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Inequalities for Power Series

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Abstract: The aim of this paper is to give several inequalities for power series starting from a generalization of Young's inequality for sequences of complex numbers. Then some inequalities deduced from some variants of the arithmetic-geometric mean inequality will be given. Thus by Theorem 1, Theorem 2 and Theorem 3 several refinements of Young's inequality for functions defined by power series with real coefficients are given and by Theorem 4 a generalization of a sharp Hölder's inequality for functions defined by power series with real coefficients is presented. Then a generalization of Young's inequality for *m* pair of complex numbers in the case of the functions defined by power series is given in Remark 1, and a variant of Muirhead's inequality for functions defined by power series with real coefficients is given in Proposition 3. There are a lot of examples related to some fundamental complex functions such as the exponential, logarithm, trigonometric and hyperbolic functions and also there are applications for some special functions such as polylogarithm, hypergeometric and Bessel functions for the first kind. Finally, we present an application related to the average information.

Keywords: Power series, Young's inequality, Muirhead's inequality, arithmetic-geometric mean inequality

1 Introduction

Power series is a special type of series of a function. The applications to power series can be found in mathematics, computer science, physics and in information theory. We will study the power series related to inequalities. Using a refinement of the Cauchy-Bunyakovsky-Schwarz inequality, Cerone and Dragomir in [12], established some inequalities concerning functions defined by convergent power series with real or nonnegative coefficients. The technique to find other inequalities of functions using power series was given by Ibrahim and Dragomir in [3], Mortici in [11] and Ibrahim, Dragomir and Darus in [4]. This method is important because can be improved and extended some of the known inequalities, which have applications in many fields.

We consider an analytic function defined by the power series

$$f(z)=\sum_{n=0}^{\infty}a_nz^n$$

with real coefficients and convergent on the disk $D(0,R)$, $R>0$. As in [4] the weighted version of Hölder's inequality can be stated as below:

$$|f(xy)|=\left|\sum_{n=0}^{\infty}a_nx^ny^n\right|\leq\left(\sum_{n=0}^{\infty}|a_n||x|^{pn}\right)^{\frac{1}{p}}\left(\sum_{n=0}^{\infty}|a_n||x|^{qn}\right)^{\frac{1}{q}}$$

$$=f_A^{\frac{1}{p}}(|x|^p)\,f_A^{\frac{1}{q}}(|y|^q)$$

for any $x,y\in\mathbf{C}$ with xy, $|x|^p,|y|^q\in D(0,R)$ and $f_A(z)$ is a power series defined by $\sum_{n=0}^{\infty}|a_n|z^n$. The power series $f_A(z)$ have the same radius of convergence as the original power series $f(z)$.

In the case when all coefficients of the series $f(z)$ are positive we have $f(z)=f_A(z)$.

Next, we present several results related to inequalities, that will be useful in our study.

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We consider the following inequality, which represents an improvement of Young's inequality:

Lemma 1. ([8]) For $0 < a, b \leq 1$ and $\lambda \in (0,1)$ we have

$$r(\sqrt{a}-\sqrt{b})^2 + A(\lambda)ab\log^2\left(\frac{a}{b}\right) \leq \lambda a + (1-\lambda)b - a^{\lambda}b^{1-\lambda}$$

$$\leq (1-r)(\sqrt{a}-\sqrt{b})^2 + B(\lambda)ab\log^2\left(\frac{a}{b}\right)$$

where $r = \min\{\lambda, 1-\lambda\}$, $A(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{r}{4}$ and $B(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{1-r}{4}$.

If we take here $\lambda = \frac{1}{p}$ and replace a^{λ} by a and $b^{1-\lambda}$ by b then $1-\lambda = \frac{1}{q}$ and we obtain:

$$ab + r(a^{\frac{p}{2}} - b^{\frac{q}{2}})^2 + A(\frac{1}{p})a^p b^q \log^2\left(\frac{a^p}{b^q}\right) \leq \frac{a^p}{p} + \frac{b^q}{q}$$

$$\leq ab + (1-r)(a^{\frac{p}{2}} - b^{\frac{q}{2}})^2 + B(\frac{1}{p})a^p b^q \log^2\left(\frac{a^p}{b^q}\right). \quad (1)$$

We also need the inequality from below which is given in [5], Lemma 2.

Lemma 2. For $a_{ij} \geq 0$, $p_j > 0$, $i \in \{1,2,...,n\}$ and $j \in \{1,2,...,m\}$ such that $\frac{1}{p_1} + \frac{1}{p_2} + .. + \frac{1}{p_m} \geq 1$ we have

$$\sum_{i=1}^{n} a_{i1}a_{i2}...a_{im} \leq \left(\sum_{i=1}^{n} a_{i1}^{p_1}\right)^{\frac{1}{p_1}} \left(\sum_{i=1}^{n} a_{i2}^{p_2}\right)^{\frac{1}{p_2}} ... \left(\sum_{i=1}^{n} a_{im}^{p_m}\right)^{\frac{1}{p_m}}.$$

Next inequality is given in [2], Proposition 5.1 and will be used in Theorem 4.

Proposition 1. ([2]) Let $a_1, ..., a_n \geq 0$ and $p_1, ..., p_n \geq 0$ with $\sum_{j=1}^{n} p_j = 1$ we have

$$\sum_{i=1}^{n} p_i a_i - a_1^{p_1}...a_n^{p_n} \geq n\lambda\left(\frac{1}{n}\sum_{i=1}^{n} a_i - a_1^{\frac{1}{n}}...a_n^{\frac{1}{n}}\right),$$

with equality if and only if $a_1 = ... = a_n$, where $\lambda = \min\{p_1, ..., p_n\}$.

Using the above results in this paper we give by Theorem 1, Theorem 2 and Theorem 3 several refinements of Young's inequality presented in [4] for functions defined by power series with real coefficients and by Theorem 4 a generalization of a sharp Hölder's inequality for functions defined by power series with real coefficients is presented. Then motivated by some results from [6,7], a generalization of Young's inequality for m pair of complex numbers in the case of the functions defined by power series is given in Remark 1, and a variant of Muirhead's inequality for functions defined by power series with real coefficients is given in Proposition 3.

These results are important due to their applications for special functions such as polylogarithm, hypergeometric and Bessel and modified Bessel functions for the first kind. Moreover, in information sciences, many applications of Hölder's inequality have also been studied by many authors as [22]. In section 3 an application related to the average information is presented.

2 Main results

The following three results were obtained using a refinement of Young's inequality given in [8] for two positive numbers a and b in $(0,1)$ for power series with real coefficients, and the same method as in [4], Theorem 1, 2 and 3.

