_\alpha(X_1) = \frac{1}{(1-\alpha)}\left\{\frac{\left(\beta\lambda^{\frac{1}{\beta}}\right)^{\alpha-1}\Gamma\{\alpha-\frac{\alpha-1}{\beta}\}}{\{\alpha\}^{\alpha-\frac{\alpha-1}{\beta}}} - 1\right\}.$$

For $\beta = 1$, (22) reduces to (14), the entropy for j^{th} record of exponential distribution.

4 Characterization Problem

In this section, we show that the distribution function F can be uniquely specified up to a location change by the equality of Tsallis entropy of record values. First we state the following lemma, due to Goffman and Pedrick (1965).

Lemma 41*A complete orthogonal system for the space $L_2(0,\infty)$ is given by sequence of Laguerre function*

$$\phi_n(x) = \frac{1}{n!}e^{-\frac{x}{2}}L_n(x),\ n \geq 0.$$

where $L_n(x)$ is the Laguerre polynomial, defined as the sum of coefficients of e^{-x} in the nth derivative of $x^n e^{-x}$, that is

$$L_n(x) = e^x\frac{d^n}{dx^n}(x^n e^{-x}) = \sum_{k=0}^{n}(-1)^k n(n-1)\cdots(k+1)x^k. \tag{23}$$

The completeness of Laguerre functions in $L_2(0,\infty)$ means that if $f \in L_2(0,\infty)$ and $\int_0^\infty f(x)e^{-\frac{x}{2}}L_n(x)dx = 0$, $\forall n \geq 0$, then f is zero almost everywhere.

Theorem 41*Let X and Y be two random variables with pdfs $f(x)$ and $g(x)$ and absolutely continuous cdfs $F(x)$ and $G(x)$ respectively, with $E[\log f(x)]^2 < \infty$ and $E[\log g(x)]^2 < \infty$. Then F and G belong to the same location family of distribution, if, and only if*

$$H_\alpha(X_j) = H_\alpha(Y_j)\ ,\ \forall j \geq 1, \tag{24}$$

where X_j and Y_j are the j^{th} upper records of X and Y respectively.

Proof The necessary part is obvious. We only need to prove the sufficiency part. Let

$$H_\alpha(X_j) = H_\alpha(Y_j)\ ,\ \forall j \geq 1.$$

We Know that

$$H_\alpha(X_j) = \frac{1}{(1-\alpha)\{\Gamma(j)\}^\alpha}\{\int_0^\infty\{-\ln\bar{F}(x)\}^{(j-1)\alpha}f^\alpha(x)dx - \{\Gamma(j)\}^\alpha\}. \tag{25}$$

Substituting $\{-\ln\bar{F}(x)\}^\alpha = u$, and hence, $x = F^{-1}\left\{1-\exp\left(-u^{\frac{1}{\alpha}}\right)\right\}$, in (25) we get

$$H_\alpha(X_j) = \frac{1}{(1-\alpha)\{\Gamma(j)\}^\alpha}\{\int_0^\infty u^{j-1}u^{\frac{1}{\alpha}-1}e^{-(u)^{\frac{1}{\alpha}}} f^{\alpha-1}\{F^{-1}(1-e^{-(u)^{\frac{1}{\alpha}}})\}du - \{\Gamma(j)\}^\alpha\}.$$

Similarly, we get

$$H_\alpha(Y_j) = \frac{1}{(1-\alpha)\{\Gamma(j)\}^\alpha}\{\int_0^\infty u^{j-1}u^{\frac{1}{\alpha}-1}e^{-(u)^{\frac{1}{\alpha}}} g^{\alpha-1}\{G^{-1}(1-e^{-(u)^{\frac{1}{\alpha}}})\}du - \{\Gamma(j)\}^\alpha\}.$$

