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A comparison of Paralindelof Type Properties in Bitopological Spaces

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Abstract: An implication of (i, j)-paralindelof spaces with (i, j)-normal and (i, j)-collectionwise normal space has been studied. We introduce the new definitions of the normal concept in bitopological setting namely (i, j, j)-normal spaces and $(i, j)_j$-normal spaces. When the pairwise paralindelof space implies pairwise Lindelof is investigated. Furthermore, the concepts of para-*m*-Lindelof and *m*-Lindelof have been extended to bitopological setting. Also, we study the relation of (i, j)-para-*m*-Lindelof and (i, j)-*m*-Lindelof.

Keywords: Bitopological space, (i, j)-paralindelof, (i, j)-collectionwise normal, (i, j)-Lindelof, *i*-*m*-locally countable

1 Introduction

The theory of bitopological spaces is associated with the paper of J. C. Kelly in 1969 [4]. In his fundamental paper, the concept of bitopological space (X, τ_1, τ_2) is formulated as a set X with two topological structures τ_1 and τ_2 given on it. Since then several papers devoted to bitopological spaces have been published at the present time. The majority of them contain generalizations of various concepts and assertions of the theory of topological spaces to bitopological spaces in the sense of Kelly. The notion of paralindelofness is a wide open field, with very little known about which implications hold between covering properties, as Lindelof, or separation axioms, as normal and collectionwise normal. During this work, we discuss some properties and characterization of (i, j)-paralindelof [8]. Comparing the concept of (i, j)-paraLindelof with (i, j)-normal and (i, j)-collectionwise normal [8] is studied and presented. We introduce the new definitions of the normal concept in bitopological setting namely (i, j, j)-normal spaces and $(i, j)_j$-normal spaces. For one of the most important covering properties, (i, j)-Lindelof space, we investigate when (i, j)-paralindelof implies (i, j)-Lindelof. In addition, the concepts of para-*m*-Lindelof and *m*-Lindelof have been extended to bitopological spaces. we study the relation of (i, j)-para-*m*-Lindelof and (i, j)-*m*-Lindelof.

2 Preliminaries

Throughout this paper, (X, τ_1, τ_2) (or X) denotes a bitopological space on which no separation axioms are assumed unless otherwise mentioned. For a subset A of X, we shall denote the closure of A and the interior of A with respect to τ_i by $i\text{-}cl(A)$ and $i\text{-}int(A)$ respectively for $i = 1, 2$. A subset A of (X, τ_1, τ_2) is said to be open (RESP. closed) if it is both 1-open and 2-open (resp. 1-closed and 2-closed), or equivalently, $A \in (\tau_1 \cap \tau_2)$ in X. A bitopological space X is said to be (i, j)-normal space [4] if, given a τ_i-closed set A and a τ_j-closed set B with $A \cap B = \emptyset$, there exist a τ_j-open set U and a τ_i-open set V such that $A \subseteq U, B \subseteq V$, and $U \cap V = \emptyset$. X is called pairwise normal if it is both $(1, 2)$-normal and $(2, 1)$-normal.

Definition 1.[8] *A bitopological space (X, τ_1, τ_2) is said to be (i, j)-collectionwise Hausdorff if every i-closed discrete collection of points has expansion of disjoint collection of j-open sets in X, i.e., if $D = \{d_\alpha : \alpha \in \Delta\}$ is i-closed discrete collection of points, then there exists a disjoint collection $\{U_\alpha : \alpha \in \Delta\}$ of j-open sets such that $d_\alpha \in U_\alpha$ for all $\alpha \in \Delta$. X is called pairwise collectionwise Hausdorff (or p-cwH) if it is $(1, 2)$-collectionwise Hausdorff and $(2, 1)$-collectionwise Hausdorff.*

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Definition 2.[8] *A bitopological space (X,τ_1,τ_2) is said to be (i,j)-collectionwise normal if every discrete collection $\{F_s : s\in S\}$ of i-closed subsets of X, then there exists a pairwise disjoint collection $\{U_s : s\in S\}$ of j-open sets such that $F_s\subseteq U_s$ for each $s\in S$. X is called pairwise collectionwise normal (or p-cwn) if it is $(1,2)$-collectionwise normal and $(2,1)$-collectionwise normal.*

*Remark.*It is clear that in $i-T_1$ space, every (i,j)-cwn is (i,j)-CwH. In the following example, we can see the relation between these concepts in a clearer point of view.

*Example 1.*Let $X=\mathbb{R}$. Let τ_1 be a cofinite topology on X and let τ_2 be a discrete topology on X. Then, $(\mathbb{R},\tau_{cof},\tau_{dis})$ is (τ_{cof},τ_{dis})-cwn. Also, Since $\mathbb{R}$ is τ_{cof}-T_1, $\mathbb{R}$ is (τ_{cof},τ_{dis})-CwH (see [6]).

