Applied Mathematics & Information Sciences
*An International Journal*

http://dx.doi.org/10.12785/amis/090448

# A Multi-Criteria Decision-Making Approach based on TODIM and Choquet Integral within a Multiset Hesitant Fuzzy Environment

*Juan-juan Peng*[1,2], *Jian-qiang Wang*[1,*], *Huan Zhou*[1] *and Xiao-hong Chen*[1]

[1] School of Business, Central South University, 410083 Changsha, China
[2] School of Economics and Management, Hubei University of Automotive Technology, 442002 Shiyan, China

Received: 18 Nov. 2014, Revised: 18 Feb. 2015, Accepted: 19 Feb. 2015
Published online: 1 Jul. 2015

**Abstract:** Hesitant fuzzy sets (HFSs), which were generalized from fuzzy sets, constrain the membership degree of an element to be a set of possible values between zero and one; furthermore if two or more decision-makers select the same value, it is only counted once. However, a situation where the evaluation value is repeated several times differs from one where the value appears only once. Multiset hesitant fuzzy sets (MHFSs) can deal effectively with a case where some values are repeated more than once in an HFS. In this paper, the new comparison method and corresponding distance of multiset hesitant fuzzy elements (MHFEs) are introduced. Then, based on the traditional TODIM and Choquet integral methods, a novel approach for multi-criteria group decision-making (MCGDM) problems, where the criteria are interdependent or interactive and the decision makers have a bounded rationality, is proposed for ranking alternatives. Finally, an example is provided in order to verify the developed approach and demonstrate its validity and feasibility. Furthermore, comparative analysis is presented by utilizing the same example as well.

**Keywords:** Multi-criteria group decision-making, multiset hesitant fuzzy sets, TODIM, Choquet integral

## 1 Introduction

In many cases, it is difficult for decision-makers to precisely express a preference when attempting to solve multi-criteria decision-making (MCDM) problems with inaccurate, uncertain or incomplete information. Zadehs fuzzy sets (FSs)[1], where the membership degree is represented by a real number between zero and one, are regarded as an important tool to solve not only MCDM problems [2,3] , but also for working with fuzzy logic and approximate reasoning [4], and pattern recognition [5]. However, a major drawback of FSs is that single values cannot convey information precisely.

In fact, the information regarding alternatives, when referring to a fuzzy concept, may be incomplete, i.e., the sum of the membership and non-membership degree of element in the universe can be less than one. The FS theory fails when it comes to managing the insufficient understanding of membership degrees. Thus, Atanassov's intuitionistic fuzzy sets (IFSs) and interval-valued intuitionistic fuzzy sets (IVIFSs), both extensions of Zadehs FSs, were introduced [6,7,8]. To date, IFSs and IVIFSs and their extensions have been widely applied in solving MCDM problems [9,10,11,12,13,14,15,16,17] ; however, in actual decision-making problems, the degrees in FSs, IFSs and IVIFSs can be a set of real numbers or intervals instead of only one.

To manage situations where people are hesitant in expressing their preference regarding the relevant objects in a decision-making process, hesitant fuzzy sets (HFSs), another extension of traditional FSs, provide a useful reference. HFSs were originally defined by Torra [18,19] and allow a membership degree to have different possible precise values between zero and one. Recently, HFSs have been the subject of a great deal of research and have been widely applied to MCDM or multi-criteria group decision-making (MCGDM) problems. For example, some work on the aggregation operators of HFSs have been undertaken [20,21,22,23,24,25,26] and the correlation coefficient, distance and correlation measures for HFSs were developed [27,28,29,30]. Furthermore, Zhang and Wei [31] developed the E-VIKOR method to

* Corresponding author e-mail: jqwang@csu.edu.cn

solve MCDM problems with HFSs. Zhang and Xu [32] proposed the TODIM method, which was based on measured functions with HFSs. Qian and Wang [33] generalized HFSs and utilized the aggregation operators to solve MCDM problems. Zhu et al. [34,35] proposed dual HFSs and outlined their operations and properties. However, in any associated distance measures, two hesitant fuzzy elements (HFEs) should be of equal length and must be arranged in ascending order. If the two HFEs being compared have different lengths, then the value of the shorter one should be increased until both are equal. In order to address these shortcomings, Wang et al. [36] proposed an outranking approach with HFSs to solve MCDM problems. Furthermore, Chen et al. [37] proposed interval-valued hesitant fuzzy sets (IVHFSs) and some aggregation operators and applied them to MCGDM problems. Wei and Zhao [38] introduced Einstein operations to IVHFSs and applied them to MCDM problems. Farhadinia [39] discussed the correlation for dual IVHFSs and Peng et al. [40] introduced a MCDM approach with hesitant interval-valued intuitionistic fuzzy sets (HIVIFSs), which is an extension of dual IVHFSs. Having reviewed the extant research, Rodriguez et al. [41] summarized the current state of, and proposed future directions for, HFSs. However, four main shortcomings of the existing methods of dealing with HFSs have emerged from the research to date. (1) The aggregation operators that are involved in those methods are related to different operations, which also lead to different rankings. Moreover, it is very difficult for decision-makers to confirm their judgments when using operators that have similar characteristics and which always need a large amount of computation. (2) Both distance measures and similarity measures should satisfy the condition that all HFEs must be arranged in ascending order and be of equal length as we discussed earlier. However, in such cases, different methods of extension could produce different results. (3) The existing comparison methods have certain problems when reflecting the preferences of decision-makers. (4) The existing methods do not clarify: how to solve a situation where there is a repeated value in the evaluation of alternatives; and in particular, whether decision makers can give more than one value (possible membership degrees of an element) for each criterion or not. At the same time, the situation where the evaluation value is repeated more than once is actually different from that where a value appears only once. For example, decision-makers may deem that the possible membership degrees by which an alternative is assessed against the criterion excellent are 0.5, 0.6 and 0.6, which is expressed by 0.5, 0.6 in the form of an HFE. However, the set of evaluation values 0.5, 0.6 are different from 0.5, 0.6, 0.6, which can lead to information loss in the data collection process. Fortunately, as they are generalized from HFSs, multiset hesitant fuzzy sets (MHFSs) can overcome these shortcomings and deal with the case where some values may be repeated more than once in an HFS. Furthermore, in those decision-making methods mentioned above, most of the criteria are assumed to be independent of one another. However in real life decision-making problems, the criteria of the problems are often interdependent or interactive. This phenomenon is referred to as correlated criteria in this paper. The Choquet integral [42] is a powerful tool for solving MCDM and MCGDM problems with correlated criteria and has been widely used for this purpose [43,44,45,46,47,48,49,50]. For example, Yager [43] extended the idea of order induced aggregation to the Choquet aggregation and introduced the induced Choquet ordered averaging (I-COA) operator. Meyer and Roubens [44] proposed the fuzzy extension of the Choquet integral and applied it to MCDM problems. Yu et al. [45] used the Choquet integral to propose a hesitant fuzzy aggregation operator and applied it to MCDM problems within a hesitant fuzzy environment. Tan and Chen [46] introduced the intuitionistic fuzzy Choquet integral operator. Tan [47] defined the Choquet integral-based Hamming distance between interval-valued intuitionistic fuzzy values and applied it to MCGDM problems. Bustince et al. [48] proposed a new MCDM method for interval-valued fuzzy preference relation, which was based on the definition of interval-valued Choquet integrals. Wei et al. [49] developed a generalized triangular fuzzy correlated averaging (GTFCA) operator based on the Choquet integral and OWA operator. Finally, Wang et al. [50] developed some Choquet integral aggregation operators with interval 2-tuple linguistic information and applied them to MCGDM problems.

