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New Aspects of Nonautonomous Discrete Systems Stability

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Abstract: We prove that a discrete evolution family $\mathbf{U} := \{U(n,m) : n \geq m \in \mathbb{Z}_+\}$ of bounded linear operators acting on a complex Banach space X is uniformly esponentially stable if and only if for each forcing term $(f(n))_{n\in\mathbb{Z}_+}$ belonging to $AP_0(\mathbb{Z}_+,X)$, the solution of the discrete Cauchy Problem

$$\begin{cases} x(n+1) = A(n)x(n) + f(n), \; n \in \mathbb{Z}_+ \\ x(0) = 0 \end{cases}$$

belongs also to $AP_0(\mathbb{Z}_+,X)$, where the operators-valued sequence $(A(n))_{n\in\mathbb{Z}_+}$ generates the evolution family $\mathbf{U}$. The approach we use is based on the theory of discrete evolution semigroups associated to this family.

Keywords: Exponential stability, Non-autonomous discrete problems, discrete evolution families of bounded linear operators, discrete evolution semigroups

1 Introduction

During the last decades, the theory of difference equations has gained a lot of attention by researchers. In fact, these equations are involved in different areas such as biology (the study of competitive species in population dynamics), physics (the study of motions of interacting bodies), control theory, the study of control theory, etc. See ([1], [12])

Difference equations usually describe the evolution of certain phenomena over the course of time. For instance, if we are interested in the study of a poluation dynamic, assuming that this population has discrete generations, the size of the $(n+1)^{th}$ generation, denoted by $x(n+1)$, is a function of the size of the n^{th} generation. In other words, for some function f, we are able to formulate this problem as the form of a certain difference equation $x(n+1) = f(x(n))$, for all integer number $n \in \mathbb{Z}$. Here, we denote by $\mathbb{Z}$ the set of all integer numbers.

One of the most common problem is the study of the stability of such difference equations. Indeed, recently, the issue of the different kinds of stability has become of great interest for researchers in a large number of fields such as social sciences ([17]), numerical analysis ([3]), neural networks ([15], [16], [19]). For the autonomous systems, many spectral characterizations can be revealed to show their stability, or more generaly to asymptotic behaviour of the semigroups appeared for ths kind of systems. Unfortunately, these spectral characterizations are no longer valid for the nonautonomous systems. This is principaly due to the fact that the term generated by the forcing term cannot be seen as as classical convolution. This matter of stability can be avoided using an evolution semgoups approach. The method of evolution semigoups is an efficient method for this aim, [6], [7].

Another topic which has been progressed in the last decades is the almost periodicity, [4], [5]. In fact, due to the interesting properties and to the worth structure of the spaces of almost periodic functions, the notion of almost periodicity seems to be a powerful tool in the study of differential and difference equations. Many contributions in the study of continuous and discrete versions of almost periodic functions have been done in this direction, showing the importance of almost periodicity. For more details, we can refer the reader to [18], [8], [9], [13].

However, the litterature linking the difference equations and the discrete almost periodic functions remains really scare. We find among others few works

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dealing with these notions. We set, for example, [14], [2], [20].

In this paper, we aim to study the asymptotic behaviour of linear non-autonomous difference equation having the form

$$x(n+1) = A(n)x(n) + f(n), n \in \mathbb{Z}_+,$$

where $(A(n))_{n\in\mathbb{Z}_+}$ is a $\mathscr{L}(X)$-valued sequence. Here, X stands for a complex Banach space and we denote by $\mathscr{L}(X)$ the Banach algebra of all linear and bounded operators acting on X.

More precisely, we prove that the evolution family associated to this equation is uniformly exponentially stable if and only if for each almost periodic forcing term $f := (f(n))_{n\in\mathbb{Z}_+}$, the solution of this equation is also almost periodic. See the next section for the corresponding definitions.