Definition 3.*A bitopological space X is said to be (i,j)-P-space if countable intersection of i-open sets in X is j-open. X is called pairwise P-space if it is $(1,2)$-P-space and $(2,1)$-P-space.*

3 Pairwise Paralindelof Spaces

Definition 4.[8] *A bitopological space (X,τ_1,τ_2) is (i,j)-paralindelof if every i-open cover of X has j-locally countable j-open refinement. X is called pairwise paralindelof (or FHP-pairwise paralindelof) if it is both $(1,2)$-paralindelof and $(2,1)$-paralindelof.*

Proposition 1.*Let a bitopological space (X,τ_1,τ_2) be i-P-space and $\mathscr{U}=\{U_\alpha:\alpha\in\Delta\}$ be i-locally countable family in X. Then, $\{i-cl(U_\alpha):\alpha\in\Delta\}$ is also i-locally countable.*

Proof. Since the collection $\{U_\alpha:\alpha\in\Delta\}$ is i-locally countable, then for each $x\in X$ there is a neighborhood O_x such that $U_\alpha\bigcap O_x=\emptyset$ for all but at most countably many α. So $U_\alpha\bigcap O_x=\emptyset\Rightarrow U_\alpha\subseteq X-O_x\Rightarrow i-cl(U_\alpha)\subseteq X-O_x\Rightarrow i-cl(U_\alpha)\cap O_x=\emptyset$. This completes the proof.

Proposition 2.*If a bitopological space (X,τ_1,τ_2) is $i-P$-spaces, then every i-locally countable family of subsets is i-closure preserving.*

Proof Suppose $\mathscr{U}=\{U_\alpha:\alpha\in\Delta\}$ be i-locally countable family. Set $\mathscr{V}\subset\mathscr{U}$, then $\mathscr{V}$ is also i-locally countable. It is clear that $\bigcup\{i-cl(U):U\in\mathscr{V}\}\subset i-cl(\bigcup\{U:U\in\mathscr{V}\})$. We need to prove the converse. Let $x\in i-cl(\bigcup\{U:U\in\mathscr{V}\})$, due to locally countability of $\mathscr{V}$, there exists i-open neighborhood of x and countable subset Δ' of Δ such that $O_x\bigcap U_\alpha\neq\emptyset$ for $U_\alpha\in\Delta'$. If $x\notin\bigcup\{i-cl(U):U\in\mathscr{V}\}$, then $x\notin i-cl(U_\alpha)$ for every $\alpha\in\Delta'$, and hence there exists i-open neighborhood O_{x_α} of x such that $U_\alpha\subset X-O_{x_\alpha}$. Since Δ' is countable and X is i-P-space, $V=O_x\bigcap(\bigcap_{\alpha\in\Delta'}O_{x_\alpha})$ is also i-open neighborhood of x and $U_\alpha\subset X-V$ for every $\alpha\in\Delta'$. So $\bigcup_{\alpha\in\Delta'}U_\alpha\subset X-V$, and hence

$$x\in i-cl(\bigcup\{U:U\in\mathscr{V}\})\subseteq i-cl(X-V)=X-V.$$

That is to say, $x\notin V$, this contradicts with being V i-open neighborhood of x. Therefore $x\in\bigcup\{i-cl(U):U\in\mathscr{V}\}$, so $i-cl(\bigcup\{U:U\in\mathscr{V}\})=\bigcup\{i-cl(U):U\in\mathscr{V}\}$.

Theorem 1.*Let a bitopological space X be (i,j)-regular and $j-P$-space. If X is (i,j)-paralindelof space, then every i-open covering of X has j-closure-preserving j-closed refinement cover.*

Proof. Let $\mathscr{U}$ be any i-open cover of X. Let $x\in X$. Then, there exists some $U\in\mathscr{U}$ containing x. Since X is (i,j)-regular, there is i-open set V_x such that $x\in V_x\subseteq j-cl(V_x)\subseteq U_\alpha$. Now, $\mathscr{V}=\{V_x:x\in X\}$ is i-open cover of X, therefore it has j-locally countable family $\mathscr{W}=\{W_\lambda:\lambda\in\Lambda\}$ of j-open sets in X refining $\mathscr{V}$ and covering X. Let $\mathscr{H}=\{j-cl(W):\alpha\in\Delta\}$. Since X is j-P-space, $\mathscr{H}$ is also j-locally countable by Proposition 1. Since $\mathscr{W}$ is refinement of $\mathscr{V}$, then for each $W_\alpha\in\mathscr{W}$ there is a $V_{x(\alpha)}\in\mathscr{V}$ such that $W_\alpha\subseteq j-cl(W_\alpha)\subseteq j-cl(V_{x(\alpha)})\subseteq U$ for some $U\in\mathscr{U}$. Therefore, $\mathscr{H}$ is refinement of $\mathscr{U}$ and covers X. Again, since X $j-P$-space, then, by Proposition 2, $\mathscr{H}$ j-closure-preserving j-closed refinement cover.