Theorem 1. Let $f(z) = \sum_{n=0}^{\infty} p_n z^n$, $g(z) = \sum_{n=0}^{\infty} q_n z^n$ be the power series with real coefficients and convergent on the open disk $D(0,R)$, $0 < R < 1$. If p, q are real numbers with $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $a, b \in \mathbf{C}$, $a, b \neq 0$, $|a| < 1$, $|b| < 1$ so that $|ab|$, $|a|^p$, $|a|^q$, $|b|^p$, $|b|^q$, $|a|^{\frac{q}{2}}|b|^{\frac{p}{2}}$, $|a|^{\frac{p}{2}}|b|^{\frac{q}{2}}$, $|a|^p|b|^q$, $|a|^q|b|^p \in D(0,R)$, then we have

$$|f(ab)g(ab)| + rM_1 + A(\frac{1}{p})T_1$$

$$\leq f_A(|ab|)g_A(|ab|) + rM_1 + A(\frac{1}{p})T_1$$

$$\leq \frac{1}{p}f_A(|a|^p)g_A(|b|^p) + \frac{1}{q}f_A(|b|^q)g_A(|a|^q) \quad (2)$$

$$\leq f_A(|a||b|)g_A(|a||b|) + (1-r)M_1 + B(\frac{1}{p})T_1,$$

and

$$|f(a|b|^{p-1})g(a|b|^{q-1})| + rM_2 + A\left(\frac{1}{p}\right)\log^2\frac{|a|}{|b|}T_2 \quad (3)$$

$$\leq f_A(|a||b|^{p-1})g_A(|a||b|^{q-1}) + rM_2 + A\left(\frac{1}{p}\right)\log^2\frac{|a|}{|b|}T_2$$

$$\leq \frac{1}{p}f_A(|a|^p)g_A(|b|^q) + \frac{1}{q}f_A(|b|^p)g_A(|a|^q)$$

$$\leq f_A(|a||b|^{p-1})g_A(|a||b|^{q-1}) + (1-r)M_2 + B\left(\frac{1}{p}\right)\log^2\frac{|a|}{|b|}T_2,$$

if in this case $|a| < |b|$, and $|a||b|^{p-1}$, $|a||b|^{q-1}$, $|a|^p$, $|a|^q$, $|b|^p$, $|b|^q$, $|a|^{\frac{p}{2}}|b|^{\frac{p}{2}}$, $|a|^{\frac{q}{2}}|b|^{\frac{q}{2}} \in D(0,R)$, where

$$M_1 = f_A(|a|^p)g_A(|b|^p) + f_A(|b|^q)g_A(|a|^q) -$$

$$-2f_A(|a|^{\frac{p}{2}}|b|^{\frac{q}{2}})g_A(|a|^{\frac{q}{2}}|b|^{\frac{p}{2}}),$$

$$M_2 = f_A(|a|^p)g_A(|b|^q) + f_A(|b|^p)g_A(|a|^q) -$$
$$-2f_A(|a|^{\frac{p}{2}}|b|^{\frac{p}{2}})g_A(|a|^{\frac{q}{2}}|b|^{\frac{q}{2}}),$$
$$T_1 = g_A(|a|^q|b|^p)S_1(|a|^p|b|^q)\log^2\frac{|a|^p}{|b|^q} +$$
$$+f_A(|a|^p|b|^q)S_2(|a|^q|b|^p)\log^2\frac{|a|^q}{|b|^p} - 2[pq(\log^2|a| + \log^2|b|) -$$
$$-(p^2+q^2)\log|a|\log|b|]S_3(|a|^p|b|^q)S_4(|a|^q|b|^p),$$

$$T_2 = p^2 g_A(|a|^q)S_1(|a|^p) + q^2 f_A(|a|^p)S_2(|a|^q) -$$
$$-2pqS_3(|a|^p)S_4(|a|^q),$$
$$S_1(x) = xf_A'(x) + x^2 f_A''(x),\ S_2(x) = xg_A'(x) + x^2 g_A''(x),$$
$$S_3(x) = xf_A'(x),\ S_4(x) = xg_A'(x).$$

Proof.

In the first case we replace a by $|a|^j|b|^k$, and b by $|a|^k|b|^j$, $j,k \in \{0,1,\ldots,n\}$ in (1) and then we have

$$|a|^j|b|^k|a|^k|b|^j + r[(|a|^j|b|^k)^{\frac{p}{2}} - (|a|^k|b|^j)^{\frac{q}{2}}]^2 +$$
$$+A(\frac{1}{p})\log^2\left(\frac{|a|^{jp}|b|^{kp}}{|a|^{kq}|b|^{jq}}\right)|a|^{jp}|b|^{kp}|a|^{kq}|b|^{jq} \le$$
$$\le \frac{|a|^{jp}|b|^{kp}}{p} + \frac{|a|^{kq}|b|^{jq}}{q} \le$$
$$\le |a|^j|b|^k|a|^k|b|^j + (1-r)[(|a|^j|b|^k)^{\frac{p}{2}} - (|a|^k|b|^j)^{\frac{q}{2}}]^2 +$$
$$+B(\frac{1}{p})\log^2\left(\frac{|a|^{jp}|b|^{kp}}{|a|^{kq}|b|^{jq}}\right)|a|^{jp}|b|^{kp}|a|^{kq}|b|^{jq}$$

for any $j,k \in \{0,1,2,\ldots,n\}$. We take into account that $|a^jb^ka^kb^j| = |a^jb^j||b^ka^k| = |a|^j|b|^k|a|^k|b|^j$ and if we multiply the inequality with positive quantities $|p_j||q_k|$ and sum over j and k from 0 to n, we obtain

$$\sum_{j=0}^{n}|p_j||ab|^j\sum_{k=0}^{n}|q_k||ab|^k + r\sum_{j=0}^{n}|p_j|\sum_{k=0}^{n}|q_k|[|a|^{jp}|b|^{kp} +$$
$$+|a|^{kq}|b|^{jq} - 2|a|^{j\frac{p}{2}}|a|^{k\frac{q}{2}}|b|^{k\frac{p}{2}}|b|^{j\frac{q}{2}}] +$$
$$+A(\frac{1}{p})\sum_{j=0}^{n}|p_j|\sum_{k=0}^{n}|q_k|\log^2\left(\frac{|a|^{jp-kq}}{|b|^{jq-kp}}\right)(|a|^p|b|^q)^j(|a|^q|b|^p)^k \le$$
$$\le \sum_{j=0}^{n}|p_j|\sum_{k=0}^{n}|q_k|\left(\frac{|a|^{jp}|b|^{kp}}{p} + \frac{|a|^{kq}|b|^{jq}}{q}\right) \le \tag{4}$$
$$\le \sum_{j=0}^{n}|p_j||ab|^j\sum_{k=0}^{n}|q_k||ab|^k + (1-r)\sum_{j=0}^{n}|p_j|\sum_{k=0}^{n}|q_k|[|a|^{jp}|b|^{kp} +$$
$$+|a|^{kq}|b|^{jq} - 2|a|^{j\frac{p}{2}}|a|^{k\frac{q}{2}}|b|^{k\frac{p}{2}}|b|^{j\frac{q}{2}}] +$$
$$+B(\frac{1}{p})\sum_{j=0}^{n}|p_j|\sum_{k=0}^{n}|q_k|\log^2\left(\frac{|a|^{jp-kq}}{|b|^{jq-kp}}\right)(|a|^p|b|^q)^j(|a|^q|b|^p)^k.$$