2 Definitions and Preliminaries

The norms on X and $\mathscr{L}(X)$ will be denoted by the same symbol, namely $\|\cdot\|$. The set nonnegative integer numbers is denoted by $\mathbb{Z}_+$. The linear space of all X-valued sequences will be denoted by $X^{\mathbb{Z}_+}$.

A sequence $x := (x(n))_{n\in\mathbb{Z}_+} \in X^{\mathbb{Z}_+}$ is said to be almost periodic if for each real number $\varepsilon > 0$, there exists a positive integer $l_\varepsilon \in \mathbb{Z}_+ \setminus \{0\}$ such that each discret integer interval of length l_ε contains at least an integer $p \in \mathbb{Z}_+$ verifying $\|x(n+p) - x(n)\| \leq \varepsilon$ for any $n \in \mathbb{Z}_+$, [10].

The space of almost periodic X-valued sequences defined on $\mathbb{Z}_+$ is denoted by $AP(\mathbb{Z}_+,X)$. It is not difficult to show that this space is a linear subspace of $\mathscr{B}(X)$.

Now, let us define $AP_0(\mathbb{Z}_+,X)$ the linear subspace of $AP(\mathbb{Z}_+,X)$ as

$$AP_0(\mathbb{Z}_+,X) := \{x = (x(n))_{n\in\mathbb{Z}_+} \in AP(\mathbb{Z}_+,X) \,:\, x(0) = 0\}.$$

It is clear that $AP_0(\mathbb{Z}_+,X)$ is a closed subspace of $AP(\mathbb{Z}_+,X)$.

Endowed with the supremum norm, the space $AP(\mathbb{Z}_+,X)$ becomes Banach space. It follows that $AP_0(\mathbb{Z}_+,X)$ is a Banach space, too.

Let Y be a Banach space. A family $\mathbf{T} := \{T(j) : j \in \mathbb{Z}_+\}$ of linear bounded operators acting on Y is called a discrete semigroup if

1. $T(0) = I$, where I denotes the identity operator on Y, and
2. $T(j+k) = T(j) \circ T(k)$, for all $j,k \in \mathbb{Z}_+$.

It follows that for all $j \in \mathbb{Z}_+$, we have $T(j) = T(1)^j$. $T(1)$ is called so the algebraic generator of the semigroup $\mathbf{T}$. The infinitesimal generator of the semigroup $\mathbf{T}$ is defined as $G := T(1) - I$.

The Taylor formula of order one for discrete semigroups may be written as:

$$T(j)f - f = \sum_{k=0}^{j-1} T(k)Gf, \ \forall j \in \mathbb{Z}_+ : \ j \geq 1, \ f \in Y \quad (1)$$

Indeed,

$$\sum_{k=0}^{j-1} T(k)Gf = \sum_{k=0}^{j-1} [T(k+1) - T(k)]f = T(j)f - f.$$

The discrete semigroup $\mathbf{T} := \{T(j) : j \in \mathbb{Z}_+\}$ is uniformly exponentially stable if there exist $N \geq 1$ and $\mu > 0$ such that $\|T(j)\| \leq Ne^{-\mu j}$ for each $j \in \mathbb{Z}_+$.

A discrete evolution family on the Banach space X is a family of linear bounded operators acting on X , $\mathbf{U} := \{U(n,m) \in \mathscr{L}(X) : \ n \geq m \in \mathbb{Z}_+\}$ having the properties

1. $U(m,m) = I$, where I is the identity operator on X, and
2. $U(m,p)U(p,n) = U(n,m =$, for all $n \geq m \in \mathbb{Z}_+$.

The discrete evolution family $\mathbf{U}$ is said to have an exponential growth if there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|U(n,m)\| \leq Me^{\omega(n-m)}$, for each pair $(n,m) \in \mathbb{Z}_+$ with $n \geq m$.

While the semigroups have automatically an exponential growth, the discrete evolution families do not necesseraly have an exponential growth.

