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On the Spectrum of Eigenparameter-Dependent Quantum Difference Equations

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Abstract: We consider a boundary value problem (BVP) consisting of a second-order quantum difference equation and boundary conditions depending on an eigenvalue parameter. Discussing the point spectrum and using the uniqueness theorem of analytic functions, we present a condition that guarantees that this BVP has a finite number of eigenvalues and spectral singularities with finite multiplicities.

Keywords: Quantum difference equation, discrete spectrum, spectral analysis, spectral singularities, eigenvalues

1 Introduction

Consider the BVP consisting of the Sturm–Liouville equation

$$\begin{cases} -y'' + Q(x)y = \lambda^2 y, & 0 \le x < \infty \\ y'(0) - hy(0) = 0, \end{cases} \tag{1}$$

where Q is a complex-valued function, $h \in \mathbb{C}$, and λ is a spectral parameter. Spectral analysis of BVP (1) was investigated by Naĭmark [12]. He showed that the spectrum of BVP (1) is composed of the eigenvalues, the continuous spectrum and the spectral singularities. The spectral singularities are poles of the resolvent's kernel which are imbedded in the continuous spectrum and are not eigenvalues. Boundary value problems for difference equations have been intensively studied in last decade in order to investigate problems in engineering, economics and control theory. Spectral theory of difference equations has been investigated by some authors in connection with the classical moment problem [4, 5, 11]. Some problems of spectral theory for difference equations were also treated in [1, 6–9]. Furthermore, spectral analysis of q-difference equations with spectral singularities has been investigated in [2, 3]. In this paper, we let $q > 1$ and use the notation $q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0\}$, where $\mathbb{N}_0$ denotes the set of nonnegative integers. Let us consider the nonselfadjoint BVP consisting of the second-order q-difference equation

$$qa(t)y(qt) + b(t)y(t) + a\left(\frac{t}{q}\right)y\left(\frac{t}{q}\right) = \lambda y(t), \quad t \in q^{\mathbb{N}} \tag{2}$$

and the boundary conditions

$$\begin{gathered} (\gamma_0 + \gamma_1\lambda)y(q) + (\beta_0 + \beta_1\lambda)y(1) = 0, \\ \gamma_0\beta_1 - \gamma_1\beta_0 \neq 0, \quad \gamma_1 \neq \frac{\beta_0}{a(1)}, \end{gathered} \tag{3}$$

where $\{a(t)\}_{t\in q^{\mathbb{N}_0}}$ and $\{b(t)\}_{t\in q^{\mathbb{N}}}$ are complex sequences, λ is a spectral parameter, $a(t) \neq 0$ for all $t \in q^{\mathbb{N}_0}$, and $\gamma_i, \beta_i \in \mathbb{C}$, $i = 0, 1$.

The set up of this paper is summarized as follows: Section 2 discusses the Jost solution and Jost function of the BVP (2)–(3). Also, we give the Green function and resolvent of this BVP in this section. In Section 3, we investigate the eigenvalues and the spectral singularities of the BVP (2)–(3) and get some properties of the eigenvalues and the spectral singularities of this BVP under the condition

$$\begin{gathered} \sup_{t\in q^{\mathbb{N}}} \left\{ \exp\left[\varepsilon\left(\frac{\ln t}{\ln q}\right)^{\delta}\right] (|1 - a(t)| + |b(t)|) \right\} < \infty, \\ \varepsilon > 0, \quad \tfrac{1}{2} \le \delta \le 1. \end{gathered} \tag{4}$$

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In Section 4, we deal with the condition (4) for $\delta=1$ and $\delta\neq 1$. For both cases, we prove that the BVP (2)–(3) has a finite number of eigenvalues and spectral singularities with finite multiplicities. Since the second case is weaker than the first, we have to use a different way for each to prove the theorem.

