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Analytical Solution of Dirac-Morse Problem with Tensor Potential by Asymptotic Iteration Method

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Abstract: In the present paper, we obtained an analytical solution of Dirac equation for modified Morse potential with tensor interaction term by similarity transformation and Asymptotic Iteration Method (AIM). The tensor potential is used to probe nuclear properties and provides a theoretical tool to study the degeneracy of the problem. Similarity transformation is able to convert the Dirac equation to a simple form applicable to the Asymptotic Iteration Method. Thus, the exact solution of the Dirac-Morse problem can be obtained by the systematic of AIM. Our results are compared with the results of other authors and a good agreement is obtained.

Keywords: Dirac equation, Similarity transformation, Asymptotic Iteration Method, Morse potential, Tensor potential

1 Introduction

The study of exact solution is an important topical research in quantum mechanics. This solution acquires its importance because it is a useful tool to improve theoretical models and check the validity of numerical methods. The description of a spin-1/2 particle motion can be achieved by Dirac equation, which plays a fundamental role in relativistic quantum mechanics. One task of Dirac equation is to solve problems in high-energy physics [1].

Recent years have witnessed several techniques that have been used to solve the Dirac equation (see Refs. [2, 3, 4, 5, 6, 7, 8, 9, 10, 11]). For example, the super-symmetry (SUSY) technique [2], shape invariance [3, 4], asymptotic iteration method (AIM) [5, 6, 7, 8, 9], factorization method [10], and the Nikiforov–Uvarov (NU) technique [11]. Nevertheless, there are very few exactly solvable potentials.

The AIM [5] has been used to find solution of the second-order ordinary differential equation. This method gives exact and approximate solution for many problems in physics [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. For solutions of Schrödinger and Klein-Gordon equations, the AIM method is widely used with great success, but for solutions of Dirac equation, it is not used in their original form until the recent appearance of similarity transformation [24, 25].

One of the interesting approaches for solving the Dirac equation is to transform it to a second-order differential equation and then try to convert the resultant equation to the Schrödinger-like one. If this technique is success, then the solution of Dirac equation can be deduced from the known form of the Schrödinger equation. However, this is not always success because in most cases the resultant equation cannot be a Schrödinger-like equation. Hence, it is very hard to find solutions by this technique.

A simple similarity transformation was used in Ref. [26] to transform the radial wave equation of Dirac-Coulomb problem to a form nearly identical to those of the Schrödinger equation. Recently, this technique was used in [25] for Dirac equation in the presence of a vector Coulomb plus scalar linear potential as well as with pure linear potential. Depending on the potential models, similarity transformation has an ability to attain a great simplification to the problem by simple choice of the parameters of this technique.

The aim of this work is to use the AIM to obtain an analytical solution for the Dirac equation with Morse modified potential including tensor interaction term by similarity transformation. Morse potential is one of the suitable models for diatomic molecules potential energy. It can be used to model interactions between an atom and a surface [27]. This potential has been studied by various methods, such as confluent hyper-geometric functions [28], the algebraic method [29], the super-symmetric

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approach [30,31], Nikiforov–Uvarov method under spin symmetry [32,33], Schrödinger-Morse problem by AIM [34], complex scaling method [35]. Besides that, the tensor potential has been introduced into the Dirac equation with the substitution $p \rightarrow p - i\beta\overrightarrow{\alpha}.\hat{r}\ T(r)$ [36, 37]. It is also used to probe nuclear properties [38] and provides a theoretical tool to study the degeneracy of the problem [39].

This paper is organized is as follows. In Sect. 2, we present a brief introduction of the Dirac formalism and explain how the simple similarity transformation can convert the Dirac equation to a second order differential equation with useful parameters. Theses parameters are chosen in such a way that the resulting equation can be transformed to a form nearly identical to the Schrödinger-like equation. In Sect. 3, we give technical details of obtaining the exact solution and the relativistic energy eigenvalue for Dirac-Morse problem by asymptotic iteration method. In Sect. 4, we present and discuss our results. Finally, we conclude our results in Sect. 5.

2 Dirac equation under Similarity Transformation

The relativistic motion of a particle of mass m under the influence of vector potential $V(r)$ and tensor interaction term $i\beta\overrightarrow{\alpha}\cdot\hat{r}T(r)$ can be presented by Dirac equation (in units of $\hbar = c = 1$) [1]

$$H(\overrightarrow{r})\psi(\overrightarrow{r}) = E\psi(\overrightarrow{r}) \tag{1}$$

with

$$H(\overrightarrow{r}) = \overrightarrow{\alpha}\cdot\overrightarrow{p} + \beta m + V(\overrightarrow{r}) - i\beta\overrightarrow{\alpha}\cdot\hat{r}T(\overrightarrow{r}) \tag{2}$$

where $\overrightarrow{\alpha}$ and β have their meaning as 4x4 Dirac matrices.