A further problem is that the aforementioned MCDM and MCGDM methods that are used within a hesitant fuzzy environment are based on rational choices. However, a decision-maker is usually influenced by his or her personality, psychological state and risk preference as well as by environmental and other factors. Therefore in the actual decision-making process, decision-makers have a bounded rationality. As a means of overcoming this shortcoming, prospect theory (PT) was firstly developed by Kahneman and Tversky [51] in 1979. Subsequently cumulative prospect theory (CPT), which introduced capacity probability and contributed a solution for the problem of strong dominance and any number of outcomes not resolved by PT, was also developed by Kahneman and Tversky in 1992 [52]. To date, PT and CPT theories and applied research have been widely used in the asset pricing model [53], behavioral finance [54], tax decisions [55] and risk investment [56]. Subsequently, Gomes and Lima [57,58] proposed the TODIM (an acronym in Portuguese of interactive and multi-criteria decision-making) method based on PT, which could solve MCDM and MCGDM problems where the criteria values were in the form of precise values. More recently, the TODIM method has been widely applied to various fields, such as the evaluation of residential properties [59], the selection of natural gass destination [60], oil spills in the sea [61] and project investment [62].

However, TODIM also plays an important role in solving MCGDM problems where decision-makers have bounded rationality, whilst the Choquet integral has a critical role in handing MCGDM problems with correlated criteria. Therefore, developing a method of combining these two methods in order to solve multiset hesitant fuzzy MCGDM problems with correlated criteria is seen as a valuable research topic.

The rest of this paper is organized as follows. In Section 2, the Choquet integral is reviewed and a definition, as well as the properties, of HFSs is provided. Moreover, the accuracy function and comparison method are also introduced. Section 3 provides a definition of MHFSs and introduces a related operation and novel comparison method. In Section 4 the extended TODIM method based on the Choquet integral within a multiset hesitant fuzzy environment is developed and applied to MCGDM problems. In Section 5 an example to illustrate the practical application of the developed approach is provided. Finally, some conclusions are drawn in Section 6.

## 2 Preliminaries

In this section, fuzzy measure, the Choquet integral and the definition of HFSs are reviewed. Some operations and comparison laws of HFSs, which will be utilized in the latter analysis, are also presented.

### *2.1 Fuzzy measure and the Choquet integral*

Let $X=\{x_1,x_2,\dots,x_n\}$ be the set of the criteria, $P(X)$ be the power set of $X$, then the fuzzy measure $\mu$ is defined as follows.

**Definition 1[63].** A fuzzy measure $\mu$ on the set $X$ is a set function $\mu: P(X)\to[0,1]$ and satisfies the following axioms:
(1) $\mu(\varnothing)=0, \mu(X)=1$;
(2) $if B\subseteq C\subseteq X$, then $\mu(B)\le\mu(C)$;
(3) $\mu(B\cup C)=\mu(B)+\mu(C)+\rho(B)\mu(C)$, for $\forall B,C\subseteq X, B\cup C=\varnothing$, where $\rho\in(-1,+\infty)$.
In Definition 1, if $\rho=0$, then the third condition is reduced to the additive measure:
for $\forall B,C\subseteq X$, and $B\cap C=\varnothing, \mu(B\cup C)=\mu(B)+\mu(C)$.
If the elements of $B$ are independent, then for $\forall B\subseteq X$,

$$\mu(B)=\sum_{x_i\in B}\mu(x_i). \quad (1)$$

In Definition 1, if $\rho=0$, then the fuzzy measure is a probability measure and the elements are independent; if $-1<\rho<0$, then a redundant relation exists among elements; if $\rho>0$, then a complementary relation exists among elements.

**Definition 2[42].** Let $\mu$ be a fuzzy measure on $(X,P(X)), f:X\to[0,+\infty)$, then the Choquet integral $f$ on with respect to $\mu$ can be defined as follows:

$$\int_X f d\mu=\int_0^{+\infty}\mu(\{x:f(x)>t\})dt,$$

where $\{x:f(x)>t\}\in P(X)$ for $\forall t\in R^+$. If $X=\{x_1,x_2,\dots,x_n\}$ is a finite set, then the discrete Choquet integral can be described as:

$$\int_X f d\mu=\sum_1^n f(x_{\sigma(i)})(\mu(A_{\sigma(i)}-\mu(A_{\sigma(i+1)})), \quad (2)$$

or

$$\int_X f d\mu=\sum_1^n (f(x_{\sigma(i)})-f(x_{\sigma(i-1)}))\mu(A_{\sigma(i)}). \quad (3)$$

Where $(\sigma(1),\sigma(2),\dots,\sigma(n))$ is a permutation of $(1,2,\dots,n)$, such that $0\le f(x_{\sigma(1)})\le f(x_{\sigma(2)})\le\dots\le f(x_{\sigma(n)}), f(x_{\sigma(0)})=0, A_{\sigma(i)}=\{x_{\sigma(i)},x_{\sigma(i+1)},\dots,x_{\sigma(n)}\}$ and $\mu(A_{\sigma(n+1)})=0$.