2 Jost Solution and Jost Function

Assume (4). Then (2) has the solution

$$e(t,z)=\alpha(t)\frac{e^{i\frac{\ln t}{\ln q}z}}{\sqrt{\mu(t)}}\left(1+\sum_{r\in q^{\mathbb{N}}}A(t,r)e^{i\frac{\ln r}{\ln q}z}\right), \quad t\in q^{\mathbb{N}_0}, \tag{5}$$

for $\lambda=2\sqrt{q}\cos z$, where $\alpha(t)$, $A(t,r)$ are expressed in terms of $\{a(t)\}$ and $\{b(t)\}$, $z\in\overline{\mathbb{C}}_+:=\{z\in\mathbb{C}:\operatorname{Im}z\geq 0\}$ and $\mu(t)=(q-1)t$ for all $t\in q^{\mathbb{N}_0}$ [2]. Moreover, $A(t,r)$ satisfies

$$|A(t,r)|\leq C\sum_{s\in\left[tq^{\lfloor\frac{\ln r}{2\ln q}\rfloor},\infty\right)\cap q^{\mathbb{N}}}(|1-a(s)|+|b(s)|), \tag{6}$$

where $\lfloor\frac{\ln r}{2\ln q}\rfloor$ is the integer part of $\frac{\ln r}{2\ln q}$ and $C>0$ is a constant. Therefore, $e(\cdot,z)$ is analytic with respect to z in $\mathbb{C}_+:=\{z\in\mathbb{C}:\operatorname{Im}z>0\}$ and continuous in $\overline{\mathbb{C}}_+$. Using (5) and the boundary condition (3), we define the function f by

$$\begin{aligned}f(z)=&(\gamma_0+2\sqrt{q}\gamma_1\cos z)e(q,z)\\&+(\beta_0+2\sqrt{q}\beta_1\cos z)e(1,z).\end{aligned} \tag{7}$$

The function f is analytic in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}}_+$, and $f(z)=f(z+2\pi)$. Analogously to the Sturm–Liouville differential equation, the solution $e(\cdot,z)$ and the function f are called the Jost solution and Jost function of (2)–(3), respectively [13]. Let $\varphi(\lambda)=\{\varphi(t,z)\}$ $t\in q^{\mathbb{N}_0}$, be the solution of (2) satisfying the initial conditions

$$\varphi(1,\lambda)=-(\gamma_0+\gamma_1\lambda),\quad \varphi(q,\lambda)=(\beta_0+\beta_1\lambda).$$

If we define

$$\phi(t,z)=\varphi(2\sqrt{q}\cos z)=\{\varphi(t,2\sqrt{q}\cos z)\}_{t\in q^{\mathbb{N}_0}},$$

then ϕ is an entire function and $\phi(z)=\phi(z+2\pi)$. Let us define the semi-strips $P_0=\{z\in\mathbb{C}_+:-\frac{\pi}{2}\leq\operatorname{Re}z\leq\frac{3\pi}{2}\}$ and $P=P_0\cup\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]$. For all $z\in P$ with $f(z)\neq 0$, we define the Green function of the BVP (2)–(3) by

$$G_{t,z}(z):=\begin{cases}-\frac{\phi(r,z)e(t,z)}{qa(1)f(z)}, & r=tq^{-k},\quad k\in\mathbb{N}_0\\ -\frac{e(r,z)\phi(t,z)}{qa(1)f(z)}, & r=tq^{k},\quad k\in\mathbb{N}.\end{cases} \tag{8}$$

It is obvious that

$$(Rh)(t):=\sum_{r\in q^{\mathbb{N}}}G(t,r)h(r),\qquad h\in\ell_2(q^{\mathbb{N}}) \tag{9}$$

is the resolvent of the BVP (2)–(3), where $\ell_2(q^{\mathbb{N}})$ is the Hilbert space of complex-valued functions with the inner product

$$\langle f,g\rangle_q:=\sum_{t\in q^{\mathbb{N}}}\mu(t)f(t)\overline{g(t)},\quad f,g:q^{\mathbb{N}}\to\mathbb{C}.$$