If for two cdfs F and G, these differences coincide, we can conclude that

$$\int_0^\infty u^{\frac{1}{\alpha}-1}e^{-(u)^{\frac{1}{\alpha}}}[f^{\alpha-1}\{F^{-1}(1-e^{-(u)^{\frac{1}{\alpha}}})\} - g^{\alpha-1}\{G^{-1}(1-e^{-(u)^{\frac{1}{\alpha}}})\}]u^{n-1}L_n(u)du = 0, \tag{26}$$

for all $n \geq 1$, By (26), we can conclude that

$$\int_0^\infty u^{\frac{1}{\alpha}-1}e^{\frac{u}{2}-(u)^{\frac{1}{\alpha}}}[f^{\alpha-1}\{F^{-1}(1-e^{-(u)^{\frac{1}{\alpha}}})\} - g^{\alpha-1}\{G^{-1}(1-e^{-(u)^{\frac{1}{\alpha}}})\}]e^{-\frac{u}{2}}u^{n-1}L_n(u)du = 0, \tag{27}$$

for all $n \geq 1$, where $L_n(u)$ is Laguerre polynomial given in Lemma 4.1. Using the assumption $E[\log f(x)]^2 < \infty$ and $E[\log g(x)]^2 < \infty$, and Minkowski inequality, we can conclude that

$$u^{\frac{1}{\alpha}-1}e^{\frac{u}{2}-(u)^{\frac{1}{\alpha}}}\{f^{\alpha-1}\{F^{-1}(1-e^{-(u)^{\frac{1}{\alpha}}})\} - g^{\alpha-1}\{G^{-1}(1-e^{-(u)^{\frac{1}{\alpha}}})\}\} \in L^2(0,\ 1).$$

Hence, by the completeness property of Lemma 4.1, we conclude that

$$f(F^{-1}(v)) = g(G^{-1}(v)),\ \forall\ v \in (0,1).$$

As, $\frac{d(F^{-1}(v))}{dv} = \frac{1}{f(F^{-1}(v))}$. Therefore, we have

$$F^{-1'}(v) = G^{-1'}(v),\ \forall\ v \in (0,1)$$

or, $F^{-1}(v) = G^{-1}(v) + c$,

where c is a constant. This concludes the proof.

5 Generalized Residual Entropy of Record Values

In reliability theory and survival analysis, X usually denotes a duration such as the lifetime of a component. The residual lifetime of the system when it is still operating at time t is $(X_t = X - t|X > t)$ which has the probability density $f(x;t) = \frac{f(x)}{\bar{F}(t)}$, $x \geq t > 0$, where $\bar{F}(t) = 1 - F(t) > 0$. Ebrahimi (1996) proposed the entropy of the residual lifetime X_t as

$$H(X;t) = -\int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} dt \ , t > 0. \tag{28}$$

This measures the uncertainty of the residual lifetime of the system when it is still operating at time t. The role of residual entropy as a measure of uncertainty in order statistics and record values has been studied by many researchers, refer to, Zarezadeh and Asadi (2010), Bartapour et al. (2007, 2008). The generalized residual entropy of order α is defined as

$$H_\alpha(X;t) = \frac{1}{(1-\alpha)}\left[\frac{\int_t^\infty f^\alpha(x)dx}{\bar{F}^\alpha(t)} - 1\right];\ \alpha > 0,\ \alpha \neq 1\,. \tag{29}$$

For more details and application of this dynamic information measure refer to Nanda and Paul (2005), and Kumar and Taneja (2011). Obviously, when $t = 0$, (28) and (29) reduce to information measures (3) and (4) respectively.

Let $X_1, X_2, \cdots X_n$ are i.i.d. random variables with an absolutely continuous distribution F and density function f, denoting the lifetime of n components. Then $Z = min(X_1, X_2, ..., X_n)$ represents the lifetime of the system, whose components are connected in series. The residual entropy measure (29) of the series system is independent of t, when X_i's are exponentially variate. In this context, we prove the following theorem.