**Example 1.** Let $X=\{x_1,x_2,x_3\}, x_1<x_2<x_3$, and $f(x)=2^x$, then $f(x_1)<f(x_2)<f(x_3)$, so $\sigma(1)=1,\sigma(2)=2,\sigma(3)=3, A_1=\{x_1,x_2,x_3\}, A_2=\{x_2,x_3\}, A_3=\{x_3\}$. Suppose $\mu(x_1)=0.3,\mu(x_2)=0.25,\mu(x_3)=0.37,\mu\{x_1,x_2\}=0.52,\mu\{x_1,x_3\}=0.65,\mu\{x_2,x_3\}=0.45,\mu\{x_1,x_2,x_3\}=1$; if they are calculated by using Eq. (3), then the following is obtained:
$\int_X f d\mu=(f(x_1)-f(x_{\sigma(0)}))\mu(A_1)+(f(x_2)-f(x_1))\mu(A_2)+(f(x_3)-f(x_2))\mu(A_3)=(2^{x_1}-0)\times 1+(2^{x_2}-2^{x_1})\times 0.45+(2^{x_3}-2^{x_2})\times 0.37$.
If $x_1=1,x_2=2,x_3=3$, then we have $\int_X f d\mu=4.38$.

### *2.2 HFSs and their operations*

**Definition 3[18,19].** Let $X$ be a universal set; an HFS on $X$ is in terms of a function that when applied to $X$ returns a subset of $[0,1]$, which can be represented as follows:

$$E=\{\langle x,h_E(x)\rangle|x\in X\}, \quad (4)$$

where $h_E(x)$ is a set of values in $[0,1]$, denoting the possible membership degrees of the element $x\in X$ to the set $E$. $h_E(x)$ is called a hesitant fuzzy element (HFE) [20], and $E$ is the set of all HFEs.

Torra [18,19] defined some operations on HFEs, and Xia and Xu [20] defined some new operations on HFEs as well as the score functions.

**Definition 4[20].** Let $h_1,h_2$ and $h$ be three HFEs, $\lambda\ge 0$, and four associated operations can be defined as follows:
(1) $\bigoplus$-union: $h_1\bigoplus h_2=\cup_{\gamma_1\in h_1,\gamma_2\in h_2}\{\gamma_1+\gamma_2-\gamma_1\times\gamma_2\}$;
(2) $\bigotimes$-intersection: $h_1\bigotimes h_2=\cup_{\gamma_1\in h_1,\gamma_2\in h_2}\{\gamma_1\times\gamma_2\}$;
(3) Multiplication: $\lambda h=\cup_{\gamma\in h}\{1-(1-\gamma)_\lambda\}$;
(4) Exponentiation: $h_\lambda=\cup_{\gamma\in h}\{\gamma_\lambda\}$.

**Example 2.** Let $h_1=\{0.1,0.2\}$ and $h_2=\{0.1,0.3,0.5\}$ be two HFEs, $\lambda=2$, and then we could get:
(1) $h_1^2=\{0.1^2,0.2^2\}=\{0.01,0.04\}$;
(2) $2h_1=\{0.19,0.36\}$;
(3) $h_1\bigoplus h_2=\{0.28,0.37,0.55,0.44,0.60\}$;
(4) $h_1\bigotimes h_2=\{0.02,0.03,0.05,0.06,0.1\}$.

**Definition 5[20].** For a HFE $h$, $s(h)=\frac{1}{l(h)}\sum_{\gamma\in h}\gamma$ is called the score function of $h$, where $l(h)$ is the number of elements in $h$. For two HFEs $h_1$ and $h_2$, if $s(h_1)>s(h_2)$, then $h_1>h_2$; if $s(h_1)=s(h_2)$, then $h_1=h_2$.

The shortcoming of using Definition 5 when comparing two HFEs, is illustrated in the following example.

**Example 3.** Let $h_1=\{0.5\}, h_2=\{0.2,0.8\}$ and $h_3=\{0.2,0.5,0.8\}$ be two HFEs. It becomes clear that $h_1\neq h_2\neq h_3$. However, by applying Definition 5, $s(h_1)=s(h_2)=s(h_3)$ can be obtained, and thus $h_1=h_2=h_3$, which is contradictory to our intuition.

Moreover, it is noted that different aggregation operators based on different operations could lead to different aggregation results, which also lead to different rankings [36].

In order to overcome this counterintuitive problem, Farhadinia [64] defined a new score function.

**Definition 6[64].** Let $h=\cup_{\gamma\in h}=\{\gamma_j|j=1,2,\ldots,l(h)\}$ be an HFE, where $l(h)$ is the number of elements in $h$. Then the score function of $h$ is defined as

$$S(h)=\frac{\sum_{j=1}^{l(h)}\delta(j)\gamma_j}{\sum_{j=1}^{l(h)}\delta(j)}, \tag{5}$$

where $\{\delta(j)|j=1,2,\ldots,l(h)\}$ is a positive-valued monotonic increasing sequence of the index $j$.

Compared to the score function in Ref. [20], the new score function can overcome the counterintuitive problem. However, this new score function was always defined based on the assumption that: the values in the concerned HFEs are arranged in an ascending order; and if two HFEs are not of equal length, then the shorter one should be extended by adding its largest number until both HFEs are the same. Therefore, this extension method has the same drawback to those approaches discussed earlier.

**Definition 7[65].** Let $h_1$ and $h_2$ be two HFEs on X, and then the following comparison method exists:

$h_1\leq h_2$ iff $\gamma_{h_1}^{\sigma(j)}\leq\gamma_{h_2}^{\sigma(j)}, 1\leq j\leq l_{h_i}$. Note that all elements in HFEs are arranged in ascending order, and $\gamma_{h_1}^{\sigma(j)}$ is referred to as the $j$-th largest value in $h_1$. Two HFEs $h_1$ and $h_2$ should have the same length. If there are fewer elements in $h_1$ than in $h_2$, an extension of $h_1$ can be created by subjectively repeating its maximum element until it is of equal length to $h_2$.