Proposition 3.*Every closed subspace of an (i,j)-paralindelof space is (i,j)-paralindelof.*

Proof. Let X be an (i,j)-paralindelof space and let $(A,\tau_1|A,\tau_2|A)$ be a closed subspace of X. So, if $\{U_\alpha:\alpha\in\Delta\}$ is $\tau_i|A$-open cover of A, then, for each α, we can find an τ_i-open set V_α in X with $V_\alpha\bigcap A=U_\alpha$. Hence, the collection $\mathscr{V}=\{V_\alpha:\alpha\in\Delta\}\bigcup\{X-A\}$ is τ_i-open cover of X so that it has j-locally countable family $\mathscr{W}=\{W_\beta:\beta\in B\}\bigcup\{X-A\}$ of j-open subsets of X refining $\mathscr{V}$ and covering X. But A and $X-A$ are disjoint; so the collection $\{W_\beta:\beta\in B\}$ covers A. So, the family $\{W_\beta\bigcap A:W_\beta\in B\}$ is also $\tau_j|A$-locally countable $\tau_j|A$-open refinement of $\mathscr{V}$. Then, A is (i,j)-paralindelof.

Theorem 2.*In pairwise Hausdorff and (j,i)-P-space X, every (i,j)-paralindelof space is (i,j)-normal.*

Proof. Let A and B be i-closed set and j-closed set respectively. Set $x\in A$ and $y\in B$. By Hausdorffness of X, there are i-open set U_x and j-open set V_x such that $x\in U_x, y\in V_x$ and $U_x\bigcap V_x=\emptyset$. The family $\mathscr{U}=\{U_x:x\in A\}\bigcup\{X-A\}$ forms i-open cover of X. By hypothesis, $\mathscr{U}$ has j-open refinement $\mathscr{W}=\{W_\alpha:\alpha\in\Delta\}$ which is j-locally countable and covers X. Set $U=\bigcup\{W_\alpha:W_\alpha\bigcap A\neq\emptyset\}$. Then U is j-open set containing A. For each $y\in B$, we get j-open neighborhood H_y of y which meets at most countably members of $\mathscr{W}$, say $W_{\alpha_1}(y),W_{\alpha_2}(y),W_{\alpha_3}(y),...,W_{\alpha_n}(y),...$,for $n=1,2,....$ where $W_{\alpha_i}\bigcap A=\emptyset$, since if some n,W_{α_n} meets A i.e. $W_{\alpha_n}\bigcap A=\emptyset$, then $W_{\alpha_n}\subseteq X-A$ which is impossible. Set $G_y=H_y\bigcap(\bigcap_{n\in\mathbb{N}}V_{x_n})=\bigcap_{n\in\mathbb{N}}(H_y\bigcap V_{x_n})$. Then G_y is i-open set containing y because X is (j,i)-P-space. Let $V=\bigcup_{y\in B}G_y$. Therefore, V is i-open set containing B and $U\bigcap V=\emptyset$. Thus, X is (i,j)-normal.

Definition 5. *A bitopological space (X,τ_1,τ_2) is said to be (i,j,j)-normal if for every two disjoint i-closed sets A and B there exists two disjoint j-open sets U and V such that $A \subseteq U, B \subseteq V$. X is called p_1-normal if it is $(1,2,2)$-normal and $(2,1,1)$-normal.*

Remark. According to **Definition 2**, every (i,j)-collectionwise normal is (i,j,j)-normal. The next theorem shows when the converse holds.

Theorem 3. *If a bitopological space X is (i,j)-paralindelof, (i,j,j)-normal and P-space, then X is (i,j)-collectionwise normal.*