Throughout this paper, for a given operator A acting on the Banach space X, we use the following notations:

1. $\rho(A)$ for the resovent set of A, that is, the set of all complex scalars $z \in \mathbb{C}$ for which the operator $zI - A$ is not invertible
2. $\sigma(A) := \mathbb{C} \setminus \rho(A)$ for the spectrum of A, and
3. $r(A) := \sup\{|z|;\, z \in \sigma(A)\}$ for the spectral radius of A.

It is well-known, [11], that

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}. \quad (2)$$

It follows from (2) that a discrete semigroup $\mathbf{T} := \{T(j) : j \in \mathbb{Z}_+\}$ is uniformly exponentially stable if and only if the spectral radius $r(T(1))$ is less than one.

3 Criterion for Uniform Exponential Stability of Evolution Famillies on $\mathbb{Z}_+$

Let $\mathbf{U} = \{U(n,m) \in \mathscr{L}(X) : \ n \geq m \in \mathbb{Z}_+\}$ be a discrete evolution family of bounded linear operators acting on a Banach space X having exponential growth.

Before investigating the evolution semigroup associated to the family $\mathbf{U}$, let us define the space $\mathscr{A}_0(\mathbb{Z}_+,X)$ as the set of all the sequences $x = (x(n))_{n\geq 0}$ for which there is an integer $n_x \geq 0$ and a sequence $F_x := (F_x(n))_{n\geq 0}$ with the property $x(n) = 0$ if $0 \geq 0 \geq n_x$ and $x(n) = F_x(n)$ if $n \geq n_x$.

The space $\mathscr{A}_0(\mathbb{Z}_+,X)$ is evidently a subspace of $AP(\mathbb{Z}_+,X)$.

Lemma 3.1. The space $\mathscr{A}_0(\mathbb{Z}_+,X)$ is dense in $AP(\mathbb{Z}_+,X)$.

Proof. Let $f := (f(n))_{n\in\mathbb{Z}_+} \in AP(\mathbb{Z}_+,X)$ and define the sequence $(g_j)_{j\in\mathbb{Z}_+}$ given by

$$g_j(n) = \begin{cases} f(n) & \text{for all } 0 \leq n \leq j-1 \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that the sequence $(g_j)_{j\in\mathbb{Z}_+}$ belongs to $AP(\mathbb{Z}_+,X)$ and converges to f as j tends to infinity. □

Now, we define the evolution semigroup associated to the family $\mathbf{U}$ for each $f\in AP(\mathbb{Z}_+,X)$ as

$$[T(j)f](n)=\begin{cases}U(n,n-j)f(n-j) & \text{for all } n\geq j\\ 0 & \text{otherwise.}\end{cases} \quad (3)$$

Lemma 3.2. The semigroup $\{T(j):\ j\geq 0\}$ defined in 3 acts on $AP(\mathbb{Z}_+,X)$.

Proof. Let us first show that $T(j)f\in\mathscr{A}_0(\mathbb{Z}_+,X)$ for each sequence $f\in\mathscr{A}_0(\mathbb{Z}_+,X)$. Given $f\in\mathscr{A}_0(\mathbb{Z}_+,X)$, we can find, by definition, an integer $n_f\geq 0$ and sequence $F_f:=F_(f(n))_\in AP(\mathbb{Z}_+,X)$ such that

$$f(n)=\begin{cases}F_f(n) & \text{if } n\geq n_f\\ 0 & \text{if } 0\leq n\leq n_f.\end{cases}$$

Let us choose the integer $n_{T(j)f}:=j+n_f$ and the sequence $(F_{T(j)f}(n))_n:=(U(n,n-j)F_f(n-j))_{n\in\mathbb{Z}_+}$. Then, we have

$$[T(j)f](n)=\begin{cases}F_{T(j)f}(n) & \text{if } n\geq n_{T(j)f}\\ 0 & \text{if } 0\leq n\leq n_{T(j)f}.\end{cases}$$