Applying a similarity transformation to Eq. (1), one can gets [25,26]

$$H'\psi'(r) = E\psi'(r) \tag{3}$$

with

$$H' = FHF^{-1}, \psi'(r) = F\psi(r) \text{ and } F = a + ib\beta\overrightarrow{\alpha}\cdot\hat{r}, \tag{4}$$

where $\hat{r}$ is the unit vector $\overrightarrow{r}/r$ and a and b are real constants to be determined later. The transformed wave function is given by

$$\psi'_{nk}(\overrightarrow{r}) = \begin{pmatrix} f_{nk}(\overrightarrow{r}) \\ g_{nk}(\overrightarrow{r}) \end{pmatrix} = \begin{pmatrix} iR_{nk}(r)\Phi^{l}_{jm}(\theta,\phi) \\ Q_{nk}(r)\overrightarrow{\sigma}\cdot\hat{r}\Phi^{l}_{jm}(\theta,\phi) \end{pmatrix} \tag{5}$$

In a straightforward manner one can calculate

$$FHF^{-1}\psi' = E\psi' \tag{6}$$

to obtain two coupled equations for $R_{nk}(r)$ [the upper component] and $Q_{nk}(r)$ [the lower component] as follows

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}\begin{pmatrix} R_{nk}(r) \\ Q_{nk}(r) \end{pmatrix} = E\begin{pmatrix} R_{nk}(r) \\ Q_{nk}(r) \end{pmatrix} \tag{7}$$

with

$$H_{11} = \sinh\theta\left(\frac{d}{dr} + \frac{1}{r}\right) + V(r) + m\cosh\theta \tag{8}$$

$$H_{12} = -\left[\cosh\theta\left(\frac{d}{dr} + \frac{1}{r}\right) - \frac{k}{r} + m\sinh\theta + T(r)\right] \tag{9}$$

$$H_{21} = \cosh\theta\left(\frac{d}{dr} + \frac{1}{r}\right) + \frac{k}{r} + m\sinh\theta - T(r) \tag{10}$$

$$H_{22} = -\left[\sinh\theta\left(\frac{d}{dr} + \frac{1}{r}\right) - V(r) + m\cosh\theta\right] \tag{11}$$

where $k = \widetilde{\omega}(j+1/2)$ denotes the quantum number for states with $l = j + \widetilde{\omega}$ and $\widetilde{\omega} = \pm 1$, whereas $\cosh\theta = (a^2+b^2)/(a^2-b^2)$, $\sinh\theta = 2ab/(a^2-b^2)$ and n denotes the radial quantum number.

The two coupled equations for $R(r)$ and $Q(r)$ are given by

$$H_{11}R_{nk}(r) + H_{12}Q_{nk}(r) = ER_{nk}(r) \tag{12}$$

$$H_{21}R_{nk}(r) + H_{22}Q_{nk}(r) = EQ_{nk}(r). \tag{13}$$

Multiplying Eq. (12) by $\sinh\theta$, Eq. (13) by $\cosh\theta$ and subtracting the resultant equations, we obtain

$$Q_{nk}(r) = \frac{1}{\zeta_1(r)}D_1(r)R_{nk}(r) \tag{14}$$

where

$$D_1(r) = \partial_r + Z_1(r), \tag{15}$$

$$Z_1(r) = \frac{1}{r} + \frac{k}{r}\cosh\theta - T(r)\cosh\theta - V(r)\sinh\theta + E\sinh\theta, \tag{16}$$

and

$$\zeta_1(r) = m - V(r)\cosh\theta + \frac{k}{r}\sinh\theta - T(r)\sinh\theta + E\cosh\theta. \tag{17}$$

Further, multiplying Eq. (12) by $\cosh\theta$, Eq. (13) by $\sinh\theta$ and subtracting the getting equations, we find

$$R_{nk}(r) = \frac{1}{\zeta_2(r)}D_2(r)Q_{nk}(r), \tag{18}$$

where

$$D_2(r) = \partial_r + Z_2(r), \tag{19}$$

$$Z_2(r) = \frac{1}{r} + \frac{k}{r}\cosh\theta + T(r)\cosh\theta + V(r)\sinh\theta - E\sinh\theta, \tag{20}$$

and

$$\zeta_2(r) = m + V(r)\cosh\theta - \frac{k}{r}\sinh\theta + T(r)\sinh\theta - E\cosh\theta\,. \tag{21}$$