Proof. Let X be (i,j)-paralindelof space. Let $\mathscr{A} = \{A_\alpha : \alpha \in \Delta\}$ be any discrete collection of i-closed subsets of X such that if $\alpha, \beta \in \Delta$ with $\alpha \neq \beta$ then $A_\alpha \neq A_\beta$. For each $x \in X$, fix O_x be i-open such that $x \in O_x$ and O_x meets at most one element of $\mathscr{A}$. Let $\mathscr{O} = \{O_x : x \in X\}$ be formed as i-open cover of X. By the paralindeloffness of X, $\mathscr{O}$ has j-locally countable j-open refinement $\mathscr{V}$. For each $\alpha \in \Delta$, set $V_\alpha = \bigcup\{V \in \mathscr{V} : V \bigcap A_\alpha \neq \emptyset\}$. Then $\{V_\alpha : \alpha \in \Delta\}$ is j-locally countable collection of j-open subsets of X such that for all $\alpha \in \Delta$, $A_\alpha \subseteq V_\alpha \subseteq X - \bigcup(\mathscr{A} - A_\alpha)$ and because X is i-P-space, the set $\bigcup(\mathscr{A} - A_\alpha)$ is i-closed. Since X is (i,j,j)-normal, there are two disjoint j-open subsets G_α and W_α such that $A_\alpha \subseteq G_\alpha$ and $\bigcup(\mathscr{A} - A_\alpha) \subseteq W_\alpha$ for all $\alpha \in \Delta$.

For all $\alpha \in \Delta$, let $H_\alpha = G_\alpha \bigcap V_\alpha$ and $U_\alpha = H_\alpha - j - cl(\bigcup\{H_\beta : \beta \in \Delta - \alpha\})$. The collection $\mathscr{U} = \{U_\alpha : \alpha \in \Delta\}$ is a pairwise disjoint collection of j-open subsets of X. For all $\alpha \in \Delta$, the collection $\mathscr{U}$ is j-locally countable and $U_\alpha \subseteq X - \bigcup(\mathscr{A} - A_\alpha)$.

Now, we shall show that $A_\alpha \subseteq U_\alpha$. Note that for each $\alpha \in \Delta$, $A_\alpha \subseteq H_\alpha$. Since $\{H_\alpha : \alpha \in \Delta\}$ is j-locally countable collection in $j - P$-space, we have $j - cl(\bigcup\{H_\alpha : \alpha \in \Delta\}) = \bigcup\{j - cl(H_\alpha) : \alpha \in \Delta\}$. Suppose that $\alpha \in \Delta, x \in A_\alpha$ and $\lambda \in \Delta - \alpha$. So, $x \in \bigcup(\mathscr{A} - A_\alpha) \subseteq W_\lambda$. From $H_\lambda \subseteq G_\lambda$ and $G_\lambda \bigcap W_\lambda = \emptyset$, we have $x \notin j - cl(H_\lambda)$. Therefore, $A_\alpha \bigcap j - cl(\bigcup\{H_\beta : \beta \in \Delta - \alpha\}) = \emptyset$ and so $A_\alpha \subseteq U_\alpha$. Therefore, X is (i,j)-collectionwise normal.

According to Theorem 3, we introduce the next theorem.

Theorem 4. *If a bitopological space X is (i,j)-paralindelof, (i,j,j)-normal and P-space, then X is (i,j)-collectionwise Hausdorff.*

Definition 6. *A bitopological space X is $(i,j)_j$-normal if, given i-closed set A and j-closed set B with $A \bigcap B = \emptyset$, there exist two j-open sets U and V such that $A \subseteq U$ and $B \subseteq V$ and $U \bigcap V = \emptyset$. X is called p_p-normal if it is both $(1,2)_2$-normal and $(2,1)_1$-normal.*

Theorem 5. *A space (X,τ_1,τ_2) is $(i,j)_j$-normal if and only if, given i-closed set A and j-open set U such that $A \subseteq U$, there is j-open set V such that $A \subseteq V \subseteq j - cl(V) \subseteq U$.*

Proof. Suppose that X is $(i,j)_j$-normal. Let A be an i-closed set and U be j-open set such that $A \subseteq U$. Then, $B = X - U$ is j-closed set with $B \bigcap A = \emptyset$. By hypothesis, there exist two j-open sets V and P such that $A \subseteq V, B \subseteq P$ and $V \bigcap P = \emptyset \Longrightarrow j - cl(V) \subseteq j - cl(X - P) = X - P \subseteq X - B$. Therefore, $A \subseteq V \subseteq j - cl(V) \subseteq X - B = U$. Conversely, suppose that the condition holds. Let A be i-closed set and B j-closed set with $A \bigcap B = \emptyset$ so that $U = X - B$ is j-open set with $A \subseteq U$. By the condition, there exists j-open set V such that $A \subseteq V \subseteq j - cl(V) \subseteq U$. It leads to that $A \subseteq V, B = X - U = X - j - cl(V)$ and $V \bigcap X - j - cl(V) = \emptyset$. Then, X is $(i,j)_j$-normal.

