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On Oscillatory Solution of Delay Differential Equation and Sufficient Condition using Sumudu Transform

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Abstract: In this study we develop sufficient condition in order to determine the solution of delay differential equation is oscillatory or not oscillatory further we also obtain a polynomial approximation to determine the stability of the solutions.

Keywords: delay differential equation, oscillatory solution, Volterra integral equation, stability of solution, Sumudu transform.

1 Introduction

Delay differential equations (DDEs) appear in the models throughout the several applications, see for example [1]. Much of the work that has been done treats DDEs with one or a few discrete delays. A number of realistic physiological models however include distributed delays and a problem of particular interest is to determine the stability of the steady state solutions. For applications in physiological systems, see [2, 3, 4]. The results concerning existence, uniqueness and continuous dependence of Eq (4) can be found in [5, 6, 7] and the asymptotic behavior of the solutions has been studied elsewhere, see, e.g., [8]. From the theoretical point of view the most important class of functions $f(t)$ for which the Sumudu transform is defined is the set of exponentially bounded functions. A function $f(t)$, defined on $[0,\infty)$, is exponentially bounded there if there is a positive $K \in \mathbb{R}^+$ and a real number, $\frac{1}{\gamma}$, such that $|f(t)| < Ke^{\frac{x}{\gamma}}$, $t \in [0,\infty)$. It is straightforward to see, for example, that all of the generalized exponential functions lie in this class of functions.

In this study we prove a criteria as sufficient condition in order to determine whether solution is oscillatory or non oscillatory for delay differential equation. The present approach is based on the method of Sumudu transform which was not used yet to study oscillation of delay differential equations. Further we also obtain a polynomial as an approximation to determine the stability of the solutions. First of all we need the some preliminaries.

Proposition 1. *If the function $f(t)$ is exponentially bounded, i.e., if for some $K > 0$ and some real $\frac{1}{\gamma}$ we have $|f(t)| < Ke^{\frac{t}{\gamma}}$, $t \in [0,\infty)$ then the corresponding Sumudu integral*

$$F(u) = S[f(t)] = \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}} f(t)\,dt$$

converges, and thus the Sumudu transform is defined, for all $u > \gamma$ or $\frac{1}{\alpha} < \frac{1}{\gamma}$.

Proof. See [9]

Remark. For related current studies with Sumudu transform we refer to ([10] - [14]). We further note that, under these circumstances, $F(u)$ is defined for all complex $u = \frac{1}{\alpha} + \frac{i}{\beta}$ for which $Reu = \frac{1}{\alpha} < \frac{1}{\gamma}$ so that $F(u)$ is defined in the whole right half complex plane $Reu < \frac{1}{\gamma}$ The smallest value of $\frac{1}{\gamma}$ for which $|f(t)| < Ke^{\frac{t}{\gamma}}$, $t \in [0,\infty)$ for some $K > 0$ is called the abscissa of convergence of $F(u)$.

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Consider the system of

$$\frac{dy}{dt} = py + \int_0^t D(t-s)y(s)ds + F(t), \qquad (1)$$

and

$$\frac{dy}{dt} = py + \int_0^t D(t-s)y(s)ds, \qquad (2)$$

in which p is an $n\times n$ constant matrix, $D(t)$ an $n\times n$ matrix of functions continuous on $[0,\infty)$, and $F : [0,\infty) \to R^n$ continuous. We suppose further that $|F(t)|$ and $|D(t)|$ may be bounded by a function Ke^{at} for $K>0$ and $a>0$. That is, F and D are said to be of exponential order.

Theorem 1. *Let $Z(t)$ be the $n\times n$ matrix whose columns are solutions of Eq (2) with $Z(0) = I$. Then the solution of Eq (1) satisfying $y(0) = y_0$ is given by*

$$y(t) = Z(t)y_0 + \int_0^t Z(t-s)F(s)ds. \qquad (3)$$

Proof. Notice that $Z(t)$ satisfies Eq (2) thus

$$\frac{dZ}{dt} = pZ(t) + \int_0^t D(t-s)Z(s)ds.$$

We first suppose that F and D are in $L^1[O,\infty)$. If we convert Eq (1) into an integral equation, we have

$$y(t) = y(0) + \int_0^t F(s)ds + \int_0^t \left[p + \int_s^t D(x-s)dx\right] y(s)ds,$$