3 Eigenvalues and Spectral Singularities

We will denote the set of all eigenvalues and spectral singularities of BVP (2)–(3) by σ_d and σ_{ss}, respectively. Using (8), (9), and the definition of the eigenvalues and the spectral singularities [13], we get

$$\sigma_d=\{\lambda\in\mathbb{C}:\lambda=2\sqrt{q}\cos z,\ z\in P_0,\ f(z)=0\}, \tag{10}$$

$$\sigma_{ss}=\left\{\lambda\in\mathbb{C}:\lambda=2\sqrt{q}\cos z,\ z\in\left[-\frac{\pi}{2},\frac{3\pi}{2}\right],\ f(z)=0\right\}\setminus\{0\}. \tag{11}$$

From (5) and (7), we find

$$\begin{aligned}f(z)=&\alpha(1)\sqrt{\frac{q}{q-1}}\beta_1e^{-iz}+\alpha(q)\frac{\gamma_1}{\sqrt{q-1}}+\alpha(1)\frac{\beta_0}{\sqrt{q-1}}\\&+\left(\alpha(q)\frac{\gamma_0}{\sqrt{q(q-1)}}+\alpha(1)\sqrt{\frac{q}{q-1}}\beta_1\right)e^{iz}\\&+\alpha(q)\frac{\gamma_1}{\sqrt{q-1}}e^{2iz}\\&+\sum_{r\in q^{\mathbb{N}}}\alpha(1)\sqrt{\frac{q}{q-1}}\beta_1A(1,r)e^{i\left(\frac{\ln r}{\ln q}-1\right)z}\\&+\sum_{r\in q^{\mathbb{N}}}\left(\alpha(q)\frac{\gamma_1}{\sqrt{q-1}}A(q,r)+\alpha(1)\frac{\beta_0}{\sqrt{q-1}}A(1,r)\right)e^{i\frac{\ln r}{\ln q}z}\\&+\sum_{r\in q^{\mathbb{N}}}\left(\alpha(q)\frac{\gamma_0}{\sqrt{q(q-1)}}A(q,r)+\alpha(1)\sqrt{\frac{q}{q-1}}\beta_1A(1,r)\right)e^{i\left(\frac{\ln r}{\ln q}+1\right)z}\\&+\sum_{r\in q^{\mathbb{N}}}\alpha(q)\frac{\gamma_1}{\sqrt{q-1}}A(q,r)e^{i\left(\frac{\ln r}{\ln q}+2\right)z}.\end{aligned}$$

If we define

$$F(z):=f(z)e^{iz}, \tag{12}$$

then we get

$$\begin{aligned}F(z)=&\alpha(1)\sqrt{\frac{q}{q-1}}\beta_1\\&+\left(\alpha(q)\frac{\gamma_1}{\sqrt{q-1}}+\alpha(1)\frac{\beta_0}{\sqrt{q-1}}\right)e^{iz}\end{aligned}$$

$$+\left(\alpha(q)\frac{\gamma_0}{\sqrt{q(q-1)}}+\alpha(1)\sqrt{\frac{q}{q-1}}\beta_1\right)e^{2iz}$$
$$+\alpha(q)\frac{\gamma_1}{\sqrt{q-1}}e^{3iz}$$
$$+\sum_{r\in q^{\mathbb{N}}}\alpha(1)\sqrt{\frac{q}{q-1}}\beta_1 A(1,r)e^{i\frac{\ln r}{\ln q}z}$$
$$+\sum_{r\in q^{\mathbb{N}}}\left(\alpha(q)\frac{\gamma_1}{\sqrt{q-1}}A(q,r)+\alpha(1)\frac{\beta_0}{\sqrt{q-1}}A(1,r)\right)e^{i\left(\frac{\ln r}{\ln q}+1\right)z}$$
$$+\sum_{r\in q^{\mathbb{N}}}\left(\alpha(q)\frac{\gamma_0}{\sqrt{q(q-1)}}A(q,r)+\alpha(1)\sqrt{\frac{q}{q-1}}\beta_1 A(1,r)\right)e^{i\left(\frac{\ln r}{\ln q}+2\right)z}$$
$$+\sum_{r\in q^{\mathbb{N}}}\alpha(q)\frac{\gamma_1}{\sqrt{q-1}}A(q,r)e^{i\left(\frac{\ln r}{\ln q}+3\right)z}. \quad (13)$$