Denoting by P_1 the quantity

$$\sum_{j=0}^{n}|p_j|\sum_{k=0}^{n}|q_k|\log^2\left(\frac{|a|^{ip-kq}}{|b|^{jq-kp}}\right)(|a|^p|b|^q)^j(|a|^q|b|^p)^k$$

by computation we have,

$$P_1 = \sum_{j=0}^{n}|p_j|\sum_{k=0}^{n}|q_k|[(jp-kq)\log|a| -$$
$$-(jq-kp)\log|b|]^2(|a|^p|b|^q)^j(|a|^q|b|^p)^k =$$
$$= \sum_{j=0}^{n}|p_j|\sum_{k=0}^{n}|q_k|[(jp-kq)^2\log^2|a| + (jq-kp)^2\log^2|b| -$$
$$-2(jp-kq)(jq-kp)\log|a|\log|b|](|a|^p|b|^q)^j(|a|^q|b|^p)^k =$$
$$= \sum_{j=0}^{n}|p_j|\sum_{k=0}^{n}|q_k|[j^2(p\log|a| - q\log|b|)^2 +$$
$$+k^2(q\log|a| - p\log|b|)^2 - 2jk(pq(\log^2|a| + \log^2|b|) -$$
$$-(p^2+q^2)\log|a|\log|b|)](|a|^p|b|^q)^j(|a|^q|b|^p)^k =$$
$$= \sum_{j=0}^{n}|p_j|\sum_{k=0}^{n}|q_k|[j^2\log^2\frac{|a|^p}{|b|^q} + k^2\log^2\frac{|a|^q}{|b|^p} -$$
$$-2jk(pq(\log^2|a| + \log^2|b|) - (p^2+q^2)\log|a|\log|b|)]\cdot$$
$$\cdot(|a|^p|b|^q)^j(|a|^q|b|^p)^k.$$

All the series whose partial sums which appear here in inequality (4) are convergent on the disk $D(0,R)$ therefore we can take the limit when n tends to ∞ in (4) and obtain the inequality (2) taking into account that because T_1 is the limit when n tends of ∞ of P_1.

In the second case, if we replace in (1) a by $\frac{|a|^j}{|b|^j}$ and b by $\frac{|a|^k}{|b|^k}$ then we have

$$\frac{|a|^j|a|^k}{|b|^j|b|^k} + r\left[\frac{|a|^{pj}}{|b|^{pj}} + \frac{|a|^{qk}}{|b|^{qk}} - 2\frac{|a|^{\frac{jp}{2}}|a|^{\frac{qk}{2}}}{|b|^{\frac{jp}{2}}|b|^{\frac{qk}{2}}}\right] +$$
$$+A(\frac{1}{p})\log^2\left(\frac{|a|^{jp}|b|^{kq}}{|b|^{jp}|a|^{kq}}\right)\frac{|a|^{jp}|a|^{kq}}{|b|^{jp}|b|^{kq}} \le$$
$$\le \frac{1}{p}\frac{|a|^{jp}}{|b|^{jp}} + \frac{1}{q}\frac{|a|^{qk}}{|b|^{qk}} \le \tag{5}$$
$$\le \frac{|a|^j|a|^k}{|b|^j|b|^k} + (1-r)\left[\frac{|a|^{pj}}{|b|^{pj}} + \frac{|a|^{qk}}{|b|^{qk}} - 2\frac{|a|^{\frac{jp}{2}}|a|^{\frac{qk}{2}}}{|b|^{\frac{jp}{2}}|b|^{\frac{qk}{2}}}\right] +$$
$$+B(\frac{1}{p})\log^2\left(\frac{|a|^{jp}|b|^{kq}}{|b|^{jp}|a|^{kq}}\right)\frac{|a|^{jp}|a|^{kq}}{|b|^{jp}|b|^{kq}}$$

for any $|b|^j,\ |b|^k \neq 0,\ j,\ k \in \{0,1,2,\ldots,n\}$.

Simplifying (5) we get

$$|a|^j|a|^k|b|^{j(p-1)}|b|^{k(q-1)} + r[|a|^{pj}|b|^{qk} + |a|^{qk}|b|^{jp} -$$

$$-2|a|^{j\frac{p}{2}+k\frac{q}{2}}|b|^{j\frac{p}{2}+k\frac{q}{2}}]+A(\frac{1}{p})\log^2\left(\frac{|a|^{jp-kq}}{|b|^{jp-kq}}\right)|a|^{jp}|a|^{kq}\leq$$

$$\leq\frac{1}{p}|a|^{jp}|b|^{qk}+\frac{1}{q}|a|^{qk}|b|^{jp}\leq \tag{6}$$

$$\leq|a|^{j}|a|^{k}|b|^{j(p-1)}|b|^{k(q-1)}+(1-r)[|a|^{pj}|b|^{qk}+|a|^{qk}|b|^{jp}-$$

$$-2|a|^{j\frac{p}{2}+k\frac{q}{2}}|b|^{j\frac{p}{2}+k\frac{q}{2}}]+B(\frac{1}{p})\log^2\left(\frac{|a|^{jp-kq}}{|b|^{jp-kq}}\right)|a|^{jp}|a|^{kq}$$

for any $j,k\in\{0,1,2,...,n\}$.

Now we multiply (6) by $|p_j||q_k|\geq 0$, $j,k\in\{0,1,2,...,n\}$ and summing over j and k from 0 to n, we have

$$\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|(|a||b|^{p-1})^j(|a|b|^{q-1})^k+$$

$$+r\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|[|a|^{pj}|b|^{qk}+|a|^{qk}|b|^{jp}-2|a|^{j\frac{p}{2}+k\frac{q}{2}}|b|^{j\frac{p}{2}+k\frac{q}{2}}]+$$

$$+A(\frac{1}{p})\log^2\frac{|a|}{|b|}\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|(jp-kq)^2|a|^{jp}|a|^{kq}\leq$$

$$\leq\frac{1}{p}\sum_{j=0}^{n}|p_j||a|^{jp}\sum_{k=0}^{n}|q_k||b|^{qk}+\frac{1}{q}\sum_{k=0}^{n}|q_k||a|^{qk}\sum_{j=0}^{n}|p_j||b|^{jp} \tag{7}$$

$$\leq\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|(|a||b|^{p-1})^j(|a|b|^{q-1})^k+(1-r)\cdot$$

$$\cdot\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|[|a|^{pj}|b|^{qk}+|a|^{qk}|b|^{jp}-2|a|^{j\frac{p}{2}+k\frac{q}{2}}|b|^{j\frac{p}{2}+k\frac{q}{2}}]+$$

$$+B(\frac{1}{p})\log^2\frac{|a|}{|b|}\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|(jp-kq)^2|a|^{jp}|a|^{kq}.$$

In this case

$$P_2=\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|(jp-kq)^2|a|^{jp}|a|^{kq}=$$

$$=\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|(j^2p^2+k^2q^2-2pqjk)|a|^{jp}|a|^{kq}.$$

Taking into account that all the series whose partial sums are involved in previous inequality are convergent on the disk $D(0,R)$, and letting n to ∞ in the inequality (7), we notice that the desired inequality (3) takes place, because T_2 is the limit when n tends of ∞ of P_2. □.