and as D and F are in L^1, we have

$$|y(t)| \le |y(0)| + k + k\int_0^t |y(s)|\,ds,$$

some $k>0$ and $0\le t<\infty$. By Gronwall's inequality we have

$$|y(t)| \le [|y(0)| + k]\,e^{kt}.$$

Thus both $y(t)$ and $Z(t)$ are of exponential order, so then the transform exists and taking their Sumudu transforms, we have

$$\frac{dZ(t)}{dt} = pZ + \int_0^t D(x-s)Z(s)ds$$

and upon transforming both sides, we obtain

$$\frac{Z(u) - Z(0)}{u} = pZ(u) + uZ(u)D(u)$$

further we have

$$Z(0) = \left(I - up - u^2 D(u)\right) Z(u)$$

and because the right side is nonsingular, so it follows that $\left(I - up - u^2D(u)\right)$ is also nonsingular for appropriate u. (Actually, $Z(u)$ is an analytic function of s in the half-plane $Reu \ge a$, where $|Z(t)| \le ke^{kt}$. Then we have

$$Z(u) = \left(I - up - u^2D(u)\right)^{-1}.$$

Now, by taking Sumudu transform for both sides of Eq(1)

$$\frac{Y(u) - y(0)}{u} = pY(u) + uY(u)D(u) + F(u),$$

or

$$\left(I - up - u^2D(u)\right) Y(u) = y(0) + uF(u)$$

so that we get

$$\begin{aligned} Y(u) &= Z(u)y(0) + uZ(u)F(u) \\ &= Z(u)y(0) + S\left(\int_0^t Z(t-s)F(s)ds\right) \\ &= S\left(Z(t)y(0) + \int_0^t Z(t-s)F(s)ds\right). \end{aligned}$$

Since y, Z, and F are of exponential order and continuous. Thus, the proof is complete for D and F being in $L^1[0,\infty)$.

We provide some sufficient conditions under which oscillation phenomenon occurs for the linear Volterra integral equation of convolution type with delay

$$y(t) = f(t) + \int_0^t \sum_{i=1}^n a_i(t-s)y(s-r_i)ds, t\ge 0 \qquad (4)$$

where $f \in C(\mathbb{R}^+,\mathbb{R})$, $a_i(.) \in L^1_{loc}(\mathbb{R}^+)$, $r_i \in \mathbb{R}$, for $i = 1,2,3,\ldots n$. Our approach is based on the method of Sumudu transform.

In this section, we establish some results which we need in the proofs of our main result. In order to guarantee the existence of Sumudu transforms of solutions of Eq(4), we assume that for the function f, there exist two real numbers $K\in\mathbb{R}^+$ and $\frac{1}{b}\in\mathbb{R}$ such that

$$|f(t)| \le Ke^{\frac{t}{b}},\ \ t\ge 0 \qquad (5)$$

$$|a_i(t)| \le Ke^{\frac{t}{b}},\ \ t\ge 0,\quad i = 1,2,3,\ldots,n. \qquad (6)$$

Then we can state the following lemma.

Lemma 1. *Assume that Eq(5) and Eq(6) are holds. Then every solution of Eq(4) has Sumudu transform.*

Proof. Consider a solution y of Eq(4) with initial function $\phi \in C([-r,0],\mathbb{R})$; by using Eq(4) and Eq(5), we have

$$\begin{aligned} |y(t)| \le Ke^{\frac{t}{b}} &+ \sum_{i=1}^n \int_{-r_i}^0 |a_i(t-s-r_i)|\,|\phi(s)|\,ds \\ &+ \sum_{i=1}^n \int_0^t |a_i(t-s-r_i)|\,|x(s)|\,ds. \end{aligned}$$

Multiplying both sides of this inequality by $e^{-\frac{t}{b}}$, and taking into account Eq(5) and Eq(6) we obtain

$$e^{-\frac{t}{b}}|y(t)| \leq K + K\sum_{i=1}^{n} e^{-\frac{r_i}{b}} \int_{-r_i}^{0} e^{-\frac{s}{b}} |\phi(s)|\, ds + K\sum_{i=1}^{n} e^{-\frac{r_i}{b}} \int_{0}^{t} e^{-\frac{s}{b}} |x(s)|\, ds.$$

By Gronwall's inequality, it follows

$$|y(t)| \leq \eta_1 e^{\left(\frac{1}{b}+\eta_2\right)t},\ t>0$$

where

$$\eta_1 = K\left(1+\sum_{i=1}^{n} e^{-\frac{r_i}{b}} \int_{-r_i}^{0} e^{-\frac{s}{b}} |\phi(s)|\, ds\right),$$
$$\eta_2 = K\sum_{i=1}^{n} e^{-\frac{r_i}{b}},$$

which is a sufficient condition for the existence of the Sumudu transform of y.