However, if HFEs are extended in the way outlined above, the initial evaluation values of decision-makers will be changed.

## 3 MHFSs and their operations

In this section, the definition of MHFSs, along with some associated operations and a novel comparison method, is introduced.

**Definition 8[18].** Let $X$ be a universal set; an MHFS on $X$ is in terms of a function that returns a multi-subset of $[0,1]$ when applied to $X$. It can be represented in the following way:

$$E=\{\langle x,H_E(x)\rangle|x\in X\}, \tag{6}$$

where $H_E(x)$ is a set of values in $[0,1]$, denoting the set of the possible membership degrees of the element $x\in X$ to the set $E$. In any $H_E(x)$, the values are allowed to be repeated several times. $H_E(x)$ is called a multiset hesitant fuzzy element (MHFE), and $E$ is the set of all HFEs.Seemingly, any HFS is a special case of a MHFS. Apparently, the operations on HFEs in Definition 4 can also be suited for MHFEs.

**Example 4.** Let $H_A=\{0.1,0.2,0.1,0.3\}$ and $H_B=\{0.2,0.3,0.3\}$ be two MHFEs and $\lambda=2$. According to Definition 4,the following results can be obtained:

(1) $H_A^2=\{0.1^2,0.2^2,0.1^2,0.3^2\}=\{0.01,0.04,0.01,0.09\}$;

(2) $2H_A=\{0.19,0.36,0.19,0.51\}$;

(3) $H_A\bigoplus H_B=\{0.28,0.37,0.44,0.51,0.37,0.51,0.28,0.37,0.36,0.44,0.37,0.44\}$;

(4)$H_A\bigotimes H_B=\{0.02,0.03,0.04,0.06,0.03,0.06,0.02,0.03,0.06,0.09,0.03,0.09\}$.

**Definition 9.** For a MHFE $H_A$, $a(H_A)=\frac{1}{l(H_A)-1}\sum_{\gamma_{H_A}\in H_A}(s_{H_A}-\gamma_{H_A})^2$ can be defined as an accuracy function of $H_A$. Where $s_{H_A}$ is the score function defined in Definition 5 and $l(H_A)$ is the number of elements in $H_A$.

The accuracy function is similar to the sample variance in statistics and can reflect the fluctuation of evaluation values of MHFEs; the greater the amplitude of fluctuation is, the larger the hesitant degree. Then the ranking of any two MHFEs can be obtained by combining the score function and the accuracy function.

Based on Definitions 5, 7 and 9, some new comparison methods for MHFEs are defined as follows.

**Definition 10.** Let $H_A$ and $H_B$ be two MHFEs on X, all elements in MHFEs be arranged in ascending order, and $\gamma_{H_A}^{\sigma(j)}$ and $\gamma_{H_B}^{\sigma(j)}$ be referred to as the $j$-th largest value in $H_A$ and $H_B$ respectively. Then the following comparison methods can be given.

(1) $H_A\leq H_B$ iff $\gamma_{H_A}^{\sigma(j)}\leq\gamma_{H_B}^{\sigma(j)}$ and $\gamma_A^{\sigma(l_{H_A})}\leq\gamma_B^{\sigma(l_{H_B})}$, where $\gamma_{H_A}^{\sigma(j)}\in H_A, \gamma_{H_B}^{\sigma(j)}\in H_B, j=1,2,\ldots,l_H$, and $l_H=\min(l_{H_A},l_{H_B})$ ($l_{H_A}$ and $l_{H_B}$ represent the number of elements in $H_A$ and $H_B$ respectively);

(2) $H_A=H_B$ iff $H_A\leq H_B$ and $H_B\leq H_A$;

(3) $H_A\nleq H_B$ and $H_A\prec H_B$ iff $s(H_A)<s(H_B)$ or iff $s(H_A)=s(H_B)$ and $a(H_A)>s(H_B)$.

Here $s(\cdot)$ and $a(\cdot)$ respectively represent the score function referred to in Definition 5 and the accuracy function referred to in Definition 9. Note that $\prec$ means inferior to. Apparently, if $H_A<H_B$, then $H_A\prec H_B$.

**Example 5.** Let $H_A=\{0.2,0.4,0.4,0.6\}$ and

$H_B = \{0.3, 0.4, 0.5\}$ be two MHFEs.
(1) By applying the score function in Definition 5, $s(H_A) = 0.4$ and $s(H_B) = 0.4$. Here $s(H_A) = s(H_B)$, and thus $H_A = H_B$.
(2) According to the novel score function in Definition 6, $S(H_A) = 0.46$ and $S(H_B) = 0.46$, where $H_B$ becomes $\{0.3, 0.4, 0.5, 0.5\}$ as required. Then $S(H_A) = S(H_B)$ can be obtained, and thus $H_A = H_B$.
(3) According to the comparison method in Definition 7, we have $\gamma_{H_A}^{\sigma(4)} \nleq \gamma_{H_B}^{\sigma(4)}$ and $H_A \nleq H_B$, where $H_B$ becomes $\{0.3, 0.4, 0.5, 0.5\}$ as required.
(4) According to the proposed comparison method in Definition 10, we have $l_H = \min(l_{H_A}, l_{H_B}) = l_{H_B} = 3$ and $\gamma_{H_A}^{\sigma(1)} \leq \gamma_{H_B}^{\sigma(1)}, \gamma_{H_A}^{\sigma(2)} \leq \gamma_{H_B}^{\sigma(2)}$ and $\gamma_{H_A}^{\sigma(3)} \leq \gamma_{H_B}^{\sigma(3)}$, but $\gamma^{\sigma(l_{H_A}))A \nleq \gamma^{\sigma(l_{H_B}))B}}$, i.e., $0.6 \nleq 0.5$, so we have $H_A \nleq H_B$. Then $a(H_A) = 0.0267$, and $a(H_B) = 0.01$. Here $a(H_A) > s(H_B)$, and thus $H_A \prec H_B$.