Theorem 2. Let $f(z)=\sum_{n=0}^{\infty}p_nz^n$, $g(z)=\sum_{n=0}^{\infty}q_nz^n$ be the power series with real coefficients and convergent on the open disk $D(0,R)$, $0<R<1$. If p,q are real numbers with $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $a,b\in\mathbf{C}$, $a,b\neq 0$, $|a|<1$, $|b|<1$ such that $|a||b|$, $|a|^2$, $|a|^q$, $|b|^p$, $|b|^2$, $|a|^{\frac{q}{2}}|b|^{\frac{p}{2}}$, $|a|^{\frac{2}{p}}|b|^{\frac{2}{q}}\in D(0,R)$, then we have

$$|f(ab)g(|a|^{\frac{2}{p}}|b|^{\frac{2}{q}})|+rM_3+A(\frac{1}{p})T_3\leq$$

$$\leq f_A(|ab|)g_A(|a|^{\frac{2}{p}}|b|^{\frac{2}{q}})+rM_3+A(\frac{1}{p})T_3\leq$$

$$\leq\frac{1}{p}f_A(|b|^p)g_A(|a|^2)+\frac{1}{q}f_A(|a|^q)g_A(|b|^2)\leq \tag{8}$$

$$\leq f_A(|a||b|)g_A(|a|^{\frac{2}{p}}|b|^{\frac{2}{q}})+(1-r)M_3+B(\frac{1}{p})T_3,$$

where

$$M_3=f_A(|a|^2)g_A(|b|^p)+f_A(|a|^q)g_A(|b|^2)-$$

$$-2f_A(|a|^{\frac{q}{2}}|b|^{\frac{p}{2}})g_A(|a||b|),$$

$$T_3=4\log^2\frac{|a|}{|b|}\cdot f_A(|a|^q|b|^p)S_1(|a|^2|b|^2)+$$

$$+\log^2\frac{|b|^p}{|a|^q}g_A(|a|^2|b|^2)S_2(|a|^q|b|^p)+$$

$$+4\log\frac{|a|}{|b|}\log\frac{|b|^p}{|a|^q}S_3(|a|^q|b|^p)S_4(|a|^2|b|^2).$$

Proof.

Now, we replace a by $|a|^{k\frac{2}{p}}|b|^j$, and b by $|a|^j|b|^{k\frac{2}{q}}$ in inequality (1), we multiply by $|p_j||q_k|\geq 0$ and then summing over j and k from 0 to n we get

$$\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k||a|^{k\frac{2}{p}}|b|^j|a|^j|b|^{k\frac{2}{q}}+$$

$$+r\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|[|a|^{2k}|b|^{jp}+|a|^{jq}|b|^{2k}-2|a|^k|b|^{j\frac{p}{2}}|a|^{j\frac{q}{2}}|b|^k]$$

$$+A(\frac{1}{p})\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|\log^2\left(\frac{|a|^{2k}|b|^{jp}}{|a|^{jq}|b|^{2k}}\right)|a|^{2k}|b|^{jp}|a|^{jq}|b|^{2k}$$

$$\leq\frac{1}{p}\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k||a|^{2k}|b|^{jp}+\frac{1}{q}\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k||a|^{jq}|b|^{2k} \tag{9}$$

$$\leq\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k||a|^{k\frac{2}{p}}|b|^j|a|^j|b|^{k\frac{2}{q}}+$$

$$+(1-r)\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|[|a|^{2k}|b|^{jp}+|a|^{jq}|b|^{2k}-$$

$$-2|a|^k|b|^{j\frac{p}{2}}|a|^{j\frac{q}{2}}|b|^k]+$$

$$+B(\frac{1}{p})\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|\log^2\left(\frac{|a|^{2k}|b|^{jp}}{|a|^{jq}|b|^{2k}}\right)|a|^{2k}|b|^{jp}|a|^{jq}|b|^{2k}$$

where P_3 is the quantity

$$\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|\log^2\left(\frac{|a|^{2k}|b|^{jp}}{|a|^{jq}|b|^{2k}}\right)|a|^{2k}|b|^{jp}|a|^{jq}|b|^{2k}.$$

By computation, we find,

$$P_3=\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|\log^2\left(\frac{|a|^{2k-jq}}{|b|^{2k-jp}}\right)|a|^{2k}|b|^{jp}|a|^{jq}|b|^{2k}=$$

$$=\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|(2k\log\frac{|a|}{|b|}-jq\log|a|+$$

$$+jp\log|b|)^2|a|^{2k}|b|^{jp}|a|^{jq}|b|^{2k}=$$

$$=\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|(4k^2\log^2\frac{|a|}{|b|}+j^2\log^2\frac{|b|^p}{|a|^q}+$$

$$+4jk\log\frac{|a|}{|b|}\log\frac{|b|^p}{|a|^q})|a|^{2k}|b|^{jp}|a|^{jq}|b|^{2k}.$$

Since all the series whose partial sums are involved in the inequality (9) are convergent on the disk $D(0,R)$, letting n tend to ∞ in (9), we deduce the desired inequality, because T_3 is the limit when n tends to ∞ of P_3. □

Theorem 3. Let $f(z)$ and $g(z)$ be as in Theorem 1. If $|a|^2,|b|^p,|b|^q,|a|^{\frac{2}{p}}|b|,|a|^{\frac{2}{q}}|b|,|a||b|^{\frac{q}{2}},|a||b|^{\frac{p}{2}} \in D(0,R)$ then one has the following inequality

$$|f(|a|^{\frac{2}{p}}b)g(|a|^{\frac{2}{q}}b)|+rM_4+A(\frac{1}{p})T_4\le$$

$$\le f_A(|a|^{\frac{2}{p}}|b|)g_A(|a|^{\frac{2}{q}}|b|)+rM_4+A(\frac{1}{p})T_4\le$$

$$\le\frac{1}{p}f_A(|a|^2)g_A(|b|^p)+\frac{1}{q}f_A(|b|^q)g_A(|a|^2)\le$$

$$\le f_A(|a|^{\frac{2}{p}}|b|)g_A(|a|^{\frac{2}{q}}|b|)+(1-r)M_4+B(\frac{1}{p})T_4,$$

where

$$M_4=f_A(|a|^2)g_A(|b|^p)+f_A(|b|^q)g_A(|a|^2)-$$

$$-2f_A(|a||b|^{\frac{q}{2}})g_A(|a||b|^{\frac{p}{2}}),$$

$$T_4=\log^2\left(\frac{|a|^2}{|b|^q}\right)g_A(|a|^2|b|^p)S_1(|a|^2|b|^q)+$$

$$+\log^2\left(\frac{|b|^p}{|a|^2}\right)f_A(|a|^2|b|^q)S_2(|a|^2|b|^p)+$$

$$+2S_3(|a|^2|b|^q)S_4(|a|^2|b|^p)\log\left(\frac{|a|^2}{|b|^q}\right)\log\left(\frac{|b|^p}{|a|^2}\right).$$

