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On the Dichotomy of Non-Autonomous Systems Over Finite Dimensional Spaces

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Abstract: In this article we study the dichotomy of the q periodic system $\dot{X}(t) = A(t)X(t)$ in terms of the boundedness of the solutions of the following Cauchy problems

$$\begin{cases} \dot{X}(t) = A(t)X(t) + e^{i\mu t}Pb, & t \geq 0 \\ X(0) = 0, \end{cases}$$

and

$$\begin{cases} \dot{X}(t) = -X(t)A(t) + e^{i\mu t}(I-P)b, & t \geq 0 \\ X(0) = 0, \end{cases}$$

where $A(t)$ is a square size matrix of order m, μ is any real number, b is a non zero vector in $\mathbb{C}^m$ and P is an orthogonal projection.

Keywords: Dichotomy, Periodic system, Bounded solutions, Orthogonal projection

1. Introduction

The aim of this paper is to study the relationship between the dichotomy of the system $\dot{x}(t) = A(t)x(t)$ and boundedness of the solutions of the q-periodic $(q > 0)$ Cauchy problems. For a well-posed non-autonomous Cauchy problem

$$\begin{cases} \dot{x}(t) = A(t)x(t) + e^{i\mu t}I, & t \geq 0 \\ x(0) = 0, \end{cases} \qquad (A(t), \mu, I, 0)$$

where $A(t)$ an $m \times m$ matrix, the solution leads to an evolution family $\mathscr{U} = \{U(t,s), t \geq s \geq 0\}$, i.e. $U(t,s)U(s,r) = U(t,r)$ and $U(t,t) = I$ for all $t \geq s \geq r \geq 0$. When the Cauchy problem $(A(t), \mu, Pb, 0)$ is q-periodic, i.e. $A(t+q) = A(t)$ for all $t \geq 0$, then the family $\mathscr{U}$ is q-periodic as well, i.e. $U(t+q, s+q) = U(t,s)$ for all $t \geq s \geq 0$. It is given in [1] that the evolution family $\mathscr{U}$ is uniformly exponentially stable if and only if the spectral radius of $U(q,0)$ is less than one, i.e.

$$r(U(q,0)) := \sup\{|\lambda|,\ \lambda \in \sigma(U(q,0))\} = \inf_{n\geq 1} \|U(q,0)^n\|^{\frac{1}{n}} < 1.$$

We show that $U(q,0)$ is dichotomic if for each $\mu \in \mathbb{R}$ the matrices

$$\Phi_\mu(q) = \int_0^q U(q,s)e^{i\mu s}ds \quad \text{and} \quad \Psi_\mu(q) = \int_0^q U^{-1}(q,s)e^{i\mu s}ds$$

are invertible and there exits a projection P which commutes with $U(q,0)$, $\Phi_\mu(q)$ and $\Psi_\mu(q)$ such that for each real $\mu \in \mathbb{R}$ and each vector $b \in \mathbb{C}^m$, the solutions of the Cauchy problems $(A(t), \mu, Pb, 0)$ and $(-A(t), \mu, (I-P)b, 0)$ are bounded on $\mathbb{R}_+$. We give an example that invertibility of the matrices $\Phi_\mu(q)$ and $\Psi_\mu(q)$ is necessary condition and boundedness of the Cauchy problems $(A(t), \mu, Pb, 0)$ and $(-A(t), \mu, (I-P)b, 0)$ is not sufficient for the dichotomy of $U(q,0)$.

In [1] and [3] stability of the map $U(q,0)$ have been studied in the discrete and continuous case respectively.

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These papers give a connection between stability of the map $U(q,0)$ and boundedness of the solutions of Cauchy problems. Results regarding the dichotomy of a matrix have been discussed in [2] and [6]. For connection between stability and periodic systems see the papers [1], [3], [5] and [7]. General theory of dichotomy of infinite dimensional systems has given in the monograph [4].

The paper is organized as follows: In section 2 we recall basic well known properties of the evolution family. In section 3 we established the results regarding the connection between dichotomy of the map $U(q,0)$ and boundedness of solutions for some periodic Cauchy problems.

2. Preliminary Results

Let X be a Banach space and let $\mathscr{L}(X)$ be the space of all bounded linear operators acting on X. The norm in X and in $\mathscr{L}(X)$ is denoted by the same symbol $\|.\|$.