Theorem 6. *Let (X,τ_1,τ_2) be (i,j)-regular, $(i,j)_j$-normal and j-P-space. If X is (i,j)-pairwise paralindelof, then X is (i,j)-collectionwise normal.*

Proof. Take any discrete family $\mathscr{A} = \{A_\alpha : \alpha \in \Delta\}$ of i-closed sets in X. For every $x \in X$, pick O_x as i-open neighborhood such that $x \in O_x$ and O_x meets at most one element of $\mathscr{A}$. Now, the family $\mathscr{O} = \{O_x : x \in X\}$ forms i-open cover of X. Since X is (i,j)-regular and (i,j)-paralindelof space, by **Theorem 1**, $\mathscr{O}$ has j-closure preserving j-closed refinement $\mathscr{V}$ such that any $V \in \mathscr{V}$ intersects at most one member of $\mathscr{A}$. Let $U_\alpha = X - (\bigcup\{V \in \mathscr{V} : V \bigcap A_\alpha = \emptyset\})$, hence for every $\alpha \in \Delta$, U_α is j-open set such that $A_\alpha \subseteq U_\alpha$.

Now, the collection $\mathscr{U} = \{U_\alpha : \alpha \in \Delta\}$ is disjoint. If $\alpha \neq \beta$ and $x \in U_\alpha \bigcap U_\beta$, then pick any $V \in \mathscr{V}$ with $x \in V$ and using the definition of U_α, U_β it follows that $V \bigcap A_\alpha \neq \emptyset \neq V \bigcap A_\beta$ which leads to contradiction with the definition of $\mathscr{A}$. Since X is $(i,j)_j$-normal, then if $A = \bigcup\{A_\alpha : \alpha \in \Delta\}$, using Theorem 5, we can select j-open set W such that $A \subseteq W \subseteq j - cl(W) \subseteq \bigcup\{U_\alpha; \alpha \in \Delta\} = U$.

For every $\alpha \in \Delta$, if $U_\alpha \bigcap W = W_\alpha \in \tau_j$. Now, $\mathscr{W} = \{W_\alpha : \alpha \in \Delta\}$ is a family of disjoint j-open subsets of X such that $A_\alpha \subseteq W_\alpha$ for each $\alpha \in \Delta$. It suffices to prove that $\mathscr{W}$ is discrete family. Let $x \in X$. Assume that $x \notin U$. Thus, X-j-$cl(W)$ is j-open neighborhood of x that does not meet any element of $\mathscr{W}$. But if $x \in \mathscr{U}$, $x \in U_\alpha$ for some $\alpha \in \Delta$. Hence, U_α is j-open neighborhood of x which intersects at most one element of $\mathscr{W}$ which means that $\mathscr{W}$ is discrete. Therefore, X is (i,j)-collectionwise normal.

According to Theorem 6, we can state the following theorem.

Theorem 7. *Let (X,τ_1,τ_2) be (i,j)-regular, pairwise Hausdorff and j-P-space. If X is (i,j)-pairwise paralindelof, then X is (i,j)-collectionwise Hausdorff.*

Proposition 4. *Every i-locally countable family of non-empty subsets of i-Lindelöf space X is countable.*

Proof. Let $\mathscr{U}$ be i-locally countable family of non-empty subsets of i-Lindelöf space X. For each $x \in X$, choose i-open neighborhood V_x of x meeting only countably many members of $\mathscr{U}$. By Lindelofness of X, i-open cover $\mathscr{V} = \{V_x : x \in X\}$ of X has a countable subcover $\mathscr{V}'$. Since each $U \in \mathscr{U}$ meets a $V \in \mathscr{V}'$. Therefore $|\mathscr{U}| \leq \aleph_0$, in words, the collection $\mathscr{U}$ is countable.

Theorem 8. *Let a bitopological space (X, τ_1, τ_2) be (i, j)-regular and $j-P$-space. If X is (i, j)-paralindelof and contains j-dense subspace Y which is j-Lindelof, then it is i-Lindelof.*

Proof. Let Y be j-Lindelof and j-dense subspace and let $\mathscr{U} = \{U_\alpha : \alpha \in \Delta\}$ be i-open cover of X. By regularity of X, there is i-open cover $\mathscr{V} = \{V_x : x \in X\}$ of X such that $\{j-cl(V_x) : x \in X\}$ refines $\mathscr{U}$. Let $\mathscr{W} = \{W_\beta : \beta \in B\}$ be a j-locally countably family of j-open sets which refines $\mathscr{V}$ and covers X. By **Proposition 4**, the set $B_0 = \{\beta \in B : Y \bigcap W_\beta \neq \emptyset\}$ is countable. Since X is j-P-space and $Y = \bigcup_{\beta\in B_0}(Y \bigcap W_\beta)$, we have

$$X = j-cl(Y) = j-cl(\textstyle\bigcup_{\beta\in B_0}(Y \bigcap W_\beta)) = \bigcup_{\beta\in B_0}(j-cl(Y \bigcap W_\beta)) \subseteq \bigcup_{\beta\in B_0}(j-cl(W_\beta)) \subseteq \bigcup_{\beta\in B_0}(j-cl(V_{x(\beta)})) \subseteq \bigcup_{\beta\in B_0}(U_{\alpha(\beta)}),$$

thus $\mathscr{U}$ has a countable subcover. Therefore, X is i-Lindelof space.