Apparently, if the comparison methods in Definition 7 are utilized by subjectively adding the maximum element to MHFEs, then the initial evaluation values of decision-makers will be changed and at the same time it will influence the final ranking. However, the proposed comparison method could overcome these shortcomings and is also suitable for HFSs.

**Definition 11.** If $A = \{\langle x, H_A(x)\rangle | x \in X\}$ and $B = \{\langle x, H_B(x)\rangle | x \in X\}$ are two MHFSs on $X = \{x_1, x_2, \ldots, x_n\}$, then $A \leq B$ iff $\forall x_i \in X, H_A(x_i) \leq H_B(x_i)$.

**Example 6.**

Let $A = \{\langle x_1, \{0.1, 0.2, 0.4\}\rangle, \langle x_2, \{0.2, 0.3\}\rangle\}$ and $B = \{\langle x_1, \{0.1, 0.2, 0.5\}\rangle, \langle x_2, \{0.3, 0.3\}\rangle\}$ be two MHFSs. Based on Definition 10, $H_A(x_1) = \{0.1, 0.2, 0.4\} \leq H_B(x_1) = \{0.1, 0.2, 0.5\}$ and $H_A(x_2) = \{0.2, 0.3\} \leq H_B(x_2) = \{0.3, 0.3\}$ can be obtained. So $A \leq B$.

**Definition 12.** Let $A = \{\langle x, H_A(x)\rangle | x \in X\}$, $B = \{\langle x, H_B(x)\rangle | x \in X\}$ and $C = \{\langle x, H_C(x)\rangle | x \in X\}$ be three MHFSs on $X$, and $\widetilde{d}$ represent the distance of two MHFSs if it satisfies the following conditions:
(1) $0 \leq \widetilde{d}(A, B) = 1$;
(2) $\widetilde{d}(A, B) = \widetilde{d}(B, A)$;
(3) $\widetilde{d}(A, B) = 0$ iff $A = B$;
(4) If $A \leq B \leq C$, then $\widetilde{d}(A, B) \leq \widetilde{d}(A, C)$ and $\widetilde{d}(B, C) \leq \widetilde{d}(A, C)$.

**Definition 13.** If $A = \{\langle x, H_A(x)\rangle | x \in X\}$ and $B = \{\langle x, H_B(x)\rangle | x \in X\}$ are two MHFSs on $X = \{x_1, x_2, \ldots, x_n\}$, then a generalized multiset hesitant normalized distance between $A$ and $B$ could be defined as follows:

$$\widetilde{d}(A, B) = [\frac{1}{n}\sum_{i=1}^{n}\frac{1}{2}(\frac{1}{l_{H_A(x_i)}}\sum_{\gamma_A(x_i)\in H_A(x_i)}\min_{\gamma_B(x_i)\in H_B(x_i)}|\gamma_A(x_i) - \gamma_B(x_i)|^\rho + \frac{1}{l_{H_B(x_i)}}\sum_{\gamma_B(x_i)\in H_B(x_i)}\min_{\gamma_A(x_i)\in H_A(x_i)}|\gamma_B(x_i) - \gamma_A(x_i)|^\rho)]^{\frac{1}{\rho}}. \tag{7}$$

Here $l_{H_A(x_i)}$ and $l_{H_B(x_i)}$ denotes the number of elements in $H_A(x_i)$ and $H_B(x_i)$ respectively. Especially, a generalized multiset hesitant distance of two MHFEs $H_A(x_i)$ and $H_B(x_i)$ can be denoted as below:

$$\widetilde{d}(H_A(x_i), H_B(x_i)) = [\frac{1}{2}(\frac{1}{l_{H_A(x_i)}}\sum_{\gamma_A(x_i)\in H_A(x_i)}\min_{\gamma_B(x_i)\in H_B(x_i)}|\gamma_A(x_i) - \gamma_B(x_i)|^\rho + \frac{1}{l_{H_B(x_i)}}\sum_{\gamma_B(x_i)\in H_B(x_i)}\min_{\gamma_A(x_i)\in H_A(x_i)}|\gamma_B(x_i) - \gamma_A(x_i)|^\rho)]^{\frac{1}{\rho}}. \tag{8}$$

If $H_A(x_i) = \varnothing$ or $H_B(x_i) = \varnothing$ for all elements $x_i \in X$, then $\widetilde{d}(A, B) = [\frac{1}{n}\sum_{i=1}^{n}(\frac{1}{l_{H_B(x_i)}}\sum_{\gamma_B(x_i)\in H_B(x_i)}\gamma_B(x_i)^\rho)]^{\frac{1}{\rho}}$ or $\widetilde{d}(A, B) = [\frac{1}{n}\sum_{i=1}^{n}(\frac{1}{l_{H_A(x_i)}}\sum_{\gamma_A(x_i)\in H_A(x_i)}\gamma_A(x_i)^\rho)]^{\frac{1}{\rho}}$.
(1) Apparently, if $\rho = 1$, then Eq.(7) is reduced to the multiset hesitant normalized Hamming-Hausdorff distance between $A$ and $B$:

$$\widetilde{d}(A, B) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{2}(\frac{1}{l_{H_A(x_i)}}\sum_{\gamma_A(x_i)\in H_A(x_i)}\min_{\gamma_B(x_i)\in H_B(x_i)}|\gamma_A(x_i) - \gamma_B(x_i)| + \frac{1}{l_{H_B(x_i)}}\sum_{\gamma_B(x_i)\in H_B(x_i)}\min_{\gamma_A(x_i)\in H_A(x_i)}|\gamma_B(x_i) - \gamma_A(x_i)|). \tag{9}$$

(2) If $\rho = 2$, then Eq.(7) is reduced to the multiset hesitant normalized Euclidean-Hausdorff distance between $A$ and $B$:

$$\widetilde{d}(A, B) = [\frac{1}{n}\sum_{i=1}^{n}\frac{1}{2}(\frac{1}{l_{H_A(x_i)}}\sum_{\gamma_A(x_i)\in H_A(x_i)}\min_{\gamma_B(x_i)\in H_B(x_i)}|\gamma_A(x_i) - \gamma_B(x_i)|^2 + \frac{1}{l_{H_B(x_i)}}\sum_{\gamma_B(x_i)\in H_B(x_i)}\min_{\gamma_A(x_i)\in H_A(x_i)}|\gamma_B(x_i) - \gamma_A(x_i)|^2)]^{\frac{1}{2}}. \tag{10}$$