A family $\mathscr{U}=\{U(t,s): t\geq s\geq 0\}\subseteq \mathscr{L}(X)$ is called evolution family if the following properties are satisfied
(i) $U(t,t)=I,$ for all $t\in\mathbb{R}_+$,
(i) $U(t,s)U(s,r)=U(t,r)$ for all $t\geq s\geq r\geq 0$,
where I denote the identity operator on $\mathscr{L}(X)$. If the later condition is satisfied for all $t,s,r\in\mathbb{R}_+$ then we say that $\mathscr{U}$ is reversible evolution family on X. In this case $U(t,s)$ is invertible for all $t,s\in\mathbb{R}_+$. An evolution family $\mathscr{U}$ is called strongly continuous if for each $x\in X$ the map

$$(t,s)\to U(t,s)x:(t,s)\in\mathbb{R}^2\to X$$

is continuous for all $t\geq s\geq 0$. Such a family is called q-periodic (with some $q>0$) if

$$U(t+q,s+q)=U(t,s),\ \text{for all } t\geq s\geq 0.$$

Clearly, a q-periodic evolution family also satisfies
(i) $U(pq+v,pq+u)=U(v,u),$ for all $p\in\mathbb{N}$, for all $v\geq u\geq 0$,
(ii) $U(pq,rq)=U((p-r)q,0)=U(q,0)^{p-r}$, for all $p,r\in\mathbb{N}$, $p\geq r$.

The family $\mathscr{U}$ is called uniformly exponentially stable if there exist two positive constants N and ω such that

$$\|U(t,s)\|\leq Ne^{-\omega(t-s)},\ \text{for all } t\geq s\geq 0.$$

The set of all $m\times m$ matrices having complex entries would be denoted by $\mathscr{M}(m,\mathbb{C})$. Assume that the map $t\mapsto A(t):\mathbb{R}\mapsto\mathscr{M}(m,\mathbb{C})$ is continuous. Then the Cauchy Problem

$$\begin{cases}\dot{X}(t)=A(t)X(t), & t\in\mathbb{R}\\ X(0)=I,\end{cases}\tag{1}$$

has a unique solution denoted by $\Phi(t)$. It is well known that $\Phi(t)$ is an invertible matrix and that its inverse is the unique solution of the Cauchy Problem

$$\begin{cases}\dot{X}(t)=-X(t)A(t), & t\in\mathbb{R}\\ X(0)=I.\end{cases}\tag{2}$$

Set $U(t,s):=\Phi(t)\Phi^{-1}(s)$ for all $t,s\in\mathbb{R}$.

For a given real number μ and a given family $(A(t))$ we consider the Cauchy Problem

$$\begin{cases}\dot{X}(t)=A(t)X(t)+e^{i\mu t}I, & t\geq 0\\ X(0)=0,\end{cases}\tag{$A(t),\mu,I,0$}$$

and the differential matrix system

$$\dot{X}(t)=A(t)X(t),\quad t\in\mathbb{R}.\tag{$A(t)$}$$

Obviously, the solution of $(A(t),\mu,I,0)$ is given by

$$\Phi_\mu(t)=\int_0^t U(t,s)e^{i\mu s}ds.$$

Now we define

$$V(t,s):=U^{-1}(t,s)=\Phi(s)\Phi^{-1}(t),\ t,s\in\mathbb{R}$$

then the family $\mathscr{V}=\{V(t,s),t,s\in\mathbb{R}\}$ is an evolution family if

$$\Phi(t)\Phi^{-1}(s)=\Phi^{-1}(s)\Phi(t)\ \text{for all } t,s\in\mathbb{R}.\tag{1}$$

Throughout the paper we assume that equation (1) is satisfied for all $t,s\in\mathbb{R}$.
Consider the Cauchy problem

$$\begin{cases}\dot{Y}(t)=-Y(t)A(t)+e^{i\mu t}I, & t\geq 0\\ Y(0)=0.\end{cases}\tag{$-A(t),\mu,I,0$}$$

The solution of $(-A(t),\mu,I,0)$ is given by

$$\Psi_\mu(t)=\int_0^t V(t,s)e^{i\mu s}ds.$$

Let p_L be the characteristic polynomial associated to the matrix $L\in\mathscr{M}(m,\mathbb{C})$ and let $\sigma(L)=\{\lambda_1,\lambda_2,\ldots,\lambda_k\}$, $k\leq m$ be its spectrum.
There exist integer numbers $m_1,m_2,\ldots,m_k\geq 1$ such that

$$p_L(\lambda)=(\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\ldots(\lambda-\lambda_k)^{m_k},$$

where $m_1+m_2+\cdots+m_k=m$. Let $j\in\{1,2,\ldots,k\}$ and $Y_j:=\ker(L-\lambda_jI)^{m_j}$ then in [2] we have the following important theorem which is useful latter on.