Lemma 2.*If $Y(u)$ is the Sumudu transform of a nonnegative function $y(t)$ and has the abscissa of convergence $\frac{1}{b} > -\infty$, then $Y(u)$ has a singularity at the point $u = \frac{1}{b}$ on the complex plane $\mathbb{C}$.*

Now we shall present the main results for the oscillation of Volterra integral equation Eq(4) via the method of Sumudu transform. Let $y_c(t)$ denote $y(t+c)$, where $c \in \mathbb{R}$. Then the Sumudu transform $Y_c(u)$ of $y_c(t)$ exists and has the same abscissa of convergence as $Y(u)$ by noting the following formula

$$Y_c(u) = e^{\frac{c}{u}}\left(Y(u) - \frac{1}{u}\int_0^c y(t)e^{-\frac{t}{u}}dt\right).$$

The last integral defines an entire function of the complex variable $u \in \mathbb{C}$. It is clear that $Y(u)$ and $Y_c(u)$ have their singularities at the same points on the complex plane. On the other hand, the translation of Eq(4) along a solution y by $c \in \mathbb{R}$ is the following equation

$$y(t+c) = f(t+c) + \int_0^{t+c} \sum_{i=1}^{n} a_i(t+c-s)y(s-r_i)ds,\ \ t \geq 0. \tag{7}$$

Multiply both sides of Eq(7), by $\frac{1}{u}e^{-\frac{t}{u}}$ and integrating it from 0 to ∞, we obtain

$$Y_c(u) = F_c(u) + \frac{1}{u}\int_0^{\infty} e^{-\frac{t}{u}} \int_0^{t+c} \sum_{i=1}^{n} a_i(t+c-s)y(s-r_i)dsdt, \tag{8}$$

where $t \geq 0$ and $F_c(u)$ denotes the Sumudu transform of $f(t+c)$. Then, we find

$$\begin{aligned}
&\frac{1}{u}\int_0^{\infty}\int_0^{t+c} e^{-\frac{t}{u}} a_i(t+c-s)y(s-r_i)dsdt \\
&= \frac{1}{u}\int_0^{c} y(s-r_i)\int_0^{\infty} e^{-\frac{t}{u}} a_i(t+c-s)dtds \\
&+ \frac{1}{u}\int_0^{\infty}\int_0^{t} e^{-\frac{t}{u}} a_i(t-s)y_c(s-r_i)dsdt \\
&= K_1 + K_2.
\end{aligned}$$

It is easy to see that

$$K_1 = \Phi_i + \beta_i(u)A_i(u),\ \ K_2 = A_i(u)\left[\mu_i(u) + e^{-\frac{r_i}{u}}Y_c(u)\right], \tag{9}$$

where

$$\Phi_i = \frac{1}{u}\int_0^{c} y_c(s-r_i)e^{-\frac{(s-c)}{u}}\int_{c-s}^{0} e^{-\frac{t}{u}} a_i(t)dtds$$
$$\beta_i(u) = \int_0^{c} y_c(s-r_i)e^{-\frac{(s-c)}{u}}ds$$
$$\mu_i(u) = \int_{-r_i}^{0} \phi_c(s)e^{-\frac{(s+r_i)}{u}}ds$$

and $A_i(u)$ is the Sumudu transform of $a_i(t)$. The functions $\Phi_i(.)$, $\beta_i(.)$ and $\mu_i(.)$ are entire functions of the complex variable $u \in \mathbb{C}$. By substituting Eq(9) into Eq(8) we have

$$Y_c(u) = F_c(u) + \sum_{i=1}^{n}\Phi_i + \sum_{i=1}^{n}(\beta_i(u)+\mu_i(u))A_i(u) + \sum_{i=1}^{n} e^{-\frac{r_i}{u}}A_i(u)Y_c(u).$$

Define $H(u) = 1 - \sum_{i=1}^{n} e^{-\frac{r_i}{u}}A_i(u)$. If $H(u) = 0$ has no real roots, then we have

$$Y_c(u) = \frac{F_c(u) + \sum_{i=1}^{n}\Phi_i + \sum_{i=1}^{n}(\beta_i(u)+\mu_i(u))A_i(u)}{H(u)}. \tag{10}$$

In the following theorem we study the oscillation of delay differential equations Eq(4) by Sumudu transform method as follows.