Definition 7. *[1] A bitopological space (X, τ_1, τ_2) is said to be (i, j)-Lindelof if for every i-open cover of X can be reduce to a countable j-open cover. X is called pairwise Lindelof (or B-Lindelof) if it is both $(1,2)$-Lindelof and $(2,1)$-Lindelof.*

Remark. Clearly, every (i, j)-Lindelof space is (i, j)-paralindelof, but the converse is not true in general as the following example explains.

Example 2. Let X be an uncountable set. For a fixed point $p \in X$, the family $\{U \subseteq X : p \in U\} \bigcup \{\emptyset\}$ of subsets of X is open sets of the first topology defined on X called particular point topology denoted τ_{pp}. Consider the discrete topology on X as τ_2. Then the bitopological space $(X, \tau_{pp}, \tau_{dis})$ is (τ_{pp}, τ_{dis})-paralindelof since the τ_{pp}-open sets is also τ_{dis}-open sets in X (see [6]), but it is not (τ_{pp}, τ_{dis})-Lindelof since $\{\{x, p\} : x \in X\}$ is τ_{pp}-open cover of X has no countable τ_{dis}-open subcover.

We extend the property of discrete countable chain condition in bitopological setting as the following definition shown.

Definition 8. *A bitopological space X is said to be satisfied i-discrete countable chain condition for short i-DCCC if the topological space (X, τ_i) satisfies discrete countable chain condition. X satisfies discrete countable chain condition if it is i-discrete countable chain condition for $i = 1, 2$.*

Proposition 5. *If a bitopological space X is i-regular space, $i-P$-space, and satisfies the i-DCCC, then every i-locally countable collection of i-open sets of X is countable.*

Proof. Let $\mathscr{U} = \{U_\alpha : \alpha \in \Delta\}$ be i-locally countable collection of i-open sets of X. Suppose that a bitopological space X is well order. For each $x \in X$. set $\mathscr{U}_x = \{U : U \in \mathscr{U}$ and x is the first element in $U\}$. Since $\mathscr{U}$ is i-locally countable, $\mathscr{U}_x$ is a countable set that may be empty. So, every member in $\mathscr{U}$ is contained in one and only one $\mathscr{U}_x$ and

$$y \leq x \text{ implies } y \notin U \text{ for } U \in \mathscr{U}_x.$$

Now, we shall show that the collection $\{\mathscr{U}_x : \mathscr{U}_x \neq \emptyset, x \in X\}$ is countable.

Let U_x be a set from each nonempty $\mathscr{U}_x$. Thus the collection $\{U_x : x \in X\}$ is i-locally countable since $\mathscr{U}$ is. By regularity of X, there is i-open neighborhood V_x of x such that $x \in V_x \subseteq i-cl(V_x) \subseteq U_x$. The collection $\{V_x\}$ is i-locally countable. For each $\mathscr{U}_x \neq \emptyset$, let construct a nonempty i-open set as following:

$W_x = V_x - \bigcup\{i-cl(V_x) | x, y \in X$, and $x \leq y\}$.

The collection $\{W_x\}$ consists of disjoint nonempty i-open sets since it contains x. Since the collection $\{V_x\}$ is i-locally countable, the collection $\{W_x\}$ is i-locally countable.

Again, since X is i-regular space, we have $x \in H_x \subseteq i-cl(H_x) \subseteq W_x$. The collection $\{H_x\}$ is i-locally countable and the i-closure of its elements are disjoint. Hence $\{H_x\}$ is a discrete collection in X satisfied i-DCCC, thus it is countable.

Theorem 9. *Let (X, τ_1, τ_2) is j-regular space, $j-P$-space, and satisfies the j-DCCC. Then, every (i, j)-paralindelof space if and only if it is (i, j)-Lindelof.*

Proof. Let X is (i, j)-paralindelof space and let $\mathscr{U}$ be i-open cover of X. Then, there exists j-locally countable family $\mathscr{V}$ of j-open sets in X which covers X and refines $\mathscr{U}$. Using **Proposition 5**, $\mathscr{V}$ is countable. Therefore X is (i, j)-Lindelof. Conversely, it is clear since every (i, j)-Lindelof space is (i, j)-paralindelof.