Indeed, if $n\leq n_{T(j)f}=j+n_f$, then it is clear that $n-j\leq n_f$ which implies that $f(n-j)=0$ and so that $[T(j)f](n)=U(n,n-j)f(n-j)=0$. In addition, if $n\geq n_{T(j)f}\geq j+n_f$ i.e. $n-j\geq n_f$, then $f(n-j)=F_f(n-j)$ and therefore,

$$\begin{aligned}[T(j)f](n)&=U(n,n-j)f(n-j)\\&=U(n,n-j)F_f(n-j)\\&=F_{T(j)f}(n).\end{aligned}$$

We have then proved that $T(j)f\in\mathscr{A}_0(\mathbb{Z}_+,X)$.
Now, if $f\in AP(\mathbb{Z}_+,X)$, from the Lemma 3.1., for any $\varepsilon>0$, there exists $g\in\mathscr{A}_0(\mathbb{Z}_+,X)$ such that $\|f-g\|_{\mathscr{B}(\mathbb{Z}_+,X)}<\varepsilon$. We have

$$\begin{aligned}\|T(j)f-T(j)g\|&=\sup_{n\geq j}\|U(n,n-j)[f(n-j)-g(n-j)]\|\\&\leq Me^{\nu j}\sup_{n\geq j}\|f(n-j)-g(n-j)\|\\&\leq Me^{\nu r}\varepsilon.\end{aligned}$$

Since $T(j)g\in\mathscr{A}_0(\mathbb{Z}_+,X)$ and $\varepsilon>0$ is arbitrarily choosen, we obtain the result. □

Lemma 3.3. Let $\mathbf{U}=\{U(n,m)\in\mathscr{L}(X)\ :\ n\geq m\in\mathbb{Z}_+\}$ be a 1-periodic evolution family and $\mathbf{R}=\{R(j):\ j\geq 0\}$ the evolution semigroup associated to $\mathbf{U}$ on the space $AP_0(\mathbb{Z}_+,X)$ having $G_\mathbf{R}$ as infintesimal generator. Let consider $x:=(x(n))_{n\in\mathbb{Z}_+}$ and $f:=(f(n))_{n\in\mathbb{Z}_+}$. The following statementes are equivalent.

1.x belongs to the domain of $G_\mathbf{R}$ i.e. $x\in D(G_\mathbf{R})$ and $G_\mathbf{R}x=-f$
2.$x(n)=\sum_{k=0}^n U(n,k)f(k)$ for every $n\geq 0$.

Proof. Let show first the implication **1**. ⇒ **2**.. Using the Taylor formula 1, one has

$$R(n)x-x=\sum_{m=0}^{n-1}R(m)G_\mathbf{R}x=-\sum_{m=0}^{n-1}R(m)f.$$

This implies when applying to n on both sides

$$\begin{aligned}x(n)&=[R(m)x]_n+\sum_{m=0}^{n-1}[R(m)f](n)\\&=U(n,0)x(0)+\sum_{m=0}^{n-1}U(n,n-m)f(n-m)\\&=\sum_{k=0}^{n}U(n,k)f(k).\end{aligned}$$

For the converse implication **2**. ⇒ **1**., we have

$$\begin{aligned}G_\mathbf{R}(n)&=[R(1)-I]x(n)\\&=U(n,n-1)x(n-1)-x(n)\\&=U(n,n-1)\sum_{k=0}^{n-1}U(n-1,k)f(k)-x(n)\\&=\sum_{k=0}^{n-1}U(n,k)f(k)-\sum_{k=0}^{n}U(n,k)f(k)\\&=-f(n),\end{aligned}$$

which completes the proof of the Lemma. □

Lemma 3.4. Let $\mathbf{U}$ be a discrete evolution family of bounded linear operator family acting on X having an exponential growth. If there exists a positive constant c such that for all $n\geq j$,

$$(n-j+1)\|U(n,j)\|\leq c,$$

then the evolution family $\mathbf{U}$ is uniformly exponentially stable.