Since f is analytic in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}}_+$ and $f(z)=f(z+2\pi)$, the function F is also analytic in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}}_+$, and $F(z)=F(z+2\pi)$. It follows from (10)–(12) that

$$\sigma_d=\left\{\lambda\in\mathbb{C}:\lambda=2\sqrt{q}\cos z,\ z\in P_0,\ F(z)=0\right\}, \quad (14)$$

$$\sigma_{ss}=\left\{\lambda\in\mathbb{C}:\lambda=2\sqrt{q}\cos z,\ z\in\left[-\frac{\pi}{2},\frac{3\pi}{2}\right],\ F(z)=0\right\}\setminus\{0\}. \quad (15)$$

Definition 1. *The multiplicity of a zero of F in P is called the multiplicity of the corresponding eigenvalue or spectral singularity of BVP (2)–(3).*

Using (14) and (15), we get that in order to investigate the quantitative properties of the BVP (2)–(3), we need to discuss the quantitative properties of the zeros of F in P. Let us define

$$\begin{aligned}M_1&:=\{z\in P_0:\ F(z)=0\},\\ M_2&:=\left\{z\in\left[-\tfrac{\pi}{2},\tfrac{3\pi}{2}\right]:\ F(z)=0\right\}.\end{aligned} \quad (16)$$

We also denote the set of all limit points of M_1 by M_3 and the set of all zeros of F with infinite multiplicity in P by M_4. From (14)–(16), we get that

$$\begin{aligned}\sigma_d&=\left\{\lambda\in\mathbb{C}:\lambda=2\sqrt{q}\cos z,\ z\in M_1\right\},\\ \sigma_{ss}&=\left\{\lambda\in\mathbb{C}:\lambda=2\sqrt{q}\cos z,\ z\in M_2\right\}\setminus\{0\}.\end{aligned} \quad (17)$$

Theorem 1. *Assume (4). Then*

i) the set M_1 is bounded and countable,
ii) $M_1\cap M_3=\emptyset$, $M_1\cap M_4=\emptyset$,
iii) the set M_2 is compact and the Lebesgue measure of M_2 in the real axis is zero,
iv) $M_3\subset M_2$, $M_4\subset M_2$, the Lebesgue measure of M_3 and M_4 are also zero,
v) $M_3\subset M_4$.

Proof. Using (6) and (13), for all $z\in P_0$, we find

$$F(z)=\frac{\sqrt{q}}{\sqrt{q-1}}\beta_1\alpha(1)+O(e^{-\operatorname{Im}z}),\quad \beta_1\neq 0,\quad \operatorname{Im}z\to\infty,$$

and

$$F(z)=\frac{1}{\sqrt{q-1}}\left[\gamma_1\alpha(q)+\beta_0\alpha(1)\right]e^{iz}+O(e^{-2\operatorname{Im}z}),\quad \beta_1=0,\quad \operatorname{Im}z\to\infty.$$

These equations show the boundedness of the set M_1. Since F is a 2π-periodic function and is analytic in $\mathbb{C}_+$, we get that M_1 has at most a countable number of elements. ii)–iv) can be obtained from the boundary uniqueness theorem of analytic functions [10]. We can easily get v) using the continuity of all derivatives of F on $\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]$.

Now we can give the following theorem as a result of Theorem 1 and (17).