Definition 9. *[2] A bitopological space X is said to be (i, j)-almost Lindelof if for every i-open cover $\{U_\alpha : \alpha \in \Delta\}$ of X, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ such that $X = \bigcup_{n\in\mathbb{N}} j-cl(U_{\alpha_n})$. X is called pairwise almost Lindelof if it is both $(1,2)$-almost Lindelof and $(2,1)$-almost Lindelof.*

Theorem 10. *Let (X, τ_1, τ_2) be (i, j)-almost Lindelof space. X is (i, j)-Lindelöf if and only if it is (i, j)-paralindelöf.*

Proof. Let $\mathscr{U} = \{U_\alpha : \alpha \in \Delta\}$ be i-open cover of X. By paraLindelöfness of X, the collection $\mathscr{U}$ admits j-locally countable family $\mathscr{W} = \{W_\beta : \beta \in B\}$ of j-open sets which refines $\mathscr{U}$. For each $x \in X$ we fix a j-neighborhood V_x which intersects at most countably many sets belonging to $\mathscr{W}$. The j-open cover $\{V_x : x \in X\}$ contains a countable family $\{V_{x_n} : n \in \mathbb{N}\}$ such that $X = \bigcup_{n\in\mathbb{N}} i - cl(V_{x_n})$. Since each set from $\mathscr{W}$ intersects some V_{x_n}, it follows that the family $\mathscr{W}$ is countable (i.e., $|\mathscr{W}| \leq \aleph_0$). Therefore X is (i, j)-Lindelöf space. Conversely, it is clear since every (i, j)-Lindelof space is (i, j)-paralindelof.

*Example 3.*Let X be a set whose cardinality is 2^c, where c=card($\mathbb{R}$). Let $\tau_1 = \{\emptyset\}\bigcup\{U \subseteq X : |X-U| \leq c\}$ and τ_2 be a cofinite topology on X. Then, (X, τ_1, τ_2) is a bitopological space. Since X is not τ_1-Lindelöf, then X is not $(1,2)$-Lindelöf. But X is $(2,1)$-almost Lindelöf. Therefore, by applying Theorem 10, X is not $(1,2)$-paralindelof.

4 Pairwise Para-*m*-Lindelof Bitopological Space

Pareek and Kaul have introduced and studied the concepts of *m*-Lindelof and para-*m*-Lindelof in generality, where m denotes an infinite cardinal. Now, we shall extend and introduce the notions of *m*-Lindelofness and para-*m*-Lindelofness in bitopological space in sense of Pareek and Kaul's definitions.

Definition 10.*A bitopological space (X, τ_1, τ_2) is called (i, j)-m-Lindelof if each i-open cover of X has a j-open subcover of cardinality at most m. X is said to be B-m-Lindelof if it is $(1,2)$-m-Lindelof and $(2,1)$-m-Lindelof.*

*Remark.*If $m = \aleph_0$, then (i, j)-$\aleph_o$-Lindelof is simply (i, j)-Lindelof space. Then, every (i, j)-Lindelof space is (i, j)-*m*-Lindelof. But the converse is not always true.

*Example 4.*Let X be a set with cardinality $m = 2^c$ where $c = card(\mathbb{R})$. It is clear that the bitopological space $(X, \tau_{pp}, \tau_{dis})$ is not (τ_{pp}, τ_{dis})-Lindelof since τ_{pp}-open cover $\{\{x, p\} : x \in X\}$ gas no countable τ_{dis}-open subcover. But it is (τ_{pp}, τ_{dis})-*m*-Lindelof since any τ_{pp}-open cover of X has a τ_{dis}-open subcover of cardinality at most m.

Definition 11.*Let (X, τ_1, τ_2) be a bitopological space and let A be a subset of X. Then a point x of X is called p_1-complete accumulation point of A if $|A\bigcap U| = |A|$ for each τ_1-open (or τ_2-open) neighborhood U of x, in words, the set A and $A\bigcap U$ have the same cardinality.*

From Definition 11, we shall redefine (i, j)-*m*-Lindelof space in sense of a complete accumulation point concept.

Definition 12.*A bitopological space (X, τ_1, τ_2) is called (i, j)-m-Lindelof in the sense of complete accumulation point if each subset of X of cardinality m has complete accumulation point.*

Proposition 6.*In (i, j)-m-Lindelof (or (j,i)-m-Lindelof) space X, every subset A of cardinality m has p_1-complete accumulation point.*

Proof. Let A be a subset of X of cardinality m. Suppose A does not have p_1-complete accumulation point in X. Then for each $x \in X$ there exists i-open (or j-open) neighborhood U_x such that $|U_x\bigcap A| < |A|$, i.e., $|U_x \cap A| \leq m$. Consider the i-open (or j-open) cover $\mathscr{U} = \{U_x : x \in X\}$ of X. Since X is (i, j)-*m*-Lindelof (or (j,i)-*m*-Lindelof) there is j-open (or i-open) subcover $\mathscr{V}$ of X such that $|\mathscr{V}| \leq m$. But, then $|A| \leq |\mathscr{V}| \leq m$, a contradiction from the fact that $|A| \succ m$. This completes our proof.