**Proposition 1.** Eq. (7) satisfies all the conditions in Definition 11.

**Proof.** (1) Since $x_i \in X, \gamma_A(x_i) \in H_A(x_i), \gamma_B(x_i) \in H_B(x_i), 0 \leq |\gamma_A(x_i) - \gamma_B(x_i)| \leq 1$, then $0 \leq |\gamma_A(x_i) - \gamma_B(x_i)|^\rho \leq 1, 0 \leq \min_{\gamma_B(x_i)\in H_B(x_i)}|\gamma_A(x_i) - \gamma_B(x_i)|^\rho \leq 1$, $0 \leq \sum_{\gamma_A(x_i)\in H_A(x_i)}\min_{\gamma_B(x_i)\in H_B(x_i)}|\gamma_A(x_i) - \gamma_B(x_i)|^\rho \leq l_{H_A(x_i)}$, $0 \leq \frac{1}{l_{H_A(x_i)}}\sum_{\gamma_A(x_i)\in H_A(x_i)}\min_{\gamma_B(x_i)\in H_B(x_i)}|\gamma_A(x_i) - \gamma_B(x_i)|^\rho \leq 1$.
Similarly, $0 \leq \frac{1}{l_{H_B(x_i)}}\sum_{\gamma_B(x_i)\in H_B(x_i)}\min_{\gamma_A(x_i)\in H_A(x_i)}|\gamma_B(x_i) - \gamma_A(x_i)|^\rho \leq 1$.

So $0 \le \frac{1}{l_{H_A(x_i)}} \sum_{\gamma_A(x_i)\in H_A(x_i)} \min_{\gamma_B(x_i)\in H_B(x_i)} |\gamma_A(x_i)-\gamma_B(x_i)|^\rho + \frac{1}{l_{H_B(x_i)}} \sum_{\gamma_B(x_i)\in H_B(x_i)} \min_{\gamma_A(x_i)\in H_A(x_i)} |\gamma_B(x_i)-\gamma_A(x_i)|^\rho \le 2, 0 \le \frac{1}{2}(\frac{1}{l_{H_A(x_i)}} \sum_{\gamma_A(x_i)\in H_A(x_i)} \min_{\gamma_B(x_i)\in H_B(x_i)} |\gamma_A(x_i)-\gamma_B(x_i)|^\rho + \frac{1}{l_{H_B(x_i)}} \sum_{\gamma_B(x_i)\in H_B(x_i)} \min_{\gamma_A(x_i)\in H_A(x_i)} |\gamma_B(x_i)-\gamma_A(x_i)|^\rho) \le 1$.

Thus $0 \le \frac{1}{n}\sum_{i=1}^n \frac{1}{2}(\frac{1}{l_{H_A(x_i)}} \sum_{\gamma_A(x_i)\in H_A(x_i)} \min_{\gamma_B(x_i)\in H_B(x_i)} |\gamma_A(x_i)-\gamma_B(x_i)|^\rho + \frac{1}{l_{H_B(x_i)}} \sum_{\gamma_B(x_i)\in H_B(x_i)} \min_{\gamma_A(x_i)\in H_A(x_i)} |\gamma_B(x_i)-\gamma_A(x_i)|^\rho)) \le 1$ i.e., $0 \le [\frac{1}{n}\sum_{i=1}^n \frac{1}{2}(\frac{1}{l_{H_A(x_i)}} \sum_{\gamma_A(x_i)\in H_A(x_i)} \min_{\gamma_B(x_i)\in H_B(x_i)} |\gamma_A(x_i)-\gamma_B(x_i)|^\rho + \frac{1}{l_{H_B(x_i)}} \sum_{\gamma_B(x_i)\in H_B(x_i)} \min_{\gamma_A(x_i)\in H_A(x_i)} |\gamma_B(x_i)-\gamma_A(x_i)|^\rho))]^{\frac{1}{\rho}} \le 1$.

So $0 \le \widetilde{d}(A,B) \le 1$.

(2) $\widetilde{d}(A,A) = 0$ can be obtained.

(3) Clearly, $\widetilde{d}(A,B) = \widetilde{d}(B,A)$.

(4) For any three MHFSs, If $A \le B \le C, \forall x_i \in X, \gamma_A(x_i) \in H_A(x_i), \gamma_B(x_i) \in H_B(x_i), \gamma_C(x_i) \in H_C(x_i)$, then $\gamma_A^{\sigma(j)}(x_i) \le \gamma_B^{\sigma(j)}(x_i) \le \gamma_C^{\sigma(j)}(x_i)$, and $\gamma_A^{\sigma(l_{H_A(x_i)})}(x_i) \le \gamma_B^{\sigma(l_{H_B(x_i)})}(x_i) \le \gamma_C^{\sigma(l_{H_C(x_i)})}(x_i)$ is obtained according to Definition 10. Thus, it can be seen that $\gamma_C(x_i)-\gamma_A(x_i) \ge \gamma_B(x_i)-\gamma_A(x_i) \ge 0, |\gamma_C(x_i)-\gamma_A(x_i)| \ge |\gamma_B(x_i)-\gamma_A(x_i)|$. Therefore $|\gamma_A(x_i)-\gamma_C(x_i)| \ge |\gamma_A(x_i)-\gamma_B(x_i)|$, $\min_{\gamma_C(x_i)\in H_C(x_i)} |\gamma_A(x_i)-\gamma_C(x_i)|^\rho) \ge \min_{\gamma_B(x_i)\in H_B(x_i)} |\gamma_A(x_i)-\gamma_B(x_i)|^\rho), \frac{1}{l_{H_A(x_i)}} \sum_{\gamma_A(x_i)\in H_A(x_i)} \min_{\gamma_C(x_i)\in H_C(x_i)} |\gamma_A(x_i)-\gamma_C(x_i)|^\rho) \ge \frac{1}{l_{H_A(x_i)}} \sum_{\gamma_A(x_i)\in H_A(x_i)} \min_{\gamma_B(x_i)\in H_B(x_i)} |\gamma_A(x_i)-\gamma_B(x_i)|^\rho)$.