Theorem 2.*Assume that the following conditions are satisfied*

$$\begin{aligned}
&a, a_1, a_2, a_3, \ldots, a_n, \text{ abscissas of convergence of } F(u) \\
&A_1(u), A_2(u), A_3(u)\ldots A_n(u) \text{ respectively, and} \\
&a > \max\{a_1, a_2, a_3, \ldots, a_n,\}\, F(u)) \text{ has asingularity on} \\
&Reu = a, \text{ but is analytic at } u = a,
\end{aligned} \tag{11}$$

$$H(u) = 0 \text{ has no real root on } [a, \infty). \tag{12}$$

Then every solution of Eq(4) is oscillatory.

*Proof.*Take a solution y of Eq(4); and in the contradictory, we assume that y is not oscillatory. Then there exists a sufficiently large $T > 0$ such that either $y(t) \geq 0$ or $y(t) \leq 0$ for $t > T$.

Now if we consider the case $y(t) \geq 0$ for $t > T$. (The case $y(t) \leq 0$ for $t > T$ can also be treated in a similar way). Let us take a number $c > T$ such that $y_c(t) \geq 0$ for $t > 0$, namely, the function $y_c(t)$ is a nonnegative function. Assume that $\frac{1}{b}$ is the convergence of $Y(u)$, so $Y_c(u)$ is analytic on the half-plane $Reu > \frac{1}{b}$. By Lemma 2, $Y_c(u)$ can not be analytically continued to the point $u = \frac{1}{b}$ from the right side since there is no complex neighborhood of b on which we can find an analytic function which agrees with $Y_c(u)$ for $Reu > \frac{1}{b}$. By assumptions Eq(11) and Eq(12), we see that the function on the right side of Eq(10) is analytic for $Reu > \max\left(a, \frac{1}{b}\right)$. If $a > \frac{1}{b}$, and in the view of Eq(11), $F(u)$ has a singularity on $Reu = a$, and $A_i(u)$, $i = 1,2,3,\ldots,n$, are analytic in $Reu \geq a$. Taking the Eq(12) into account, we see that $Y_c(u)$ has a singularity $Reu = a$, which contradicts that $Y_c(u)$ is analytic in $Reu > \frac{1}{b}$. If $a < \frac{1}{b}$, by Eq(11) and Eq(12), the function on the right side of Eq(10) is an analytic in the region $Reu > a$ and at $u = a$. This implies that $Y_c(u)$ is analytic even in the strip $a < Reu \leq \frac{1}{b}$. This is a contradiction. If $a = \frac{1}{b}$, by the assumptions Eq(11) and Eq(12), we see that the function on the right side of Eq(10) is analytic in $Reu = a$, but $Y_c(u)$ has a singularity at $Reu = a = \frac{1}{b}$, which is a contradiction. The proof is complete.

Theorem 3.*Assume that the following conditions are satisfied*

$a, a_1, a_2, a_3, \ldots, a_{n,}$ abscissas of convergence of $F(u)$, $A_1(u), A_2(u), A_3(u) \ldots A_n(u)$ respectively, there is an$i \in \{1,2,3,\ldots n\}$ such that $a_i > \max\{a, a_1, a_{i-1}, a_{i+1}, \ldots, a_n\}$. $A_i(u)$ has a singularity on $Reu = a_i$, but is analytic at $u = a_i$, (13)

$$H(u) = 0 \text{ has no real root on } [a_i, \infty). \tag{14}$$

ndent Then every solution of Eq(4) is oscillatory.

The proof is similar to the proof of Theorem 2.

Now we note that in Eq(4), if $a_i(t) = c_i w(t)$, $i = 1,2,...,n$, c_i are real numbers, then $a_i(t), (i = 1,2,\ldots,n)$, have the same abscissa of $\frac{1}{d}$. If $\frac{1}{d} > a$, where a is the abscissa of convergence of $Y(u)$, then it is not possible to apply Theorems 2 and 3. To cover the latter case, we have the following theorem.