Definition 13.*A collection $\mathscr{U}$ of subsets of a bitopological space (X, τ_1, τ_2) is called i-locally-m if each point $x \in X$ has i-neighborhood intersecting at most m member of $\mathscr{U}$.*

*Remark.*If $m = \aleph_o$, then i-locally-m collection will be called i-locally countable.

Theorem 11.*Let a bitopological space X be i-Lindelof. Then, each a i-locally m collection of non-empty i-open subsets is of cardinality at most m.*

Proof. Let $\mathscr{U}$ be i-locally m collection of non-empty i-open subsets of X and let A be the image set of a choice function on $\mathscr{U}$. Assume that $|\mathscr{U}| > m$ which leads to contradiction. By hypothesis, for any $x \in A$ there is i-open neighbourhood U_x of x such that $|\{\alpha \in \Delta : U_x\bigcap U_\alpha \neq \emptyset\}| \leq m$.

Again, for any $\alpha \in \Delta$, there exists a point x in A such that $U_x\bigcap U_\alpha \neq \emptyset$. Thus, $|\{\alpha \in \Delta : U_x\bigcap U_\alpha \neq \emptyset : x \in A\}| = \Delta$ and we obtain $|A| \leq m$ which implies that $|\Delta| \leq m$. But by assuming that $|\mathscr{U}| > m$ we have $|A| > m$ which leads to a contradiction with that $\mathscr{U}$ must be i-locally m. Therefore $|\mathscr{U}| \leq m$.

Proposition 7.*Let (X, τ_1, τ_2) be a bitopological space. If $\mathscr{U}$ is i-locally-m i-open cover of X and Y is i-dense i-m-Lindelof subspace, then $\mathscr{U}$ has a subcover of cardinality at most m.*

Proof. Set $\mathscr{V} = \{U\bigcap Y : U \in \mathscr{U}\}$ so that $\mathscr{V}$ is also i-locally m collection. Thus $|\mathscr{V}| \leq m$ by **Theorem 11** and since Y is i-dense in X, $U\bigcap Y \neq \emptyset$ for all $U \in \mathscr{U}$, therefore, $|\mathscr{U}| \leq m$. The proof completes.

Definition 14.*A bitopological space (X, τ_1, τ_2) is called (i, j)-para-m-Lindelof if each i-open cover of X has a j-locally-m j-open refinement. X is called pairwise para-m-Lindelof if it is both $(1,2)$-para-m-Lindelof and $(2,1)$-para-m-Lindelof.*

Theorem 12.*Let Y be j-dense j-m-Lindelof subspace of the bitopological space (X, τ_1, τ_2). Then X is (i, j)-para-m-Lindelof if and only if it is (i, j)-m-Lindelof.*

Proof. Let X be (i,j)-para-m-Lindelof and $\mathscr{U}=\{U_\alpha:\alpha\in\Delta\}$ be i-open cover of X. Hence $\mathscr{U}$ admits j-locally m family $\mathscr{V}=\{V_\lambda:\lambda\in\Lambda\}$ of j-open sets in X covering X and refining $\mathscr{U}$. Since Y is j-dense, $\mathscr{W}=\{Y\bigcap V_\lambda\neq\emptyset:$ for all $\lambda\in\Lambda\}$ is j-locally m family , because $\mathscr{W}$ is a subfamily of j-locally m family $\mathscr{V}$, of non-empty j-open sets in j-Lindelof subspace Y. Using **Proposition 7**, $|\mathscr{W}|\leq m$, in words, $\mathscr{W}$ has a cardinality at most m. Therefore, X is (i,j)-m-Lindelof. Conversely, it is clear since every (i,j)-m-Lindelof is (i,j)-para-m-Lindelof.

Theorem 13. *If a bitopological space (X,τ_1,τ_2) is j-m-Lindelof and (Y,σ_1,σ_2) is j-m-separable, then if $(X\times Y,\rho_1,\rho_2)$ is (i,j)-para-m-Lindelof if and only if $(X\times Y,\rho_1,\rho_2)$ is (i,j)-m-Lindelof where $\rho_i=\tau_i\times\sigma_i$.*

Proof. Let D be j-dense subset of Y such that $|D|\leq m$. Set $X_D=X\times D$ which is j-homeomorphic to X and hence X_D is j-dense and j-m-Lindelof subspace of $X\times Y$. Then, by **Theorem 12**, $X\times Y$ is (i,j)-m-Lindelof. Conversely, it is clear since every (i,j)-m-Lindelof is (i,j)-para-m-Lindelof.

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