Theorem 1. *For each $z\in\mathbb{C}^m$ there exists $y_j\in Y_j$, $j=\overline{1,k}$ such that*

$$L^nz=L^ny_1+L^ny_2+\cdots+L^ny_k.$$

Moreover, if $y_j(n):=L^ny_j$ then $y_j(n)\in Y_j$ for all $n\in\mathbb{Z}_+$ and there exist a $\mathbb{C}^m$-valued polynomials $p_j(n)$ with $\deg(p_j)\leq m_j-1$ *such that*

$$y_j(n)=\lambda_j^np_j(n),\quad n\in\mathbb{Z}_+,\ j\in\{1,2,\ldots,k\}.$$

3. Results

Let us denote $\Gamma_1 = \{z \in \mathbb{C} : |z| = 1\}$, $\Gamma_1^+ := \{z \in \mathbb{C} : |z| > 1\}$ and $\Gamma_1^- := \{z \in \mathbb{C} : |z| < 1\}$. Clearly $\mathbb{C} = \Gamma_1 \cup \Gamma_1^+ \cup \Gamma_1^-$.

A matrix L is called:

(i)*stable* if $\sigma(L)$ is the subset of Γ_1^- or, equivalently, if there exist two positive constants N and T such that $\|L^n\| \le Ne^{-Tn}$ for all $n = 0,1,2\ldots,$
(ii)*expansive* if $\sigma(L)$ is the subset of Γ_1^+ and
(iii)*dichotomic* if $\sigma(L)$ does not intersect the set Γ_1.

*Remark.*If L is a dichotomic matrix then there exists $\eta \in \{1,2,\ldots,\xi\}$ such that

$$|\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_\eta| < 1 < |\lambda_{\eta+1}| \le \cdots \le |\lambda_\xi|.$$

Having in mind the decomposition of $\mathbb{C}^m$ given by (3.1) let us consider

$$X_1 = Y_1 \oplus Y_2 \oplus \cdots \oplus Y_\eta \quad \text{and} \quad X_2 = Y_{\eta+1} \oplus Y_{\eta+2} \oplus \cdots \oplus Y_\xi.$$

Then $\mathbb{C}^m = X_1 \oplus X_2$.

Recall that a linear map $P : \mathbb{C}^m \to \mathbb{C}^m$ is called projection if $P^2 = P$. In the following theorem we give our first result.

Theorem 2.*Let $q > 0$. If the matrix $U(q,0)$ is dichotomic and there exists a projection P commuting with $U(q,0)$, $\Phi_\mu(q)$ and $\Psi_\mu(q)$ then for each $\mu \in \mathbb{R}$ and each non-zero vector $b \in \mathbb{C}^m$ the solutions of the following Cauchy problems*

$$\begin{cases} \dot{X}(t) = A(t)X(t) + e^{i\mu t}Pb, & t \ge 0 \\ X(0) = 0, \end{cases} \qquad (A(t),\mu,Pb,0)$$

and

$$\begin{cases} \dot{X}(t) = -X(t)A(t) + e^{i\mu t}(I-P)b, & t \ge 0 \\ X(0) = 0, \end{cases} \qquad (-A(t),\mu,(I-P)b,0)$$

are bounded.