Proof. Using again the inequality (1) with $|a|^{j\frac{2}{p}}|b|^k$ instead of a and $|a|^{k\frac{2}{q}}|b|^j$ instead of b we obtain for any $j,k\in\{0,1,2,...,n\}$ the following inequality

$$(|a|^{\frac{2}{p}}|b|)^j(|b||a|^{\frac{2}{q}})^k+r[|a|^{2j}|b|^{pk}+|a|^{2k}|b|^{jq}-$$

$$-2|a|^j|b|^{k\frac{p}{2}}|a|^k|b|^{j\frac{q}{2}}]+$$

$$+A(\frac{1}{p})\log^2\left(\frac{|a|^{2j}|b|^{pk}}{|a|^{2k}|b|^{jq}}\right)(|a|^{2j}|b|^{qj})(|a|^{2k}|b|^{kp})\le$$

$$\le\frac{1}{p}|a|^{2j}|b|^{pk}+\frac{1}{q}|a|^{2k}|b|^{jq}\le \tag{10}$$

$$\le(|a|^{\frac{2}{p}}|b|)^j(|b||a|^{\frac{2}{q}})^k+(1-r)[|a|^{2j}|b|^{pk}+|a|^{2k}|b|^{jq}-$$

$$-2|a|^j|b|^{k\frac{p}{2}}|a|^k|b|^{j\frac{q}{2}}]+$$

$$+B(\frac{1}{p})\log^2\left(\frac{|a|^{2j}|b|^{pk}}{|a|^{2k}|b|^{jq}}\right)(|a|^{2j}|b|^{qj})(|a|^{2k}|b|^{kp}).$$

By the same method as in Theorem 1 we find the desired inequality. □.

Remark 1. Let $r_1,r_2,...,r_m\neq 0$ be real numbers such that $\frac{1}{r_1}+\frac{1}{r_2}+...+\frac{1}{r_m}=1$ and $f(z)=\sum_{n=0}^{\infty}p_nz^n$, $g(z)=\sum_{n=0}^{\infty}q_nz^n$ be the power series with real coefficients and convergent on the open disk $D(0,R)$, $0<R$. If $a_1,a_2,...,a_m,b_1,b_2,...,b_m\in\mathbf{C}$, such that $a_1a_2...a_m$, $b_1b_2...b_m$, $|a_i|^{r_i}$, $|b_i|^{r_i}\in D(0,R)$, $i\in\{1,2,...,m\}$ then we have

$$|f(a_1a_2...a_m)g_A(b_1b_2...b_m)|\le$$

$$\le f_A(|a_1||a_2|...|a_m|)g_A(|b_1||b_2|...|b_m|)\le$$

$$\le\frac{1}{r_1}f_A(|a_1|^{r_1})g_A(|b_1|^{r_1})+\frac{1}{r_2}f_A(|a_2|^{r_2})g_A(|b_2|^{r_2})+...+$$

$$+\frac{1}{r_m}f_A(|a_m|^{r_m})g_A(|b_m|^{r_m}).$$

Proof. We use the well-known inequality

$$\alpha_1x_1+\alpha_2x_2+...+\alpha_mx_m\ge x_1^{\alpha_1}x_2^{\alpha_2}...x_m^{\alpha_m}$$

which takes place for any $x_1,x_2,..,x_m>0$ and $\alpha_1,\alpha_2,...,\alpha_m$ real numbers such that $\alpha_1+\alpha_2+...+\alpha_m=1$ and replacing α_i by $\frac{1}{r_i}$ and $x_i^{\frac{1}{r_i}}$ by x_i we obtain

$$\frac{1}{r_1}x_1^{r_1}+\frac{1}{r_2}x_2^{r_2}+...+\frac{1}{r_m}x_m^{r_m}\ge x_1x_2...x_m.$$

Taking above $x_1=|a_1|^j|b_1|^k$, $x_2=|a_2|^j|b_2|^k,...$ $x_m=|a_m|^j|b_m|^k$ for $j,k\in\{0,1,2,...,n\}$ and using the same method like in Theorem 1 we find the desired inequality. □.

Proposition 2. Let a_j be complex numbers and $p_j>0$, $j\in\{1,2,...,m\}$ such that $\frac{1}{p_1}+\frac{1}{p_2}+...+\frac{1}{p_m}\ge 1$. If $f(z)=\sum_{n=0}^{\infty}p_n^{'}z^n$ is the power series with real coefficients and convergent on the open disk $D(0,R)$, $0<R$ and $a_1a_2...a_m$, $|a_1||a_2|...|a_m|$, $|a_1|^{p_1},|a_2|^{p_2},...,|a_m|^{p_m}\in D(0,R)$, and $|p_i^{'}|\ge 1$ for all $i\in\mathbf{N}$ then the following inequality holds:

$$|f(a_1a_2...a_m)|\le f_A(|a_1||a_2|...|a_m|) \tag{11}$$

$$\leq f_A^{\frac{1}{p_1}}(|a_1|^{p_1})\, f_A^{\frac{1}{p_2}}(|a_2|^{p_2}) \dots f_A^{\frac{1}{p_m}}(|a_m|^{p_m}).$$

Proof. If we consider $a_{ij} = |p_i'|^{\frac{1}{p_j}} |a_j|^i$ with $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$ in Lemma 2, see [5] page 743, the inequality

$$\sum_{i=1}^n a_{i1} a_{i2} \dots a_{im} \leq \left(\sum_{i=1}^n a_{i1}^{p_1} \right)^{\frac{1}{p_1}} \left(\sum_{i=1}^n a_{i2}^{p_2} \right)^{\frac{1}{p_2}} \dots \left(\sum_{i=1}^n a_{im}^{p_m} \right)^{\frac{1}{p_m}}$$

becomes:

$$\sum_{i=1}^n |p_i'|^{\frac{1}{p_1} + \frac{1}{p_2} + .. + \frac{1}{p_m}} |a_1|^i |a_2|^i \dots |a_m|^i \leq$$

$$\leq \left(\sum_{i=1}^n |p_i'| |a_1|^{ip_1} \right)^{\frac{1}{p_1}} \left(\sum_{i=1}^n |p_i'| |a_2|^{ip_2} \right)^{\frac{1}{p_2}} \dots \left(\sum_{i=1}^n |p_i'| |a_m|^{ip_m} \right)^{\frac{1}{p_m}}$$

or

$$\sum_{i=1}^n |p_i'| |a_1 a_2 \dots a_m|^i \leq \sum_{i=1}^n |p_i'|^{\frac{1}{p_1} + \frac{1}{p_2} + .. + \frac{1}{p_m}} |a_1|^i |a_2|^i \dots |a_m|^i \leq$$

$$\leq \left(\sum_{i=1}^n |p_i'| |a_1|^{ip_1} \right)^{\frac{1}{p_1}} \left(\sum_{i=1}^n |p_i'| |a_2|^{ip_2} \right)^{\frac{1}{p_2}} \dots \left(\sum_{i=1}^n |p_i'| |a_m|^{ip_m} \right)^{\frac{1}{p_m}}.$$