Theorem 2. *Assume (4). Then the set σ_d is bounded, has at most countable number of elements and its limit points can lie only in $[-2\sqrt{q},2\sqrt{q}]$. Also $\sigma_{ss}\subset[-2\sqrt{q},2\sqrt{q}]$ and the Lebesgue measure of the set σ_{ss} in the real axis is zero.*

4 Main Result

Let us suppose that the complex sequences $\{a(t)\}_{t\in q^{\mathbb{N}_0}}$ and $\{b(t)\}_{t\in q^{\mathbb{N}}}$ satisfy

$$\sup_{t\in q^{\mathbb{N}}}\left\{\exp\left(\varepsilon\frac{\ln t}{\ln q}\right)(|1-a(t)|+|b(t)|)\right\}<\infty,\quad \varepsilon>0. \quad (18)$$

It is clear that (4) reduces to (18) for $\delta=1$.

Theorem 3. *Assume (18). Then the BVP (2)–(3) has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.*

Proof. It follows from (6) and (18) that

$$|A(t,r)|\le C\exp\left(-\frac{\varepsilon}{4}\frac{\ln r}{\ln q}\right),\ t\in\{1,q\},\quad r\in q^{\mathbb{N}}. \quad (19)$$

By using (19), we observe that the function F has an analytic continuation to the half-plane $\operatorname{Im}z>-\frac{\varepsilon}{4}$. So, the limit points of its zeros in P cannot lie in $\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]$. From Theorem 1, we get that the bounded sets M_1 and M_2 have no limit points, i.e., the sets M_1 and M_2 have a finite number of elements. Using the analyticity of F in $\operatorname{Im}z>-\frac{\varepsilon}{4}$, we find that all zeros of F in P have finite multiplicity. Consequently, we get the finiteness of the eigenvalues and the spectral singularities of the BVP (2)–(3).

In the following, we will assume that

$$\sup_{t\in q^{\mathbb{N}}}\left\{\exp\left[\varepsilon\left(\frac{\ln t}{\ln q}\right)^{\delta}\right](|1-a(t)|+|b(t)|)\right\}<\infty,\quad \varepsilon>0,\quad \frac{1}{2}\le\delta<1, \tag{20}$$

which is weaker than (18). As is known, the condition (18) guarantees the analytic continuation of F from the real axis to the lower half-plane. So, we get the finiteness of the eigenvalues and the spectral singularities of BVP (2)–(3) as a result of this analytic continuation. It follows from (20) that the function F is analytic in $\mathbb{C}_+$ and infinitely differentiable on the real axis. But F does not have an analytic continuation from the real axis to the lower half-plane. Therefore, under the condition (20), the finiteness of the eigenvalues and the spectral singularities of the BVP (2)–(3) cannot be proved by the same technique used in Theorem 3. We will use the following uniqueness theorem [8, Lemma 4.4] for analytic functions in order to prove the next theorem.

Theorem 4. *Assume that the 2π-periodic function g is analytic in $\mathbb{C}_+$, all of its derivatives are continuous in $\overline{\mathbb{C}}_+$, and*

$$\sup_{z\in P}|g^{(k)}(z)|\le\eta_k,\quad k\in\mathbb{N}_0.$$

If the set $G\subset\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]$ with Lebesgue measure zero is the set of all zeros of the function g with infinity multiplicity in P, and if

$$\int_0^w \ln t(s)\,\mathrm{d}\mu(G_s)=-\infty,$$

where $t(s)=\inf_{k\in\mathbb{N}_0}\frac{\eta_k s^k}{k!}$ and $\mu(G_s)$ is the Lebesgue measure of the s-neighborhood of G, and $w>0$ is an arbitrary constant, then $g\equiv 0$ in $\overline{\mathbb{C}}_+$.

Lemma 1. *Assume (20). Then the inequality*

$$|F^{(k)}(z)|\le\eta_k,\quad z\in P,\quad k\in\mathbb{N}_0 \tag{21}$$

holds, where

$$\eta_k\le C4^k+Dd^k k!k^{k\left(\frac{1}{\delta}-1\right)}, \tag{22}$$

and D and d are positive constants depending on C, ε and δ.