Similarly, $\frac{1}{l_{H_C(x_i)}} \sum_{\gamma_C(x_i)\in H_C(x_i)} \min_{\gamma_A(x_i)\in H_A(x_i)} |\gamma_C(x_i)-\gamma_A(x_i)|^\rho) \ge \frac{1}{l_{H_B(x_i)}} \sum_{\gamma_B(x_i)\in H_B(x_i)} \min_{\gamma_A(x_i)\in H_A(x_i)} |\gamma_B(x_i)-\gamma_A(x_i)|^\rho)$.

Thus, $\widetilde{d}(A,C) = \widetilde{d}(A,B)$. It can be proved that $\widetilde{d}(A,C) = \widetilde{d}(B,C)$.

## 4 The extended TODIM method based on the Choquet integral for MCGDM with MHFEs

In this section, the extended TODIM method based on the Choquet integral is proposed in order to solve MCGDM problems within a multiset hesitant fuzzy environment.

The MCGDM ranking/selection problems with multiset hesitant fuzzy information consists of a group of alternatives, denoted by $A = \{a_1, a_2, \ldots, a_n\}$. The alternatives could be of any type, and each alternative is evaluated based on the criteria denoted by $C = \{c_1, c_2, \ldots, c_n\}$. $a_{ij}$ is the value of the alternative $a_i$ for the criterion $c_j$, and $a_{ij} = \{\gamma_{ij}^k | k = 1,2,\ldots,l(a_{ij})\} (i = 1,2,\ldots,n; j = 1,2,\ldots,m\}$ are in the form of MHFEs, which are given by several decision-makers. Furthermore $l(a_{ij})$ represents the number of elements in $a_{ij}$ and the corresponding weight vector $w = (w_1, w_2, \ldots, w_m)$. This method is suitable if the number of decision-makers is small. A situation could arise where decision-makers evaluate these alternatives based on the given criteria, and one decision-maker could give several evaluation values. In particular, in the case where two or more decision-makers give the same value, it is counted repeatedly. $a_{ij}$ is the set of evaluation values for all decision-makers. The approach is an integration of MHFSs and TODIM based on the Choquet integral to solve MCGDM problems mentioned above.

Step 1. Normalize the decision matrix.

For MCGDM problems, the most common criteria are of maximizing and minimizing types. In order to unify all criteria, it is necessary to normalize the evaluation values. (Note: if all the criteria are of the maximizing type and have the same measurement unit, then there is no need to normalize them). Suppose that the matrix $R = (a_{ij})_{\{}n \times m\}$, where $a_{ij} = \{\gamma_{ij}^1, \gamma_{ij}^2, \ldots, \gamma_{ij}^k\} (i = 1,2,\ldots,n; j = 1,2,\ldots,m; k = 1,2,\ldots,l(a_{ij}))$ are MHFEs, is normalized into the corresponding matrix $\widetilde{R} = (\widetilde{a}_{ij})_{\{}n \times m\}$, where $\widetilde{a}_{ij} = \{\widetilde{\gamma}_{ij}^1, \widetilde{\gamma}_{ij}^2, \ldots, \widetilde{\gamma}_{ij}^k\} (i = 1,2,\ldots,n; j = 1,2,\ldots,m; k = 1,2,\ldots,l(a_{ij}))$. $l(a_{ij})$ is the number of the elements of $a_{ij}$.

For the maximizing criteria, the normalization formula is

$$\widetilde{\gamma}_{ij}^k = \gamma_{ij}^k, k = 1,2,\ldots,l(a_{ij}); \quad (11)$$

for the minimizing criteria,

$$\widetilde{\gamma}_{ij}^k = 1-\gamma_{ij}^k, k = 1,2,\ldots,l(a_{ij}); \quad (12)$$

Seemingly, the normalization values $\widetilde{a}_{ij} = \{\widetilde{\gamma}_{ij}^1, \widetilde{\gamma}_{ij}^2, \ldots, \widetilde{\gamma}_{ij}^k\} (i = 1,2,\ldots,n; j = 1,2,\ldots,m; k = 1,2,\ldots,l(a_{ij}))$ are also MHFEs.

Step 2. Confirm the fuzzy measures of the criteria of C and the criteria sets of C.

Based on these fuzzy measures, the corresponding the weight of criteria can be obtained as follows:

$$w_{\sigma(j)} = \mu(A_{\sigma(j)}) - \mu(A_{\sigma(j+1)}), j = 1,2,\ldots,m. \quad (13)$$

Here $A_{\sigma(j)} = \{c_{\sigma(j)}, c_{\sigma(j+1)}, \ldots, c_{\sigma(m)}, c_{\sigma(m)\}=\varnothing}$,and $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1,2,\ldots,n)$. $w_{lj} = \frac{w_{\sigma(j)}}{w_{\sigma(l)}} (j = 1,2,\ldots,m)$ is the weight of the criterion $c_{\sigma(j)}$ to the reference criterion $c_l$ and $w_{\sigma(l)} = w_{\sigma(m)}$.

Step 3. Calculate the dominance degree.