*Proof.*Assume that $U(q,0)$ is dichotomic, then by Remark 3 we have a decomposition of $\mathbb{C}^m$, i.e. $\mathbb{C}^m = X_1 \oplus X_2$.
We define $P : \mathbb{C}^m \to \mathbb{C}^m$ by $Px = x_1$, where $x = x_1 + x_2$, such that $x_1 \in X_1$ and $x_2 \in X_2$. It is clear that P is a projection.
Moreover for all $x \in \mathbb{C}^m$ and all $k \in \mathbb{Z}_+$, this yields

$$\begin{aligned} PU(q,0)^k x &= P(U(q,0)^k(x_1+x_2)) \\ &= P(U(q,0)^k(x_1) + U(q,0)^k(x_2)) \\ &= U(q,0)^k(x_1) \\ &= U(q,0)^k Px. \end{aligned}$$

Hence $PU(q,0)^k = U(q,0)^k P$ for all $k \in \mathbb{Z}_+$. Also we have

$$\begin{aligned} P\Phi_\mu(q)x &= P(\Phi_\mu(q)(x_1+x_2)) \\ &= P(\Phi_\mu(q)(x_1) + \Phi_\mu(q)(x_2)) \\ &= \Phi_\mu(q)(x_1) \\ &= \Phi_\mu(q)Px \end{aligned}$$

and similarly we conclude that $P\Psi_\mu(q) = \Psi_\mu(q)P$. Now the solution of the Cauchy problem $(A(t),\mu,Pb,0)$ is given by

$$\Phi_{(\mu,P,b)}(t) = \int_0^t U(t,s)e^{i\mu s}Pbds.$$

Let n be the integer part of $\frac{t}{q}$ and let $r := (t-qn) \in [0,q)$. Then

$$\begin{aligned} &\int_0^t U(t,s)e^{i\mu s}Pbds = \int_0^{qn+r} U(t,s)e^{i\mu s}Pbds \\ &= \int_0^{qn} U(t,s)e^{i\mu s}Pbds + \int_{qn}^{qn+r} U(t,s)e^{i\mu s}Pbds \\ &= \int_{qn}^{qn+r} U(t,s)e^{i\mu s}Pbds + \sum_{k=0}^{n-1}\int_{qk}^{q(k+1)} U(qn+r,s)e^{i\mu s}Pbds \\ &= \int_{qn}^{qn+r} U(t,s)e^{i\mu s}Pbds \\ &+ U(r,0)\sum_{k=0}^{n-1}\int_{qk}^{q(k+1)} U(qn,s)e^{i\mu s}Pbds \\ &= \int_{qn}^{qn+r} U(t,s)e^{i\mu s}Pbds \\ &+ U(r,0)\sum_{k=0}^{n-1}\int_0^q U(qn,qk+\tau)e^{i\mu(qk+\tau)}Pbd\tau \\ &= \int_{qn}^{qn+r} U(t,s)e^{i\mu s}Pbds \\ &+ U(r,0)\sum_{k=0}^{n-1} e^{qi\mu k}\int_0^q U(q(n-k),\tau)e^{i\mu\tau}Pbd\tau \\ &= \int_{qn}^{qn+r} U(t,s)e^{i\mu s}Pbds \\ &+ U(r,0)\sum_{k=0}^{n-1} e^{i\mu qk}U(q,0)^{n-k-1}\int_0^q U(q,\tau)e^{i\mu\tau}Pbd\tau \\ &= I_1 + I_2. \end{aligned}$$

where

$$I_1 = \int_{qn}^{qn+r} U(t,s)e^{i\mu s}Pbds,$$

and

$$I_2 = U(r,0)\sum_{k=0}^{n-1} e^{i\mu qk}U(q,0)^{n-k-1}\Phi_\mu(q)Pb.$$

Now the family $\mathscr{U}$ has a growth bound and $0 \le t-s \le r < q$, so we have

$$\begin{aligned}\|I_1\| &= \left\| \int_{qn}^{qn+r} U(t,s)e^{i\mu s}Pb\,ds \right\| \\ &\le M\int_{qn}^{qn+r} e^{\omega(t-s)}\|Pb\| \\ &\le rMe^{q\omega}\|Pb\| \\ &\le qMe^{q\omega}\|Pb\|,\end{aligned}$$

where ω is a real number and $M \ge 1$. Hence I_1 is bounded. Next let $z_\mu = e^{i\mu q}$, and $\Phi_\mu(q)b = l \in \mathbb{C}^m$ then

$$\begin{aligned}I_2 &= U(r,0)\big(U(q,0)^{n-1}z_\mu^0 \\ &\quad + U(q,0)^{n-2}z_\mu^1 + \cdots + U(q,0)^0 z_\mu^{n-1}\big)Pl.\end{aligned}$$