Substituting by Eq. (14) in Eq. (18) and solving for $R_{nk}(r)$, one gets

$$R_{nk}(r) = \frac{1}{\zeta_2(r)} D_2(r) \frac{1}{\zeta_1(r)} D_1(r) R_{nk}(r)\,. \tag{22}$$

After a simple algebra, Eq. (22) can be rewritten as

$$R''_{nk}(r) + \left(\frac{2}{r} - \frac{\zeta_1'(r)}{\zeta_1(r)}\right) R'_{nk}(r) + \left(h(r) - \frac{Z_1(r)\zeta_1'(r)}{\zeta_1(r)}\right) R_{nk}(r) = 0\,, \tag{23}$$

with

$$h(r) = -\frac{k\cosh\theta + k^2}{r^2} + \frac{2kT(r)}{r} - m^2 - T^2(r) - \sinh\theta V'(r) - \cosh\theta T'(r) - 2EV(r) + V^2(r) + E^2\,, \tag{24}$$

where the prime notation denotes differentiation with respect to r.

3 Application of AIM to Dirac-Morse problem

In order to compare our results with those obtained by other techniques [31,34], we confine our investigations with the modified form of Morse potential [31]

$$V(r) = Ae^{-\lambda r} \tag{25}$$

where the parameters A and λ are real and $\lambda > 0$. Substituting by the Morse potential into Eq. (23) yields a complicated equation which needs to be simplified in order to be easily treated by AIM. At this stage, we can use the similarity transformation to reach a great simplification by the following choice of $\cosh\theta$ and $\sinh\theta$ in Eq. (23)

$$\cosh\theta = \frac{B}{C} \quad \text{and} \quad \sinh\theta = -\frac{A}{C} \quad \text{with} \quad C = \sqrt{B^2 - A^2} \tag{26}$$

which yields $\zeta_1'(r) = 0$ if the tensor potential $T(r) = Be^{-\lambda r} + \frac{k}{r}$ is taken where B is a real parameter. So that the only remaining term of the first order derivatives of $R(r)$ in Eq. (23) is $(2/r)R'(r)$. This term can be easily removed by assuming that $R(r) = r^{-1}\phi(r)$. Then one gets

$$\phi''_{nk}(r) + \left[E_{nk}^2 - m^2 - C^2 e^{-2r\lambda} + e^{-r\lambda}(C\lambda - 2AE_{nk})\right] \times \phi_{nk}(r) = 0. \tag{27}$$

Introducing a new variable $z = e^{-\lambda r/2}$ yields

$$\phi''_{nk}(z) + \left(\frac{-\frac{4m^2}{\lambda^2} + \frac{4E_{nk}^2}{\lambda^2}}{z^2} - \frac{4C^2z^2}{\lambda^2} - \frac{8AE_{nk}}{\lambda^2} + \frac{4C}{\lambda}\right)\phi_{nk}(z) + \frac{\phi'_{nk}(z)}{z} = 0. \tag{28}$$

The first order derivative of $\phi(z)$ can be easily removed by defining $\phi(z) = e^{u(z)}Y(z)$; $u'(z) = 1/z$ which allows us to obtain

$$Y''_{nk}(z) + \left(\frac{4C}{\lambda} - \frac{8AE_{nk}}{\lambda^2} - \frac{4C^2z^2}{\lambda^2} - \frac{\varepsilon_{nk}(\varepsilon_{nk}+1)}{r^2}\right) Y_{nk}(z) = 0, \tag{29}$$

where

$$\varepsilon_{nk}(\varepsilon_{nk}+1) = -\frac{1}{4} + \frac{4m^2}{\lambda^2} - \frac{4E_{nk}^2}{\lambda^2}. \tag{30}$$

The relativistic energy eigenvalues of Dirac-Morse problem can be found by applying the asymptotic iteration method (AIM) to Eq. (29). A well-known description of the AIM can be found in Ref. [5] and we refer the reader to this paper for more detail. In the following, we present a short summary of AIM method.