Theorem 51*If $X_1, X_2, ..., X_n$ are independent random variables having an exponential distribution with parameters $\theta_i, i = 1, 2, ..., n$, then the residual entropy (29), of the random variable $Z = min(X_1, X_2, ..., X_n)$ is independent of the parameters t.*

Proof Since $Z = min(X_1, X_2, ..., X_n)$, therefore the cumulative distribution function (c.d.f.) Z is

$$F_Z(z) = P(Z \leq z) = 1 - \prod_{i=1}^n (\exp(-\theta_i z)) = 1 - \exp(-z\sum_{i=1}^n \theta_i)\,.$$

Survival function of Z is

$$\bar{F}_Z(z) = 1 - F_Z(z) = \exp\left(-z\sum_{i=1}^n \theta_i\right).$$

Then its p.d.f. is given by

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \sum_{i=1}^n \theta_i \left[\exp\left(-z\sum_{i=1}^n \theta_i\right)\right].$$

Substituting these values in (29) and simplify, we obtain

$$H_\alpha(Z;t) = \frac{1}{1-\alpha}\left(\frac{\sum_{i=1}^n \{\theta_i\}^{\alpha-1}}{\alpha} - 1\right),$$

which is independent of t.

Corollary 51*If $X_1, X_2, ..., X_n$ are independent and identically distributed (i.i.d.) random variables, then :*

$$H_\alpha(Z;t) = \frac{1}{1-\alpha}\left(\frac{\sum_{i=1}^n \{n\theta\}^{\alpha-1}}{\alpha} - 1\right).$$

The role of residual entropy as a measure of uncertainty in order statistics and record value has been studied by Zarezadeh and Asadi (2010). Next, we derive generalized residual entropy of order α for the jth upper record value. Before the main result we state the following two lemmas which are easy to prove.

Lemma 51*Let U_j^* be the j^{th} upper record value for a sequence of observations from uniform distribution on $(0,1)$. Then*

$$H_\alpha(U_j^*;t) = \frac{1}{1-\alpha}\left\{\frac{\Gamma[(j-1)\alpha+1; -\log(1-t)]}{\Gamma^\alpha(n; -\log(1-t))} - 1\right\}. \tag{30}$$

Proof For uniform distribution, using (29) we have

$$H_\alpha(U_j^*;t) = \frac{1}{(1-\alpha)}\left[\frac{\int_t^\infty g_{X_j}^\alpha(x)dx}{G_{X_j}^\alpha(t)} - 1\right];\ \alpha > 0,\ \alpha \neq 1\,. \tag{31}$$

Putting values from (1) and (2) in (31), we get the desired result (30).

Lemma 52*Let $\bar{U}_j$ be the j^{th} upper record values for a sequence of observations from standard exponential distribution. Then*

$$H_\alpha(\bar{U}_j;t) = \frac{1}{1-\alpha}\left\{\frac{\Gamma[(j-1)\alpha+1;t]}{\Gamma^\alpha(n;t)}E\{e^{-(\alpha-1)U_t}\} - 1\right\}, \tag{32}$$

where $U_t \sim \Gamma\{(j-1)\alpha+1;t\}$

The proof follows on the same lines as in Lemma 5.1.
Now we state the main result.

Theorem 52*Let X_n, $n > 1$ be a sequence of i.i.d. continuous random variable from the distribution $F(x)$ with density function $f(x)$ and the quantile function $F^{-1}(.)$. Let U_j denote the j^{th} upper record . Then the dynamic generalized entropy (29) of j^{th} upper record value can be expressed as*

$$H_\alpha(\bar{U}_j;t) = \frac{1}{1-\alpha}\{\frac{\Gamma[(j-1)\alpha+1; -\log\bar{F}(t)]}{\Gamma^\alpha(n; -\log\bar{F}(t))} E\{f^{\alpha-1}\{F^{-1}(1-e^{-V_z})\}\} - 1\}, \tag{33}$$

where $z = -\log\bar{F}(t)$ and $V_z \sim \Gamma\{(j-1)\alpha+1; -\log\bar{F}(t)\}$ and E is the expectation.