Theorem 4.*Assume that the following conditions are satisfied*

a and $\frac{1}{d}$ are the abscissas of convergence of $F(u)$ and $D(u)$, and $\frac{1}{d} > a$ where $D(u)$ is the Sumudu transform of $w(t)$. $D(u)$ has a singularity on $Reu = \frac{1}{d}$, but is analytic at $u = \frac{1}{d}$. (15)

$$H(u) = 0 \text{ has no real root on } \left[\frac{1}{d}, \infty\right). \tag{16}$$

Then every solution of the Volterra integral equation

$$y(t) = f(t) + \int_0^t w(t-s) \sum_{i=1}^n c_i x(s - r_i) ds, \quad t \geq 0 \tag{17}$$

is oscillatory.

*Proof.*Since $a_i(t) = c_i w(t)$, we can easily see that Eq(10) has the following form

$$Y_c(u) = \frac{F_c(u) + \sum_{i=1}^n \Phi_i + D(u) \sum_{i=1}^n c_i (\beta_i(u) + \mu_i(u))}{H(u)} \tag{18}$$

where $H(u) = 1 - D(u) \sum_{i=1}^n c_i e^{-\frac{r_i}{u}}$. The rest of the proof is similar to the one of Theorem 2.

However, by the following example we show that for some Volterra integral equations, if Eq(15) or Eq(16), or both, are not true then all solutions of the equation do not need be oscillatory.

*Example 1.*Consider the Volterra integral equation

$$y(t) = 1 + \int_0^t 2y(s-1) ds, \quad t \geq 0 \tag{19}$$

by using Sumudu transform to Eq(19) we have

$$Y(u) = \frac{1 + 2e^{-\frac{1}{u}} \int_{-1}^0 e^{-\frac{t}{u}} y(t) dt}{1 - 2ue^{-\frac{1}{u}}}.$$

The abscissas of convergence of $F(u)$ and $D(u)$ are 0, namely, $a = \frac{1}{b} = 0$. Note that $D(u) = 2u$ is singular at $u = \infty$ and $D\left(\frac{1}{p}\right) = \frac{2}{p}$ is singular at the point $p = 0$. This means that Eq(15) is not satisfied. Furthermore

$$H\left(\frac{1}{p}\right) = 1 - \frac{2}{p} e^{-p} = \frac{p-2}{p} e^{-p} \quad at \ u = \frac{1}{p}$$

the function $L(p) = (p-2)e^{-p}$ has only one real root $\bar{p} \in [0, \infty)$. So Eq(16) does not hold. On the other hand, if we only consider the solutions of the delay differential equation

$$y'(t) - 2y(t-1) = 0$$

with the initial functions $\lambda \in \mathbb{C}([-1,0],\mathbb{R})$ and $\lambda(0)=1$ by using Sumudu transform we have

$$Y(u) = \frac{1+2e^{-\frac{1}{u}}\int_{-1}^{0} e^{-\frac{t}{u}} y(t)dt}{1-2ue^{-\frac{1}{u}}},$$

these solutions are also the solutions of the above Volterra integral equation. But it is clear that $y(t) = e^{t\bar{p}}$ is a nonoscillatory solution for this delay differential equation. So the Volterra integral equation has a nonoscillatory solution.

2 Polynomial approximations of the characteristic equation

Similar to the ordinary differential equations, several properties of delay differential equations can be characterized and examined by using the characteristic equations. For example, consider the first order delay differential equation given as follows.

$$\frac{dy}{dt} - ay = by(t-1). \tag{20}$$

Then applying the for example Laplace transform of both sides of Eq(20), we can obtain

$$\frac{Y(u)-y(0)}{u} - aY(u) = bS(\varphi) \tag{21}$$

where $\varphi(t)$ is an shifted initial function on $t \in [0,1)$ and y_0 is the initial condition at the point $t=0$. We note that y_0 can be different than the value $\lim_{t\to 1}\varphi(t))$. Here the Sumudu transform is defined as

$$S[f(t)] = \frac{1}{u}\int_0^{\infty} e^{-\frac{t}{u}} f(t)\,dt$$

on the finite time. Further, note that this is equivalent to applying the Sumudu transform on the function which is extended by zero on $(1,\infty)$. For the first interval $[0,1]$ the Sumudu transform of $y(t)$ can be expressed as

$$Y(u) = \frac{buS(\varphi)+y(0)}{1-au} \tag{22}$$

and can be calculated by evaluating the inverse Sumudu transform of $S(y(t))$ at $t=1$ (denoted by S_1^{-1})

$$y_1 = S^{-1}[S(y)](1) = S_1^{-1}[S(y)]. \tag{23}$$

By using the following notation

$$Y_0(u) = S(\varphi), \;\; X_1(u) = S(y), \tag{24}$$

equations Eq(22) and Eq(23) can be written as

$$\begin{aligned} Y_1(u) &= \frac{buY_0+y_0}{1-au}, \\ y_1 &= S_1^{-1}[Y_1(u)]. \end{aligned} \tag{25}$$