Consider the equation of the form

$$f''(z) = \lambda_0(z) f'(z) + s_0(z) f(z) \tag{31}$$

which can be iterated up to $(i+1)^{th}$ and to $(i+2)^{th}$ derivatives, where $i = 1,2,3,\ldots$ denote the iteration number, one gets

$$f^{(i+1)}(z) = \lambda_{i-1}(z) f'(z) + s_{i-1}(z) f(z)\,, \tag{32}$$

$$f^{(i+2)}(z) = \lambda_i(z) f'(z) + s_i(z) f(z)\,, \tag{33}$$

where

$$\lambda_i(z) = \lambda'_{i-1}(z) + s_{i-1}(z) + \lambda_0(z)\lambda_{i-1}(z)\,,$$
$$s_i(z) = s'_{i-1}(z) + s_0(z)\lambda_{i-1}(z)\,. \tag{34}$$

Taking the ratio of $(i+2)^{th}$ to $(i+1)^{th}$ derivatives, one obtains

$$\frac{f^{(i+2)}}{f^{(i+1)}} = \frac{d}{dz}\ln(f^{(i+1)}) = \frac{\lambda_i(f' + \frac{s_i}{\lambda_i}f)}{\lambda_{i-1}(f' + \frac{s_{i-1}}{\lambda_{i-1}}f)}\,. \tag{35}$$

For large i, the asymptotic aspect of the method arises. So that one can assume [5]

$$\frac{s_i}{\lambda_i} = \frac{s_{i-1}}{\lambda_{i-1}} \equiv \alpha \tag{36}$$

which yields

$$\frac{d}{dz}\ln(f^{(i+1)}) = \frac{\lambda_i}{\lambda_{i-1}}. \tag{37}$$

After integration, the solution can be obtained as [5]

$$f(z) = e^{-\int \alpha dz} \tag{38}$$

and the energy eigenvalues ε which can be determined from Eq. (34) by imposing the termination condition as [5]

$$\delta_i(z,\varepsilon) = \lambda_i(z)s_{i-1}(z) - s_i(z)\lambda_{i-1}(z) = 0\,. \tag{39}$$

Hence, the accuracy and convergence rate of AIM depends on the asymptotic behavior of the wave function Y_{nk} in Eq. (29). So that, we need to analyze the asymptotic behavior of Y_{nk} near the singular points; $z=0$ and $z=\infty$. This investigation suggests that Y_{nk} takes the form

$$Y_{nk}(z) = z^{\varepsilon_{nk}+1} e^{-\beta z^2} f_{nk}(z)\,. \tag{40}$$

where the unknown parameter β can be determined through AIM. By substituting this trial wave function into Eq. (29) and after some algebra we find

$$\begin{aligned} f''_{nk}(z) = &\left[\frac{4z^2\beta - 2(1+\varepsilon_{nk})}{z}\right] f'_{nk}(z) \\ &+ \left[6\beta + 4\beta\varepsilon_{nk} + z^2\left(\frac{4C^2}{\lambda^2} - 4\beta^2\right)\right. \\ &\left. + \frac{8AE_{nk}}{\lambda^2} - \frac{4C}{\lambda}\right] f_{nk}(z), \end{aligned} \tag{41}$$

where the introduced function $f_{nk}(z)$ satisfies a new second-order homogenous linear differential equation which is now suitable for the use of the AIM.

We can attain a valuable simplification if we assume that $\beta = -\frac{C}{\lambda}$. This choice removes the z^2 term in Eq. (41). The application of AIM is now started by rewriting Eq. (41) in the AIM format of Eq. (31) to get

$$\begin{aligned} \lambda_0(z) &= \frac{4z^2\beta - 2(1+\varepsilon_{nk})}{z}, \\ s_0(z) &= 6\beta + 4\beta\varepsilon_{nk} + \frac{8AE_{nk}}{\lambda^2} - \frac{4C}{\lambda}. \end{aligned} \tag{42}$$

4 Results and discussion

Within the framework of the AIM, the relativistic energy eigenvalues of the modified Morse potential can be calculated by means of Eq. (39). The condition $\delta_i(z,\varepsilon_{nk})$ depends on the two variables,z and ε_{nk}. We found that the termination condition in Eq. (39) is independent of the choice of z. Thus, the problem is exactly solvable through AIM. The definition of $\lambda_0(z)$ and $s_0(z)$ can be used in AIM to obtain the energy through the termination condition. The result of applying AIM for ε_{nk} with different values of n is

$$\text{for } n=0 \to \varepsilon_{0k} = -\tfrac{3}{2} - \tfrac{2AE_{0k}}{\beta\lambda^2} + \tfrac{C}{\beta\lambda}\,, \tag{43}$$

$$\text{for } n=1 \to \varepsilon_{1k} = -\tfrac{7}{2} - \tfrac{2AE_{1k}}{\beta\lambda^2} + \tfrac{C}{\beta\lambda}\,, \tag{44}$$

$$\text{for } n=2 \to \varepsilon_{2k} = -\tfrac{11}{2} - \tfrac{2AE_{2k}}{\beta\lambda^2} + \tfrac{C}{\beta\lambda}\,, \tag{45}$$

$$\ldots\ldots\ldots, etc. \tag{46}$$

The resulting ε_{nk} converges for any number of iterations. This means that for any values of n we get

$$\varepsilon_{nk} = -\frac{4n+3}{2} - \frac{2AE_{nk}}{\beta\lambda^2} + \frac{C}{\beta\lambda}, \quad \text{for } n = 0,1,2,3,\ldots \tag{47}$$