Proof Dynamic generalized entropy (29) of the j^{th} upper record value is defined as

$$H_\alpha(U_j;t) = \frac{1}{(1-\alpha)}\left[\frac{\int_t^\infty g_{X_j}^\alpha(x)dx}{G_{X_j}^\alpha(t)} - 1\right];\ \alpha > 0,\ \alpha \neq 1$$

$$= \frac{1}{(1-\alpha)}\left\{\frac{\int_t^\infty \{-\ln \bar{F}(x)\}^{(j-1)\alpha} f^\alpha(x)dx}{\Gamma^\alpha\{n; -\log F(t)\}} - 1\right\}. \qquad (34)$$

Substituting $-\log \bar{F}(x) = u$ and $x = F^{-1}(1-e^{-u})$, we have

$$H_\alpha(U_j;t) = \frac{\left\{\int_{-\log \bar{F}(t)}^\infty u^{(j-1)\alpha} e^{-u} \times \left[f^{\alpha-1}\{F^{-1}(1-e^{-u})\}\right] du\right\}}{(1-\alpha)(\Gamma^\alpha\{n; -\log F(t)\} - 1)}$$

It can be rewritten as

$$H_\alpha(\bar{U}_j;t) = \frac{1}{1-\alpha}\{\frac{\Gamma[(j-1)\alpha + 1; -\log \bar{F}(t)]}{\Gamma^\alpha(n; -\log F(t))} E\{f^{\alpha-1}\{F^{-1}(1-e^{-V_z})\}\} - 1\}.$$

So, the result follows.

Example Let X have Weibull distribution with density

$$f(x) = \lambda\beta x^{\beta-1}\exp\left\{-\lambda x^\beta\right\},\ \lambda,\ \beta > 0,\ x > 0$$

Here, $x = F^{-1}(1-e^{-u}) = \{\frac{u}{\lambda}\}^{\frac{1}{\beta}}$. Then we have

$$f^{\alpha-1}\{F^{-1}(1-e^{-u})\} = \left(\beta\lambda^{\frac{1}{\beta}}\right)^{\alpha-1}\{u\}^{\frac{(\alpha-1)(\beta-1)}{\beta}} e^{-u(\alpha-1)}$$

Therefore

$$H_\alpha(\bar{U}_j;t) = \frac{1}{1-\alpha}\left\{\frac{\beta^{(\alpha-1)}\lambda^{\frac{\alpha-1}{\beta}}\Gamma\left[j\alpha - \frac{\alpha-1}{\beta}; \lambda t^\beta\right]}{\Gamma^\alpha\{n; \lambda t^\beta\}[\alpha]^{j\alpha - \frac{\alpha-1}{\beta}}} - 1\right\}. \qquad (35)$$

Remark 1 For $b = 2$, (35) reduces to

$$H_\alpha(\bar{U}_j;t) = \frac{1}{(1-\alpha)}\left\{\frac{\{2\sqrt{\lambda}\}^{\alpha-1}\Gamma\{j\alpha - \frac{\alpha-1}{2}; \lambda t^2\}}{\Gamma^\alpha\{n; \lambda t^2\}\{\alpha\}^{\{j\alpha - \frac{\alpha-1}{2}\}}} - 1\right\}; \qquad (36)$$

the residual entropy of the jth record value from a Rayleigh distribution, that is, $X \sim Rayleigh(\lambda > 0)$.

Remark 2 For $b = 1$, (35) reduces to

$$H_\alpha(\bar{U}_j;t) = \frac{1}{(1-\alpha)}\left\{\frac{\{\lambda\}^{\alpha-1}\Gamma\{j\alpha - \alpha + 1; \lambda t\}}{\Gamma^\alpha\{n; \lambda t\}\{\alpha\}^{\{j\alpha-\alpha+1\}}} - 1\right\}; \qquad (37)$$

the residual entropy of the jth record value from a exponential distribution, that is, $X \sim \exp(\lambda)$.

6 Conclusion

Information theoretic measures of Shannon and Renyi which are additive in nature have been studied by many researchers for record values. It is of interest to study non-additive entropy measures, which find applications in many phenomena, for record values. We have seen that Havrda and Charvat entropy measure for record values characterize the underlying distribution function uniquely except for the location. Also the concept generalized residual entropy for record values has been studied which can be explored further both theoretical interest and application point of view.

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