Taking into account that $a_1 a_2 \dots a_m, |a_1||a_2| \dots |a_m|$, $|a_1|^{p_1}, |a_2|^{p_2}, \dots, |a_m|^{p_m} \in D(0,R)$, when n tends to ∞ we get inequality (11). $\square$.

Using a refinement of the weighted arithmetic-geometric mean inequality for n real numbers, see [2], we find the following:

Theorem 4. Let $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n > 0$, $p_1, p_2, \dots, p_n > 0$ with $\sum_{j=1}^n p_j = 1$ and $\lambda = \min\{p_1, \dots p_n\}$. If we assume that the multiplicity attaining λ is 1, then we have the following inequality:

$$\sum_{i=1}^n p_i f_A(a_i) g_A(b_i) - |f(a_1^{p_1} a_2^{p_2} \dots a_n^{p_n}) g(b_1^{p_1} b_2^{p_2} \dots b_n^{p_n})| \geq$$

$$\geq \sum_{i=1}^n p_i f_A(a_i) g_A(b_i) -$$

$$- f_A(a_1^{p_1} a_2^{p_2} \dots a_n^{p_n}) g_A(b_1^{p_1} b_2^{p_2} \dots b_n^{p_n}) \geq$$

$$\geq n\lambda [\frac{1}{n} \sum_{i=1}^n f_A(a_i) g_A(b_i) - f_A(a_1^{\frac{1}{n}} .. a_n^{\frac{1}{n}}) g_A(b_1^{\frac{1}{n}} \dots b_n^{\frac{1}{n}})],$$

where f, g, f_A and g_A are as in Theorem 1 and $a_1^{p_1} \dots a_n^{p_n}$, $b_1^{p_1} \dots b_n^{p_n}$, a_i, b_i, $b_1^{\frac{1}{n}} \dots b_n^{\frac{1}{n}}$, $a_1^{\frac{1}{n}} \dots a_n^{\frac{1}{n}} \in D(0,R)$.

Proof. We replace $a_i > 0$ by $a_i^j b_i^k$ for $j, k \in \{1, 2, \dots, m\}$, $i \in \{1, \dots, n\}$ in inequality from below and write again this inequality

$$\sum_{i=1}^n p_i a_i - a_1^{p_1} a_2^{p_2} \dots a_n^{p_n} \geq n\lambda \left(\frac{1}{n} \sum_{i=1}^n a_i - a_1^{\frac{1}{n}} \dots a_n^{\frac{1}{n}} \right)$$

from Proposition 5.1 (Proposition 1), see [2] obtaining:

$$\sum_{i=1}^n p_i a_i^j b_i^k - a_1^{p_1 j} b_1^{p_1 k} \dots a_n^{p_n j} b_n^{p_n k} \geq$$

$$\geq n\lambda [\frac{1}{n} (a_1^j b_1^k + \dots + a_n^j b_n^k) -$$

$$- a_1^{\frac{j}{n}} b_1^{\frac{k}{n}} a_2^{\frac{j}{n}} b_2^{\frac{k}{n}} \dots a_n^{\frac{j}{n}} b_n^{\frac{k}{n}}]$$

which by multiplication by $|p_j'||q_k|$ and summing over j and k will give the desired inequality from conclusion when m tend to infinity. $\square$.

For finite sequences of real numbers we use the majorization relation from [6]. Let $a = (a_1, a_2, .., a_n)$ and $b = (b_1, b_2, \dots, b_n)$ two finite sequences of real numbers. We say that the sequence a majorizes the sequence b and we write

$$a >> b \text{ or } b << a,$$

if after rearranging terms of the sequence a and b satisfy the following three conditions:

$$a_1 \geq a_2 \geq \dots \geq a_n \text{ and } b_1 \geq b_2 \geq \dots \geq b_n$$

$$a_1 + a_2 + \dots + a_k \geq b_1 + b_2 + .. + b_k, \text{ for each } k,\ 1 \leq k \leq n-1;$$

$$a_1 + a_2 + \dots + a_n = a_1 + b_2 + .. + b_n.$$

As in [6], Definition 2, let $F(x_1, x_2, \dots, x_n)$ be a function in n nonnegative real variables. Define

$$\sum^! F(x_1, x_2, \dots, x_n)$$

as the sum of $n!$ summands, obtained from the expression $F(x_1, x_2, \dots, x_n)$ as all the possible permutations of the sequence $x = (x_i)_{i=1}^n$.

Particularly, if for some sequence of nonnegative exponents $a = (a_i)_{i=1}^n$, the function F is of the form $F(x_1, x_2, .., x_n) = x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$, then instead of

$$\sum^! F(x_1, x_2, \dots, x_n)$$

we shall write also

$$T[a_1, a_2, \dots, a_n](x_1, x_2, \dots, x_n)$$

or just $T[a_1, a_2, \dots, a_n]$ if it is clear which is the sequence x used here.

Using the technique given in [3] for Muirhead's theorem, we find the following inequality:

Proposition 3. If $a << b$ and $y_i, z_i \in \mathbf{C}$, $y_i, z_i \neq 0$, $i \in \{1, .., n\}$ then

$$\sum^! f_A(|y_1|^{a_1} |y_2|^{a_2} \dots |y_n|^{a_n}) g_A(|z_1|^{a_1} |z_2|^{a_2} \dots |z_n|^{a_n}) \leq$$

$$\stackrel{!}{\leq} \sum f_A(|y_1|^{b_1}|y_2|^{b_2}...|y_n|^{b_n})g_A(|z_1|^{b_1}|z_2|^{b_2}...|z_n|^{b_n}),$$

where f, g, f_A and g_A are as in Theorem 1, $|y_{\sigma(1)}|^{a_1}|y_{\sigma(2)}|^{a_2}...|y_{\sigma(n)}|^{a_n}$, $|z_{\sigma(1)}|^{b_1}|z_{\sigma(2)}|^{b_2}...|z_{\sigma(n)}|^{b_n} \in D(0,R)$ for any σ, σ being an arbitrary permutation of the numbers $\{1,2,...,n\}$.