Proof. There exists a positive integer N such that $\frac{c}{n-j+1}\leq\frac{1}{2}$ for each $n-j\geq N$. Then, in virtue of the assumption, we get

$$\|U(n,j)\|\leq\frac{1}{2},$$

for all $n\geq j+N$.

Let us denote by p the integer part of $\frac{n-j}{N}$. It is ten clear that $p\geq 1$ and $n=j+pN+\rho N$ with $\rho\in(0,1)$. It follows then

$$U(n,j)=U(j+pN+\rho N,j+pN)U(j+pN,j).$$

Since the evolution family $\mathbf{U}$ has an exponential growth, we deduce that

$$\|U(n,j)\|\leq Me^{\omega N}\|U(j+pN,j)\|.$$

The evolution property
$U(j+Nm,j)=U(j+Nm,j+N(m-1))U(j+N(m-1),j+N(m-2))\cdots U(j+N,j)$ implies that

$$\|U(j+Nm,j)\|\leq\frac{1}{2^m}.$$

Finaly, we obtain that

$$\|U(n,j)\| \leq Me^{\omega N}\frac{1}{2^m} \leq Me^{\omega N}(\frac{1}{2})^{\frac{n-j}{N}-1} = Le^{-\nu(n-j)},$$

where $L = 2Me^{\omega N}$ and $\nu = \frac{\ln(2)}{N}$. □

Now, let us state the main results for this section in this paper

Theorem 3.1. Let $\mathbf{U} = \{U(n,m) : n \geq m \in \mathbb{Z}_+\}$ be a discrete evolution family acting on the space of Banach X and having an exponential growth. Let us set the following map defined for each sequence $f \in AP_0(\mathbb{Z}_+, X)$ by

$$x_{\mathbf{U},f} := \sum_{k=0}^{n} U(n,k)f(k).$$

If for all $f \in AP_0(\mathbb{Z}_+, X)$, the map $x_{\mathbf{U},f}$ belongs to $AP_0(\mathbb{Z}_+, X)$, then the evolution family $\mathbf{U}$ is uniformly exponentially stable.

Proof. We denote by K the linear operator defined by

$$K : AP_0(\mathbb{Z}_+, X) \to AP_0(\mathbb{Z}_+, X)$$

$$Kf(n) := x_{\mathbf{U},f}(n), \text{ for each } n \in \mathbb{Z}_+ \text{ and } f \in AP_0(\mathbb{Z}_+, X).$$

It is clear that the operator K is a linear one.

We first show that the operator K is bounded on $AP_0(\mathbb{Z}_+, X)$. To this end, and by virtue of the Closed Graph theorem, it is enough to prove that K is closed.

Let $(f_j)_j \in (AP_0(\mathbb{Z}_+, X))^{\mathbb{Z}_+}$ and $f, g \in AP_0(\mathbb{Z}_+, X)$ such that $(f_j)_j$ converges to f in $AP_0(\mathbb{Z}_+, X)$ and $(Kf_j)_j$ converges to g in $AP_0(\mathbb{Z}_+, X)$ as j tends to infinity.

It follows that, for each fixed $j \in \mathbb{Z}_+$, the X-valued sequence $(f_j(k))_k$ converges to $f(k)$.

The continuity of the operators $U(n,k)$ implies that for each $n \in \mathbb{Z}_+$, the sequence $((Kf_j)(n))_n$ converges to $Kf(n)$ as $j \to \infty$. We obtain then for all $n \in \mathbb{Z}_+$, $g(n) = Kf(n)$, i.e. $g = Kf$. This shows that the linear operator K is closed.

Therefore, there exists a positive constant c such that for each sequence $f \in AP_0(\mathbb{Z}_+, X)$ with $\|f\|_{AP_0(\mathbb{Z}_+,X)} \leq 1$, $\|Kf\|_{AP_0(\mathbb{Z}_+,X)} = \sup_{n\in\mathbb{Z}_+}\|Kf(n)\| \leq c$.