Proof. Using (6) and (20), we obtain

$$|A(t,r)|\le C\exp\left(-\frac{\varepsilon}{4}\left(\frac{\ln r}{\ln q}\right)^{\delta}\right),\quad t\in\{1,q\},\quad r\in q^{\mathbb{N}}. \tag{23}$$

It follows from (13) and (23) that

$$|F^{(k)}(z)|\le C4^k+D_k,\quad z\in P,\quad k\in\mathbb{N}_0,$$

where

$$D_k=C4^k\sum_{r\in q^{\mathbb{N}}}\left(\frac{\ln r}{\ln q}\right)^k e^{-\frac{\varepsilon}{4}\left(\frac{\ln r}{\ln q}\right)^{\delta}},\quad k\in q^{\mathbb{N}_0}.$$

We can also write for D_k

$$\begin{aligned}D_k&=C4^k\sum_{m=1}^{\infty}m^k e^{-\frac{\varepsilon}{4}m^{\delta}}\\&=C4^k\int_0^n t^k e^{-\frac{\varepsilon}{4}t^{\delta}}\mathrm{d}t\le C4^k\int_0^{\infty}t^k e^{-\frac{\varepsilon}{4}t^{\delta}}\mathrm{d}t.\end{aligned}$$

If we define $y=\frac{\varepsilon}{4}t^{\delta}$, then we get

$$D_k\le C4^k\left(\frac{4}{\varepsilon}\right)^{\frac{k+1}{\delta}}\frac{1}{\delta}\int_0^{\infty}y^{\frac{k+1}{\delta}-1}e^{-y}\mathrm{d}y,$$

and using the Gamma function, we obtain

$$D_k\le C4^{2k+1}\left(\frac{4}{\varepsilon}\right)^{\frac{k+1}{\delta}}(k+1)^{\frac{1}{\delta}-1}(k+1)^{\frac{k}{\delta}}. \tag{24}$$

Using (24) and the inequalities $\left(1+\frac{1}{k}\right)^{\frac{k}{\delta}}<e^{\frac{1}{\delta}}$, $(k+1)^{\frac{1}{\delta}-1}<e^{\frac{k}{\delta}}$, and $k^k<k!e^k$, we have

$$D_k\le Dd^k k!k^{k\left(\frac{1}{\delta}-1\right)},\quad k\in\mathbb{N},$$

where D and d are positive constants depending on ε and δ.

Lemma 2. *If (20) holds, then $M_4=\emptyset$.*

Proof. Since the function F is not equal to zero, we can write

$$\int_0^w \ln t(s)\,\mathrm{d}\mu(M_4,s)>-\infty \tag{25}$$

by using Theorem 4, where $t(s)=\inf_{k\in\mathbb{N}_0}\frac{\eta_k s^k}{k!}$, and $\mu(M_4,s)$ is the Lebesgue measure of s-neighborhood of M_4, and η_k is defined by (22). Substituting (22) in the definition of $t(s)$, we find

$$t(s)=D\exp\left\{-\frac{1-\delta}{\delta}e^{-1}(ds)^{\frac{-\delta}{1-\delta}}\right\}. \tag{26}$$

It follows from (25) and (26) that

$$\int_0^w s^{-\frac{\delta}{1-\delta}}\mathrm{d}\mu(M_4,s)<\infty.$$

The last inequality holds for arbitrary s if and only if $\mu(M_4,s)=0$, i.e., $M_4=\emptyset$. This completes the proof.

Theorem 5. *Under the condition (20), the BVP (2)–(3) has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.*

Proof. To be able to prove this, we have to show that the function F has a finite number of zeros with finite multiplicities in P. Using Theorem 1 and Lemma 2, we obtain that $M_3=\emptyset$. So the bounded sets M_1 and M_2 have no limit points, i.e., the function F has only finite number of zeros in P. Since $M_4=\emptyset$, these zeros are of finite multiplicity.

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