Proof. We consider in Muirhead's inequality instead of x_i, $|y_i|^j|z_i|^k$, $i \in \{1,2,...,n\}$ we multiply by $|p_j||q_k|$, and summing over $j,k \in \{0,...,m\}$ we get the desired inequality when m tends to infinity. $\square$.

3 Applications related to the average information

1. Next, we present an application related to the average information for two messages.

For $n=2$ in Theorem 4, with $f(z)=\sum_{n\geq 0}a_nz^n$, $g(z)=\sum_{n\geq 0}b_nz^n$, $a_n,b_n \geq 0$, for all $n=1,2,...$, we deduce the following inequality:

$$\lambda f(a)g(c)+(1-\lambda)f(b)g(d)-f(a^{\lambda}b^{1-\lambda})g(c^{\lambda}d^{1-\lambda}) \geq$$

$$\geq \min\{\lambda.1-\lambda\}[f(a)g(c)+f(b)g(d)-2f(\sqrt{ab})f(\sqrt{cd})]. \tag{12}$$

If we take $g(x)=1$ in inequality (12), we obtain the inequality:

$$\lambda f(a)+(1-\lambda)f(b)-f(a^{\lambda}b^{1-\lambda}) \geq$$

$$\geq \min\{\lambda,1-\lambda\}[f(a)+f(b)-2f(\sqrt{ab})]. \tag{13}$$

But, we have

$$f(a)+f(b)=\sum_{n\geq 0}a_na^n+\sum_{n\geq 0}a_nb^n=\sum_{n\geq 0}a_n(a^n+b^n) \geq$$

$$\geq 2\sum_{n\geq 0}(\sqrt{ab})^n=2f(\sqrt{ab}),$$

so we find the inequality

$$f(a)+f(b)=\sum_{n\geq 0}a_na^n+\sum_{n\geq 0}a_nb^n=\sum_{n\geq 0}a_n(a^n+b^n) \geq$$

$$\geq 2\sum_{n\geq 0}(\sqrt{ab})^n=2f(\sqrt{ab}).$$

Therefore $\lambda f(a)+(1-\lambda)f(b)-f(a^{\lambda}b^{1-\lambda}) \geq 0$ for all $\lambda \in [0,1]$ and $0<a,b<1$, i.e. f is an GA-convex function.

From Information Theory and Coding, let messages be m_1 and m_2 and they have probabilities of occurrence as p and $1-p$. Suppose that a sequence of n messages is transmitted. If n is sufficiently large, then we say that np messages of m_1 are transmitted and $n(1-p)$ messages of m_2 are transmitted.

The information due to message m_1 will be $I_1=\log_2\left(\frac{1}{p}\right)$ and the information due to message m_2 will be $I_2=\log_2\left(\frac{1}{1-p}\right)$. Then the total information carried due to the sequence of n message will be

$$I=npI_1+n(1-p)I_2=$$

$$=n[p\log_2\left(\frac{1}{p}\right)+(1-p)\log_2\left(\frac{1}{1-p}\right)].$$

Average information is the ratio between I and n, so is represented by Shannon entropy H given by

$$H=p\log_2\left(\frac{1}{p}\right)+(1-p)\log_2\left(\frac{1}{1-p}\right).$$

This function denoted by $\Omega(.)$ is also called as *Horseshoe function*, where

$$\Omega(p)=p\log_2\left(\frac{1}{p}\right)+(1-p)\log_2\left(\frac{1}{1-p}\right).$$

In inequality (13), we consider $f(x)=\ln\left(\frac{1}{1-x}\right)=\ln 2\cdot\log_2\frac{1}{1-x}$, $\lambda=p$, $a=1-p$ and $b=p$.

Therefore we obtain the following inequality

$$H=\ln 2\cdot\Omega(p) \geq$$

$$\geq \ln\frac{1}{1-(1-p)^pp^{1-p}}+r\ln\frac{(1-\sqrt{p(1-p)})^2}{p(1-p)} \geq$$

$$\geq \ln\frac{1}{1-(1-p)^pp^{1-p}}, \tag{14}$$

where $r=\min\{p,1-p\}$.

2. Using a similar method as in [19,20] we present an application of such inequalities in Information Theory. For that we consider the inequality

$$AB+r(A^p+B^q-2A^{\frac{p}{2}}B^{\frac{q}{2}}) \leq$$

$$\frac{1}{p}A^p+\frac{1}{q}B^q \leq AB+(1-r)(A^p+B^q-2A^{\frac{p}{2}}B^{\frac{q}{2}}),$$

where r is as in [8], inequality (1.5). In this inequality we put, as in the proof of the classical Hölder's inequality, $A=\frac{a_i}{(\sum_{i=1}^n a_i^p)^{\frac{1}{p}}}$ and $B=\frac{b_i}{(\sum_{i=1}^n b_i^q)^{\frac{1}{q}}}$ where a_i, $b_i>0, i\in\{1,2,...,n\}$ (here $p>1$) and summing over i from 1 to n we get

$$\frac{\sum_{i=1}^n a_ib_i}{\left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^n b_i^q\right)^{\frac{1}{q}}}+$$

$$+2r\left[1-\frac{\left(\sum_{i=1}^n a_i^{\frac{p}{2}}b_i^{\frac{q}{2}}\right)}{\left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{2}}\left(\sum_{i=1}^n b_i^q\right)^{\frac{1}{2}}}\right] \leq 1$$

$$1 \leq \frac{\sum_{i=1}^{n} a_i b_i}{\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}} +$$
$$+2(1-r)\left[1 - \frac{\left(\sum_{i=1}^{n} a_i^{\frac{p}{2}} b_i^{\frac{q}{2}}\right)}{\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{2}}}\right]$$

or

$$\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}} \{1-2r[1-$$
$$-\frac{\left(\sum_{i=1}^{n} a_i^{\frac{p}{2}} b_i^{\frac{q}{2}}\right)}{\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{2}}}]\} \geq \sum_{i=1}^{n} a_i b_i$$

$$\sum_{i=1}^{n} a_i b_i \geq \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}} \{1-2(1-r)[1-$$
$$-\frac{\left(\sum_{i=1}^{n} a_i^{\frac{p}{2}} b_i^{\frac{q}{2}}\right)}{\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{2}}}]\}. \tag{15}$$

We will replace from now r by r_1 in previous inequalities. If $a_i = h_i^r t_i^r$, $b_i = t_i^{-r}$, $p = \frac{1}{r}$, $\frac{1}{s} + \frac{1}{r} = 1$ then by calculus we obtain:

$$\sum_{i=1}^{n} h_i t_i \{1-2r_1[1-\frac{\sum_{i=1}^{n} h_i^{\frac{1}{2}} t_i^{\frac{s+1}{2}}}{\left(\sum_{i=1}^{n} h_i t_i\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} t_i^s\right)^{\frac{1}{2}}}]\}^{\frac{1}{r}} \leq$$
$$\leq \left(\sum_{i=1}^{n} h_i^r\right)^{\frac{1}{r}} \left(\sum_{i=1}^{n} t_i^s\right)^{\frac{1}{s}} \leq$$
$$\leq \sum_{i=1}^{n} h_i t_i \{1-2(1-r_1)[1-\frac{\sum_{i=1}^{n} h_i^{\frac{1}{2}} t_i^{\frac{s+1}{2}}}{\left(\sum_{i=1}^{n} h_i t_i\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} t_i^s\right)^{\frac{1}{2}}}]\}^{\frac{1}{r}},$$

where $0 < r < 1$, $s < 0$.