By our assumption we know that L is dichotomic and $|z_\mu| = 1$ thus z_μ is contained in the resolvent set of L therefore the matrix $(z_\mu I - U(q,0))$ is an invertible matrix. Hence

$$I_2 = U(r,0)(z_\mu I - U(q,0))^{-1}(z_\mu^n I - U(q,0)^n)Pl.$$

Taking norm of both sides

$$\begin{aligned}\|I_2\| &\le \|U(r,0)(z_\mu I - U(q,0))^{-1}z_\mu^n Pl\| \\ &\quad + \|U(r,0)(z_\mu I - U(q,0))^{-1}PU(q,0)^n l\| \\ &= \|U(r,0)\|\|(z_\mu I - U(q,0))^{-1}\|\|Pl\| \\ &\quad + \|U(r,0)\|\|(z_\mu I - U(q,0))^{-1}\|\|PU(q,0)^n l\|.\end{aligned}$$

Using Theorem 1, we have

$$U(q,0)^n l = \lambda_1^n p_1(n) + \lambda_2^n p_2(n) + \cdots + \lambda_\xi^n p_\xi(n),$$

thus

$$PU(q,0)^n l = \lambda_1^n p_1(n) + \lambda_2^n p_2(n) + \cdots + \lambda_\eta^n p_\eta(n),$$

where each $p_i(n)$ are $\mathbb{C}^m$-valued polynomials with degree at most $(m_i - 1)$ for any $i \in \{1,2,\dots,\xi\}$. From hypothesis we know that $|\lambda_i| < 1$ for each $i \in \{1,2,\dots,\eta\}$. So $\|PU(q,0)^n l\| \to 0$ when $n \to \infty$. Thus I_2 is bounded, hence the solution of $(A(t),\mu,Pb,0)$ is bounded.

Next, since the solution of the Cauchy problem $(-A(t),\mu,(I-P)b,0)$ is given by

$$\Psi_{(\mu,I-P,b)}(t) = \int_0^t V(t,s)e^{i\mu s}(I-P)b\,ds.$$

By similar method we obtain that

$$\Psi_{(\mu,I-P,b)}(t) = J_1 + J_2$$

where $J_1 = \int_{qn}^{qn+r} V(t,s)e^{i\mu s}(I-P)bds$ and

$$\begin{aligned}J_2 &= V(r,0)(z_\mu^0 U(q,0)^{-(n-1)} + z_\mu^1 U(q,0)^{-(n-2)} \\ &\quad + \cdots + z_\mu^{n-1}U(q,0)^0)\Psi_\mu(q)(I-P)b.\end{aligned}$$

Proceeding as before we can show that J_1 is bounded. Now for J_2 we have since $PU(q,0) = U(q,0)P$, therefore $(I-P)U(q,0) = U(q,0)(I-P)$. By our assumption we know that $U(q,0)$ is invertible and since $U(q,0)^{-1}$ is also dichotomic hence using the same arguments as above we have

$$\begin{aligned}J_2 &= V(r,0)(z_\mu I - U(q,0)^{-1})^{-1}(z_\mu^n I - U(q,0)^{-n}) \\ &\quad \times \Psi_\mu(q)(I-P)b \\ &= V(r,0)(z_\mu I - U(q,0)^{-1})^{-1}(z_\mu^n I - U(q,0)^{-n})(I-P) \\ &\quad \times \Psi_\mu(q)b.\end{aligned}$$

Taking norm of both sides we get

$$\begin{aligned}\|J_2\| &\le \|V(r,0)\|\|(z_\mu I - U(q,0)^{-1})^{-1}\| \\ &\quad \times \|(I-P)\Psi_\mu(q)b\| \\ &\quad + \|V(r,0)\|\|(z_\mu I - U(q,0)^{-1})^{-1}\| \\ &\quad \times \|U(q,0)^{-n}(I-P)\Psi_\mu(q)b\|.\end{aligned}$$