Accordingly, the exact analytical relativistic energy of the Dirac-Morse potential, E_{nk}, can be obtained from Eq. (30) as

$$E_{nk} = \frac{AC\overline{n}\lambda + C\sqrt{m^2t^2 - C\overline{n}^2\lambda^2}}{t^2} \tag{48}$$

with $t^2 \equiv A^2 + C^2$ and $\overline{n} \equiv (1+n)$. This result is in agreement with the result obtained by Alhaidari's formalism for the Dirac-Morse problem [31].

It should be noted that Eq. (41) has a convenient form of the confluent hyper-geometric differential equation. So that its solution can be obtained by any conventional method and one can gets

$$f_{nk}(z) = \mathscr{A}\, {}_1F_1(a_1;a_2;a_3z^2), \tag{49}$$

where $\mathscr{A}$ is an arbitrary constant and ${}_1F_1$ is the well-known confluent hyper-geometric function [40] with

$$\begin{aligned} a_1 &= \frac{3}{4} + \frac{\gamma}{2} + \frac{AE_{nk}}{\beta\lambda^2} - \frac{C}{2\beta\lambda} \\ a_2 &= \gamma + \frac{3}{2}, \\ a_3 &= 2\beta. \end{aligned} \tag{50}$$

By using Eqs. (40) and (41), the general solution of the wave function of Dirac-Morse problem can be written as

$$Y_{nk}(z) = \mathscr{A}\, z^{\varepsilon_{nk}+1} e^{-\beta z^2}\, {}_1F_1(a_1;a_2;a_3z^2). \tag{51}$$

On the other hand, the non-relativistic energy eigenvalue can be found. It is known that the non-relativistic Schrödinger-Morse equation for S-wave can be written as [31]

$$\psi''_n(r) + \left[b(\lambda+2a)\,e^{-r\lambda} - b^2 e^{-2r\lambda} + 2\varepsilon_n\right]\psi_n(r) = 0. \tag{52}$$

By comparing Eq. (52) with Eq. (27), one can obtain the following correspondence between nonrelativistic and relativistic parameters

$$\varepsilon_n \to (E_{nk}^2 - m^2)/2, \quad b \to C, \quad \text{and} a \to -\frac{AE_{nk}}{b}. \tag{53}$$

Hence, the energy can be written as

$$\varepsilon_n = ((ab/A)^2 - m^2)/2. \tag{54}$$

Substituting by the expression of A from Eq. (48) and from the mapping $a = -AE_{nk}/b$ yields

$$\varepsilon_n = -\frac{1}{2}(a + \overline{n}\lambda)^2. \tag{55}$$

We would like to emphasize that AIM is an effective method to obtain exact solutions for second order differential equations, like Schrödinger and Klein-Gordon equations. But for Dirac equation, AIM is not used directly. By using the technique of similarity transformation one can covert Dirac equation to a form of a second order Dirac equation. Then the application of AIM can be achieved easily where the eigenvalues and eigenfunctions can be deduced directly without the traditional technique for Dirac equation of other formalism (see, for example, Ref. [31]).

The results obtained in this work are found to be in agreement with the well-known nonrelativistic bound state spectrum for Schrödinger-Morse problem [31]. In addition, the results for the eigenvalues are obtained directly through the systematic of AIM itself, and not by the traditional procedure of inferring the solution by the similarity to other equations as given in Ref. [31].

5 Conclusion

In this work, we an analytical solution of the Dirac-Morse problem with tensor interaction term is obtained by Asymptotic Iteration Method with the aid of similarity transformation. Similarity transformation converted Dirac equation to be ready in use for AIM with the advantage that an accurate choice of parameters can simplify the resultant equation. So, that the application of the AIM method to solve the problem becomes simple. The nonrelativistic bound states spectrum for Schrödinger-Morse equation is obtained directly through the systematic of AIM itself. Comparing with the results of other authors, agreement with the results of Ref. [31] is obtained. In the latter, the eigenvalues were obtained by the traditional procedure of inferring the solution by the similarity to other equations. The combination of Asymptotic Iteration Method and similarity transformation is a valuable and powerful technique if the problem under consideration is exactly solvable. Future extensions to complex problems are also of interest.

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