Let Λ be the utility information scheme, as in Remark 3.2, see [19,20] where $X = (x_1, x_2, ..., x_n)$ is the alphabet; $P^\beta = (p_1^\beta, p_2^\beta, ..., p_n^\beta)$ is the power probability distribution; $U = (u_1, u_2, ..., u_n)$ is the utility distribution $u_k > 0$ for all $k = 1, 2, 3, ..., n$; $\beta \neq 1$, $\beta > 0$, $\sum_{k=1}^{n} p_k^\beta = 1$. Then, for every uniquely decipherable code, Singh et al. [21] obtained

$$\frac{\alpha}{1-\alpha} \log_D \left(\sum_{k=1}^{n} \frac{p_k^\beta u_k^{\frac{1}{\alpha}} D^{l_k \frac{1-\alpha}{\alpha}}}{\left(\sum_{i=1}^{n} p_i^\beta u_i\right)^{\frac{1}{\alpha}}}\right) \geq \frac{\log_2 \sum_{k=1}^{n} \left(\frac{p_k^{\beta\alpha} u_k}{\sum_{i=1}^{n} p_i^\beta u_i}\right)}{(1-\alpha)\log_2 D}$$

where $\alpha > 0$, $\alpha \neq 1$, $D \geq 2$, l_k integers, $p_k \geq 0$, $k = 1, 2, .., n$ and $\sum_{i=1}^{n} D^{-l_k} \leq 1$. According to [21] the "useful" information of order α for power distribution P^β is defined as

$$\frac{1}{1-\alpha} \log \sum_{k=1}^{n} \left(\frac{p_k^{\beta\alpha} u_k}{\sum_{i=1}^{n} p_i^\beta u_i}\right)$$

and the exponential "useful" mean lengths of codewords weighted with the function of power probabilities and utilities is defined as

$$\frac{\alpha}{1-\alpha} \sum_{k=1}^{n} p_k^\beta \left(\frac{u_k}{\sum_{i=1}^{n} p_i^\beta u_i}\right)^{\frac{1}{\alpha}} D^{\frac{(1-\alpha)l_k}{\alpha}}.$$

The last inequality is a generalization of Shannon inequality.

Theorem 5. Let $\alpha > 0$, $\beta > 0$, α, $\beta \neq 1$, $p_k \geq 0$, $k = 1, 2, 3, ..., n$ and $\sum_{k=1}^{n} p_k^\beta = 1$, let $D (D \geq 2)$ is the size of the code alphabet. If $l_k, k = 1, 2, 3, ..., n$ are the lengths of the codewords satisfying $\sum_{k=1}^{n} D^{-l_k} \leq 1$ then for every uniquely decipherable code, the "useful" α-average length of codewords satisfies

$$\frac{\alpha}{1-\alpha} \log_D \left(\sum_{k=1}^{n} \frac{p_k^\beta u_k^{\frac{1}{\alpha}} D^{l_k \frac{1-\alpha}{\alpha}}}{\left(\sum_{i=1}^{n} p_i^\beta u_i\right)^{\frac{1}{\alpha}}}\right) + \frac{\alpha}{\alpha-1} \log_D M(p_k, u_k; \alpha)$$
$$\geq \frac{\log_2 \sum_{k=1}^{n} \left(\frac{p_k^{\beta\alpha} u_k}{\sum_{i=1}^{n} p_i^\beta u_i}\right)}{(1-\alpha)\log_2 D} \tag{16}$$

where

$$M(p_k, u_k; \alpha) =$$
$$= 1-2(1-r_1)\left[1 - \frac{\sum_{k=1}^{n} D^{-\frac{l_k}{2}} p_k^{\frac{\alpha\beta}{2}} u_k^{\frac{1}{2}}}{\left(\sum_{k=1}^{n} D^{-l_k}\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} p_k^{\alpha\beta} u_k\right)^{\frac{1}{2}}}\right],$$

when $\alpha > 1$.

Proof. Using the substitution $r = \frac{\alpha-1}{\alpha} > 0$, $s = 1-\alpha < 0$,

$$h_k = p_k^{\frac{\alpha\beta}{\alpha-1}} \left(\frac{u_k}{\sum_{i=1}^{n} u_i p_i^\beta}\right)^{\frac{1}{\alpha-1}} D^{-l_k}$$

and

$$t_k = p_k^{\frac{\alpha\beta}{1-\alpha}} \left(\frac{u_k}{\sum_{i=1}^{n} u_i p_i^\beta}\right)^{\frac{1}{1-\alpha}}$$

in (15) and after suitable calculus we obtain inequality (16) when $\alpha > 1$.

4 Conclusions

This paper has proposed several inequalities concerning functions defined by convergent power series with real or nonnegative coefficients. This method is useful, because many difficult inequalities can be easily solved and often they can be extedded.

As in [4], there exist some inequalities for special functions such as polylogarithm, hypergeometric, Bessel and modified Bessel functions for the first kind. It is known that $Li_n(z)$, ${}_2F_1(a,b;c;z)$, $J_a(z)$ and $I_a(z)$ are power series with real coefficients and convergent on the open disk $D(0,1)$. Therefore, like in [4], we can think to rewrite the inequalities given before under conditions from our theorems.

In addition, as in [4], because the functions $\exp(z)$, $z \in \mathbf{C}$, $\frac{1}{1-z}$, $z \in D(0,1)$, $\ln(\frac{1}{1-z})$, $z \in D(0,1)$, $\sinh(z)$, $z \in \mathbf{C}$ are power series with real coefficients and convergent on the open disk $D(0,1)$ we can think to rewrite the inequalities given before under conditions from our theorems.

Also many inequalities involving the polylogarithm, hypergeometric, Bessel and modified Bessel functions can be found in the literature, see [13,14,15,16,17,18] and references therein, and on the other hand it is wellknown that the power series and special functions have important applications in engeneering sciences and applied mathematics as parts of Information Sciences, therefore new questions will arise from new applications.

Moreover, in Information Theory appear many inequalities and concepts such as Singh's inequality ([21]) and Shannon entropy which can be obtained from generalizations of Hölder's inequality as in Remark 3.2 and Remark 3.4 from [19,20]. Therefore considering particular suitable functions in our results, new inequalities can be deduced for some of these concepts.

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