First we prove that $U(q,0)^{-n}x \to 0$ as $n \to \infty$ for any $x \in X_2$. Since $(I-P)\Psi_\mu(q)b \in X_2$ the assertion would follows. Now since $X_2 = Y_{\eta+1} \oplus Y_{\eta+2} \oplus \cdots \oplus Y_\xi$. So any $x \in X_2$ can be written as a sum of $\xi - \eta$ vectors $y_{\eta+1}, y_{\eta+2}, \dots\, y_\xi$. It would be sufficient to prove that $U(q,0)^{-n}y_i \to 0$ as $n \to \infty$ for any $i \in \{\eta+1, \eta+2, \dots, \xi\}$. Let $Y \in \{Y_{\eta+1}, Y_{\eta+2}, \dots, Y_\xi\}$ say $Y = ker(U(q,0) - \lambda I)^\rho$, where $\rho \ge 1$ is an integer number and $|\lambda| > 1$. Consider $d_1 \in Y \setminus \{0\}$ such that $(U(q,0) - \lambda I)d_1 = 0$ and let $d_2, d_3, \dots, d_\rho$ given by $(U(q,0) - \lambda I)d_i = d_{i-1}$. Then $A := \{d_1, d_2, \dots, d_\rho\}$ is a basis in Y. So it is sufficient to prove that $U(q,0)^{-n}d_i \to 0$ as $n \to \infty$ for any $i \in \{1,2,\dots,\rho\}$. For $i = 1$, we have that $U(q,0)^{-n}d_1 = \frac{1}{\lambda^n}d_1 \to 0$ as $n \to \infty$.
For $i = 2,3,\dots,\rho$, denote $B_n := U(q,0)^{-n}d_i$. Then $(U(q,0) - \lambda I)^\rho B_n = 0$, i.e.

$$B_n - C_\rho^1 B_{n-1}\alpha + C_\rho^2 B_{n-2}\alpha^2 + \cdots + C_\rho^\rho B_{n-\rho}\alpha^\rho = 0, \tag{3.2}$$

where $n \ge \rho$ and $\alpha = \frac{1}{\lambda}$.
Passing for instance at the components, it follows that there exists a $\mathbb{C}^m$-valued polynomial P_ρ having degree at most $\rho - 1$ and verifying (3.2) such that $B_n = \alpha^n P_\rho(n)$. Thus $B_n \to 0$, *when* $n \to \infty$ i.e. $U(q,0)^{-n}d_i \to 0$ for any $i \in \{1,2,\dots,\rho\}$. Thus J_2 is bounded.

The converse statement of the above theorem is not straight forward and we need to put an extra condition i.e. the matrices $\Phi_\mu(q)$ and $\Psi_\mu(q)$ are invertible, at the end of the paper we have given an example which shows that the invertibility conditions on matrices $\Phi_\mu(q)$ and $\Psi_\mu(q)$ can not be removed. Due to this reason we put the converse statement of the above theorem as a new theorem which is stated as.

Theorem 3. *If for each real number μ and each non-zero vector $b \in \mathbb{C}^m$, the solutions of the Cauchy problems $(A(t),\mu,Pb,0)$ and $(-A(t),\mu,(I-P)b,0)$ are bounded then the map $U(q,0)$ is dichotomic, provided that there exists a* *projection P commuting with $U(q,0)$, $\Phi_\mu(q)$ and $\Psi_\mu(q)$ and for each $\mu\in\mathbb{R}$ the matrices $\Phi_\mu(q)$ and $\Psi_\mu(q)$ are invertible.*

*Proof.*Suppose on contrary that the matrix $U(q,0)$ is not dichotomic then $\sigma(U(q,0))\cap\Gamma_1\neq\phi$. Let $\omega\in\sigma(U(q,0))\cap\Gamma_1$ then there exists a non zero $y\in\mathbb{C}^m$ such that $U(q,0)y=\omega y$, it is easy to see that $U(q,0)^ky=w^ky$. Here we have two cases:
Case 1: If $Py\neq 0$. Choose $\mu_1\in\mathbb{R}$ such that $\omega=e^{i\mu_1 q}$, then $U(q,0)^ky=e^{i\mu qk}y$. Since $\Phi_{\mu_1}(q)$ is invertible so there exists $b_1\in\mathbb{C}^m$ such that $\Phi_{\mu_1}(q)b_1=y$. Then