In general

$$\begin{aligned} Y_n(u) &= \frac{buY_{n-1}+y_{n-1}}{1-au}, \\ y_n &= S_1^{-1}[Y_n(u)]. \end{aligned} \tag{26}$$

Substituting Y_{n-1} into Eq(26) we have

$$Y_n(u) = \frac{bu\frac{buY_{n-2}+y_{n-2}}{1-au}+y_{n-1}}{1-au}. \tag{27}$$

The repeated applications of this procedure terminates at y_0 and one arrives at

$$\begin{aligned} Y_n(u) &= \frac{y_{n-1}}{1-au} + \frac{bu\,y_{n-2}}{(1-au)^2} + \frac{b^2u^2\,y_{n-3}}{(1-au)^3} + \dots \\ &+ \frac{b^{n-1}u^{n-1}\,y_0}{(1-au)^n} + \frac{b^{n-1}u^{n-1}\,Y_0(u)}{(1-au)^n} \\ &= \sum_{i=0}^{n-1} \frac{b^{n-i-1}u^{n-i-1}\,y_i}{(1-au)^{n-i}} + \frac{b^{n-1}u^{n-1}\,Y_0(u)}{(1-au)^n}. \end{aligned} \tag{28}$$

By using Eq(26), we have y_n in terms of $\{y_0, y_1, \dots, y_{n-1}\}$:

$$\begin{aligned} y_n &= S_1^{-1}\left(\sum_{i=0}^{n-1} \frac{b^{n-i-1}u^{n-i-1}\,y_i}{(1-au)^{n-i}}\right) \\ &\quad + S_1^{-1}\left(\frac{u^{n-1}\,Y_0(u)}{(1-au)^n}\right) b^{n-1}. \end{aligned} \tag{29}$$

On using the linearity of the inverse transform

$$\begin{aligned} y_n &\simeq \sum_{i=0}^{n-1} S^{-1}\left(\frac{u^{n-i-1}}{(1-au)^{n-i}}\right) b^{n-i-1}\,y_i \\ &= \sum_{i=0}^{n-1} \frac{t^{n-i-1}e^{at}}{(n-i-1)!} b^{n-i-1}\,y_i, \end{aligned} \tag{30}$$

at $t=1$, the Eq(30) becomes

$$y_n = \sum_{i=0}^{n-1} \frac{e^a}{(n-i-1)!} b^{n-i-1}\,y_i. \tag{31}$$

Here we neglected the term $S_1^{-1}\left(\frac{u^{n-1}\,Y_0(u)}{(1-au)^n}\right)$. The justification for this lies in the fact that stability should not depend on the form of the initial function, i.e. φ is chosen so as to make this term negligible. Since for any positive integer n the state y_n depends on all previous terms. Thus the characteristic equation of this map will be obtained by substituting $y_i = \lambda^i$ $(\lambda = 0)$ and $j = n-i-1$, so Eq(30), can be written in the form of

$$\lambda^n - \lambda^{n-1}e^a \sum_{i=0}^{n-1} \frac{\left(\frac{b}{\lambda}\right)^j}{j!} = 0. \tag{32}$$

The sum can be recognized as the exponential function

$$\sum_{i=0}^{n-1} \frac{\left(\frac{b}{\lambda}\right)^j}{j!} = e^{\frac{b}{\lambda}} \frac{\Gamma\left(n, \frac{b}{\lambda}\right)}{\Gamma(n)}, \tag{33}$$

by substituting Eq(33) into Eq(33), we have

$$f_n(\lambda) = \lambda^n - \lambda^{n-1} e^a e^{\frac{b}{\lambda}} \frac{\Gamma\left(n, \frac{b}{\lambda}\right)}{\Gamma(n)} = 0, \qquad (34)$$

as an nth order polynomial approximation to determine the stability of equation Eq(20). Since we replaced the original stability problem with that of a difference equation, the condition for stability now can be stated as $|\lambda| < 1$.

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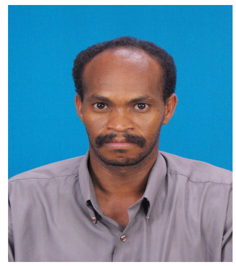

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