Now, let us prove that the evolution family $\mathbf{U}$ is uniformly bounded.

We extend the evolution family $\mathbf{U}$ by $U(n,m) = 0$ for each $n < m$. Consider the functions f_j , defined for each $j \in \mathbb{Z}_+$, $j \geq 1$ by

$$f_j(k) = \begin{cases} e^{ik}, & \text{if } k = j \\ 0, & \text{otherwise,} \end{cases}$$

It is obvious that $f_j \in AP_0(\mathbb{Z}_+, X)$ and $\|f_j\| \leq 1$. We deduce that

$$\|U(n,j)(e^{in})\| = \|Kf_j(n)\| \leq \|Kf_j\|_{AP_0(\mathbb{Z}_+,X)} \leq c,$$

for each $n \geq j \geq 1$.

On the other hand, we have

$$\|U(n,0)\| = \|U(n,1)U(1,0)\|$$
$$\leq \|U(n,1)\|\|U(1,0)\| \leq c\|U(1,0)\|.$$

Taking $c_1 := \max\{c, c\|U(1,0)\|\}$, we get that

$$\sup_{n\geq j\geq 0}\|U(n,j)\| \leq c_1 < \infty,$$

which means that the evolution family $\mathbf{U}$ is uniformly bounded.

Next, if we define the function $h_j(k) := \frac{1}{c_1}\chi_{[[j,\infty[[}(k)U(k,j)(e^{ik})$, for $j \geq 1$. Here, by $[[j,\infty[[$, we mean the discrete interval of integer numbers which are greater than or equal to j and we denote by $\chi_{[[j,\infty[[}$ the function defined as $\chi_{[[j,\infty[[}(k) = 1$, if $k \geq 1$ and $\chi_{[[j,\infty[[} = 0$, otherwise.

It follows then that h_j belongs to the space $AP_0(\mathbb{Z}_+, X)$ and that $\|h_j\|_{AP_0(\mathbb{Z}_+,X)} \leq 1$ since we have $\|U(n,j)\| \leq c_1$. This implies, using the boundedness of the operator K, that $\|Kh_j(n)\| \leq \|Kh_j\| \leq c$.

Moreover, we have

$$\begin{aligned}(Kh_j)(n) &= \frac{1}{c}\sum_{k=0}^{n} U(n,k)h_j(n) \\ &= \frac{1}{c}\sum_{k=j}^{n} U(n,j)b \\ &= \frac{1}{c}(n-j+1)U(n,j)b.\end{aligned}$$

We obtain so that for all integers $n \geq j \geq 1$, $\|(n-j+1)U(n,j)b\| \leq cc_1$, where, as above, $c_1 = \max\{c, c\|U(1,0)\|\}$.

For $j = 0$, we can write that, using the evolution property, $(n+1)\|U(n,0)\| = (n+1)\|U(n,1)U(1,0)\| \leq (n+1)\|U(n,1)\|\|U(1,0)\|$.

Since
$(n+1)\|U(n,1)\|\|U(1,0)\| = n\frac{n+1}{n}\|U(n,1)\|\|U(1,0)\| \leq 2cc_1\|U(1,0)\|$,
we get that $\|U(n,0)\| \leq 2cc_1\|U(1,0)\|$.

Then, setting $c_2 := \max\{cc_1, 2cc_1\|U(1,0)\|\}$, we have for all integers $n \geq j \geq 0$, that $\|U(n,j)\| \leq \frac{c_2}{n-j+1}$.

Finally, applying the Lemma 3.4., we ensure the existence of postive constants K and ν such that for any integers $n \geq j \geq 0$, $\|U(n,j)\| \leq Ke^{-\nu(n-j)}$. □

Now, the main resut for this section reads as follow. It gives, namely, a new criterion to characterize the exponential stability for difference non-autonomous systems in terms of almost periodic sequences.