$$\begin{aligned}\Phi_{(\mu_1,P,b_1)}(t) &= \int_{qn}^{qn+r}U(t,s)e^{i\mu_1 s}Pb_1\,ds\\ &+U(r,0)\sum_{k=0}^{n-1}e^{i\mu_1 qk}PU(q,0)^{n-k-1}y\\ &=\int_{qn}^{qn+r}U(t,s)e^{i\mu_1 s}Pb_1\,ds\\ &+U(r,0)\sum_{k=0}^{n-1}e^{i\mu_1 qk}Pe^{i\mu_1 q(n-k-1)}y\\ &=\int_{qn}^{qn+r}U(t,s)e^{i\mu_1 s}Pb_1\,ds\\ &+U(r,0)\sum_{k=0}^{n-1}e^{i\mu_1 q(n-1)}Py\\ &=\int_{qn}^{qn+r}U(t,s)e^{i\mu_1 s}Pb_1\,ds\\ &+U(r,0)ne^{i\mu_1 q(n-1)}Py.\end{aligned}$$

Now clearly $U(r,0)ne^{i\mu_1 q(n-1)}Py\to\infty$ as $n\to\infty$. Hence there exist $\mu_1\in\mathbb{R}$ and $b_1\in\mathbb{C}^m$ such that $\Phi_{(\mu_1,P,b_1)}$ is unbounded. Therefore contradiction arises.

Case 2: If $Py=0$ then surely $(I-P)y\neq 0$. Since $PU(q,0)=U(q,0)P$ therefore $(I-P)U(q,0)=U(q,0)(I-P)$. Choose $\mu_2\in\mathbb{R}$ such that $\omega=e^{-i\mu_2 q}$. In this case we note that $U(q,0)^{-k}y=e^{i\mu_2 qk}y$. Also $\Psi_{\mu_2}(q)$ is invertible so there exists $b_2\in\mathbb{C}^m$ such that $\Psi_{\mu_2}(q)b_2=y$. Now consider the solution of $(-A(t),\mu_2,b_2,0)$ we have

$$\Psi_{(\mu_2,I-P,b_2)}(t)=J_{1,\mu_2}+J_{2,\mu_2},$$

where

$$J_{1,\mu_2}=\int_{qn}^{qn+r}V(t,s)e^{i\mu_2 s}(I-P)b_2\,ds,$$

and

$$\begin{aligned}J_{2,\mu_2} &= V(r,0)\sum_{k=0}^{n-1}e^{i\mu_2 qk}U(q,0)^{-(n-k-1)}\Psi_{\mu_2}(q)(I-P)b_2\\ &=V(r,0)\sum_{k=0}^{n-1}e^{i\mu_2 qk}(I-P)U(q,0)^{-(n-k-1)}y\\ &=V(r,0)\sum_{k=0}^{n-1}e^{i\mu_2 qk}(I-P)e^{i\mu_2 q(n-k-1)}y\\ &=V(r,0)\sum_{k=0}^{n-1}e^{i\mu_2 q(n-1)}(I-P)y\\ &=V(r,0)ne^{i\mu_2 q(n-1)}(I-P)y.\end{aligned}$$

Clearly we see that $J_{2,\mu_2}=V(r,0)nz_{\mu_2}^{n-1}(I-P)y\to\infty$ as $n\to\infty$. Hence there exist $\mu_2\in\mathbb{R}$ and $b_2\in\mathbb{C}^m$ such that $\Psi_{(\mu_2,I-P,b_2)}(t)$ is unbounded. Which is again an absurd. This completes the proof.

The following theorem is taken from [1] which we used to obtained Theorem 3.5.

Theorem 4.*The matrix $U(q,0)$ is stable if and only if for each $b\in\mathbb{C}^m$, the solution of $(A(t),\mu,Pb,0)$ is bounded on $\mathbb{R}_+$ uniformly with respect to the parameter $\mu\in\mathbb{R}$, i.e.*

$$\sup_{\mu\in\mathbb{R}}\sup_{t\geq 0}\left\|\int_0^t U(t,s)e^{i\mu s}bds\right\|:=K(b)<\infty.$$

Theorem 5.*The matrix $U(q,0)$ is dichotomic if and only if there exists a projection P such that for each vector $b\in\mathbb{C}^m$, the solutions of the Cauchy problems $(A(t),\mu,Pb,0)$ and $(-A(t),\mu,(I-P)b,0)$ are uniformly bounded on $\mathbb{R}_+$ with respect to the parameter $\mu\in\mathbb{R}$, i.e.*

$$\sup_{\mu\in\mathbb{R}}\sup_{t\geq 0}\left\|\int_0^t U(t,s)e^{i\mu s}Pbds\right\|:=K_P(b)<\infty,\qquad(3.3)$$

and

$$\sup_{\mu\in\mathbb{R}}\sup_{t\geq 0}\left\|\int_0^t V(t,s)e^{i\mu s}(I-P)bds\right\|:=K_{I-P}(b)<\infty.\qquad(3.4)$$

*Proof.*Suppose the matrix $U(q,0)$ is dichotomic and let $U(q,0)_1$ and $U(q,0)_2$ be the restrictions of $U(q,0)$ on X_1 and X_2 respectively. Consider the spectral decomposition of $\mathbb{C}^m$ as given in Remark 3, that is we can write

$$\mathbb{C}^m=X_1\oplus X_2.$$

Then $U(q,0)_1$ is stable on X_1 and $U(q,0)_2^{-1}$ is stable on X_2. Define the projection $P:\mathbb{C}^m\to\mathbb{C}^m$ as $Px=x_1$ where $x=x_1+x_2$ such that $x_1\in X_1$ and $x_2\in X_2$. Then clearly $P\mathbb{C}^m=X_1$ and $(I-P)\mathbb{C}^m=X_2$.
Since $Pb\in X_1$ for each $b\in\mathbb{C}^m$, therefore Theorem 4 implies that

$$\sup_{\mu\in\mathbb{R}}\sup_{t\geq 0}\left\|\int_0^t U(t,s)e^{i\mu s}Pbds\right\|:=K_P(b)<\infty.$$

Also $(I-P)b \in X_2$ for each $b \in \mathbb{C}^m$ then again Theorem 4 implies that

$$\sup_{\mu\in\mathbb{R}}\sup_{t\geq 0}\|\int_0^t V(t,s)e^{i\mu s}(I-P)bds\| := K_{I-P}(b) < \infty.$$

Conversely let P be the projection for which (3.3) and (3.4) are satisfied. Assume that $P\mathbb{C}^m = W_1$ and $(I-P)\mathbb{C}^m = W_2$. Then clearly $\mathbb{C}^m = W_1 \oplus W_2$. So by (3.3) and using Theorem 4 we have $U(q,0)$ is stable on W_1. Similarly by (3.4) and again using Theorem 4 we obtain that $U(q,0)^{-1}$ is stable on W_2. Hence $U(q,0)$ is dichotomic on $\mathbb{C}^m$.

References

[1] S. Arshad, C. Buşe and Olivia Saierli, *Connections between exponential stability and boundedness of solutions of a couple of differential time depending and periodic systems*, Electronic Journal of Qualitative Theory of Differential Equations, (2011), No. 90, pp. 1-16.

[2] C. Buse, Akbar Zada, Dichotomy and bounded-ness of solutions for some discrete Cauchy problems, Proceedings of IWOTA- 2008, Operator Theory, Advances and Applications, (OT) Series Birkhäuser Verlag, Eds: J.A. Ball, V. Bolotnikov, W. Helton, L. Rodman, T. Spitkovsky, Vol. **203**, 165-174, (2010).

[3] S. Arshad, C. Buşe, A. Nosheen and A. Zada, *Connections between the stability of a Poincare map and boundedness of certain associate sequences*, Electronic Journal of Qualitative Theory of Differential Equations, (2011), No. 16, pp. 1-12.

[4] C. Chicone, Y.Latushkin, ”Evolution Semigroups in Dynamical Systems and Differential Equations”, Amer. Math. Soc., Math. Surv. and Monographs, No. 70 (1999).

[5] C. Buşe and A. Zada, *Boundedness and exponential stability for periodic time dependent systems*, Electronic Journal of Qualitative Theory of Differential Equations, **37**, pp 1-9 (2009).

[6] Akbar Zada, A characterization of dichotomy in terms of boundedness of solutions for some Cauchy problems, Electronic Journal of Differential Equations, Vol.(2008), No. 94, 1-5.

[7] C. Buşe, P. Cerone, S. S. Dragomir and A. Sofo, *Uniform stability of periodic discrete system in Banach spaces*, J. Difference Equ. Appl. **11**, No .12 (2005) 1081-1088.

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