Theorem 3.2. Let $\mathbf{U} := \{U(n,m) \in \mathscr{L}(X) : n \geq m \in \mathbb{Z}_+\}$ be an evolution family acting on the Banach space X having exponential growth. The following assertions are equivalent

1.U is uniformly exponentially stable.

2. The evolution semigroup **T** associated to te family **U** is uniformly exponentially stable on $AP_0(\mathbb{Z}_+,X)$.

3. The infinitesimal generator $G := T(1) - I$ of **T** is invertible.

4. The solution $x_f(n,0)$ of the non-autonomous Cauchy Problem

$$\begin{cases} x(n+1) = A(n)x(n) + f(n), & n \in \mathbb{Z}_+ \\ x(0) = 0 \end{cases} \tag{4}$$

belongs to $AP_0(\mathbb{Z}_+,X)$ for each forcing term $f \in AP_0(\mathbb{Z}_+,X)$.

Proof. From the definition of the evolution semigroup, the statement **1**. implies **2**.

If **2**. holds true, then the spectral radius $r(T(1)) < 1$. So, $1 \in \rho(A)$, which implies that $T(1) - I$ is an invertible operator and hence, **2**. implies **3**.

Now, assuming that **3**. holds true, for each $f \in AP_0(\mathbb{Z}_+,X)$, there exists a unique $x \in AP_0(\mathbb{Z}_+,X)$ such that $[T(1) - I]x = -f$. By virtue of the Lemma 3.3, this is equivalent to $x(j) = \sum_{k=0}^{j} U(j,k) f_k$ for all $j \in \mathbb{Z}_+$. Thus, $\sum_{k=0}^{j} U(j,k) f_k$ belongs to $AP_0(\mathbb{Z}_+,X)$.

On the other hand, the solution of (4) is given by

$$x_f(j,0) := \sum_{k=0}^{j} U(j,k) f_k,$$

which belongs, consequently, to $AP_0(\mathbb{Z}_+,X)$.

From the Theorem 3.1., we conclude that **4**. implies **1**. The proof of the theorem is now complete.

□

Example Let $A \in \mathscr{L}(X)$ and consider the following discrete Cauchy problem associated to A

$$\begin{cases} x(n+1) = Ax(n) + f(n), \; n \in \mathbb{Z}_+ \\ x(0) = 0. \end{cases}$$

The solution of this problem is given by $x(n,0) = \sum_{k=0}^{n} A^{n-k} f(k)$. Here, the evolution family $\mathbf{U} := \{U(k,j),\; j \geq k \geq 0\}$ associated to this system is defined as $U(k,j) := A^{j-k}$.

The solution $(x(n))_n$ is in $AP_0(\mathbb{Z}_+,X)$ if and only if the solution of the homogeneous equation

$$\begin{cases} z(n+1) = Az(n), \; n \in \mathbb{Z}_+ \\ z(0) = b, \end{cases}$$

that is $z(n) = A^n b$, decays exponentially, or similarly, this means the existence of two constants $\mu > 0$ and $M \geq 1$ such that for each vector $x \in X$, $\|T(k)x\| \leq Me^{-\mu k}\|x\|$.

4 Conclusion

We investigated the asymptotic behaviour of linear non-autonomous difference equation having the form

$$x(n+1) = A(n)x(n) + f(n), n \in \mathbb{Z}_+.$$

In general, unlike the autonomous case, the study of such behaviour is more complicated for the nonautonomous systems. In this paper, we gave a new criterion to characterize the uniform exponential stability of such systems using the almost periodic sequences. The almost periodic sequences are indeed a useful tool in such study. The method we use was based on the evolution semigroups theory. The last theory has shown a great efficiency to avoid the major difficulties appearing for the nonautonomous differential systems in both the continuous and discrete cases.

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