The dominance degree of the alternative $a_i$ over alternative $a_k$ concerning the criterion $c_j$ can be calculated using the following expression:
$\Phi_j(a_i,a_k) =$

$$\begin{cases} \sqrt[\lambda]{\frac{(\widetilde{d}(\widetilde{a}_{ij},\widetilde{a}_{kj}))^{\lambda} w_{lj}}{\sum_{j=1}^{n} w_{lj}}}, & \widetilde{a}_{ij} \succ \widetilde{a}_{kj} \text{ or } \widetilde{a}_{ij} > \widetilde{a}_{kj} \\ 0, & \widetilde{a}_{ij} > \widetilde{a}_{kj} \\ -\frac{1}{\theta}\sqrt[\lambda]{\frac{(\widetilde{d}(\widetilde{a}_{ij},\widetilde{a}_{kj}))^{\lambda} \sum_{j=1}^{n} w_{lj}}{w_{lj}}}. & \widetilde{a}_{ij} \prec \widetilde{a}_{kj} \text{ or } \widetilde{a}_{ij} < \widetilde{a}_{kj} \end{cases} \quad (14)$$

Here $\lambda \geq 1$ is regulating variable and can be determined according to the decision-makers preference. $\widetilde{d}(\widetilde{a}_{ij},\widetilde{a}_{kj})$ denotes the distance between two MHFEs $\widetilde{a}_{ij}$ and $\widetilde{a}_{kj}$ as defined in Definition 13. $\widetilde{a}_{ij}$ and $\widetilde{a}_{kj}$ can be compared by utilizing the ranking method of MHFEs in Definition 10. Furthermore, (1) if $\widetilde{a}_{ij} \succ \widetilde{a}_{kj}$, then it will represent the gain of $a_i$ over $a_k$ concerning the criterion $c_j$. (2) If $\widetilde{a}_{ij} = \widetilde{a}_{kj}$, then $\Phi_j(a_i,a_k)$ is nil. (3) If $\widetilde{a}_{ij} \prec \widetilde{a}_{kj}$ , then it will represent the loss of $a_i$ over $a_k$ concerning the criterion $c_j$.
Step 4. Calculate the overall dominance degree.

Based on Step 3, the overall dominance degree of $a_i$ over $a_k$ can be calculated according to the following expression:

$$\delta(a_i,a_k) = \sum_{j=1}^{m} \Phi_j(a_i,a_k), i = 1,2,\ldots,n; k = 1,2,\ldots,n. \quad (15)$$

Step 5. Calculate the global value.

Based on Step 4, the global value $\xi(a_i)$ of $a_i$ can be obtained using the following expression:
$\xi(a_i)$

$$= \frac{\sum_{k=1}^{n} \delta(a_i,a_k) - \min_{i\in m}\{\sum_{k=1}^{n} \delta(a_i,a_k)\}}{\max_{i\in m}\{\sum_{k=1}^{n} \delta(a_i,a_k)\} - \min_{i\in m}\{\sum_{k=1}^{n} \delta(a_i,a_k)\}} (i = 1,2,\ldots,n) \quad (16)$$

Step 6. Rank the alternatives.

The greater the value of $\xi(a_i)$, the better the alternative $a_i$.

## 5 An illustrative example

In this section, an example is adapted from Schmeidler [66] for further illustration of the feasibility of the proposed approach. There is an investment company, which wants to invest in a project. There are five possible alternatives in which to invest: $a_1$ is a car company; $a_2$ is a food company; $a_3$ is a computer company; $a_4$ is an arms company; and $a_5$ is a TV company. The investment company must make a decision according to the following four criteria: $c_1$ is the environment impact; $c_2$ is the risk; $c_3$ are the growth prospects; and $c_4$ is the social-political impact. The environmental impact refers to the impact on the companys environment and the processes used in making the product, such as the management methods and work environment. The risk involves more than one risk factor, including product risk and development environment risk. The growth prospects include increased profitability and returns. The social-political impact refers to the governments and local residents support for company. The four criteria are correlated with each other in the assessment process. The evaluation values $a_{ij}(i = 1,2,3,4,5; j = 1,2,3,4)$ should be in the form of MHFEs which are provided by two decision-makers based on their knowledge and experience. In the case where two decision-makers give the same value, then it is counted repeatedly, and $a_{ij}$ is the set of evaluation values for two decision-makers.The five possible alternatives $a_i(i = 1,2,3,4,5)$ are to be evaluated using the multiset hesitant fuzzy information of two decision makers as presented in the following.

$$R = \begin{pmatrix} 0.4,0.5,0.7 & 0.5,0.5,0.8 & 0.6,0.6,0.9 & 0.5,0.6 \\ 0.6,0.7,0.8 & 0.5,0.6 & 0.6,0.7,0.7 & 0.4,0.5 \\ 0.6,0.8 & 0.2,0.3,0.5 & 0.6,0.6 & 0.5,0.7 \\ 0.5,0.5,0.7 & 0.4,0.5 & 0.8,0.9 & 0.3,0.4,0.5 \\ 0.6,0.7 & 0.5,0.7 & 0.7,0.8 & 0.3,0.3,0.4 \end{pmatrix}$$

### *5.1 An illustration of the proposed approach*

The procedures of obtaining the optimal alternative, by using the developed method, are shown as follows.
Step 1. Normalize the decision matrix.

Because all the criteria are of the maximizing type and have the same measurement unit, there is no need for normalization and $\widetilde{R} = (\widetilde{a}_{ij})_{5\times 4}$.
Step 2. Determine the fuzzy measure of the criteria of $C$.

Suppose that $\mu(c_1) = 0.40, \mu(c_2) = 0.25, \mu(c_3) = 0.37, \mu(c_4) = 0.20, \mu(c_1,c_2) = 0.60, \mu(c_1,c_3) = 0.70, \mu(c_1,c_4) = 0.56, \mu(c_2,c_3) = 0.68, \mu(c_2,c_4) = 0.43, \mu(c_3,c_4) = 0.54, \mu(c_1,c_2,c_3) = 0.88, \mu(c_1,c_2,c_4) = 0.75, \mu(c_2,c_3,c_4) = 0.73,$ $\mu(c_1,c_3,c_4) = 0.84,$ and $\mu(c_1,c_2,c_3,c_4) = 1.$ Assume the criteria sets of $C = \{c_1,c_2,c_3,c_4\}$ are ordered as: $c_4 \prec c_2 \prec c_1 \prec c_3$, according to Eq.(13), the following results can be obtained.
$w_{\sigma(1)} = \mu(A_{\sigma(1)}) - \mu(A_{\sigma(2)}) = \mu(c_{\sigma(1)},c_{\sigma(2)},c_{\sigma(3)},c_{\sigma(4)}) - \mu(c_{\sigma(2)},c_{\sigma(3)},c_{\sigma(4)}) = \mu(c_1,c_2,c_3,c_4) - \mu(c_2,c_1,c_3) = 1 - 0.88;$
$w_{\sigma(2)} = \mu(A_{\sigma(2)}) - \mu(A_{\sigma(3)}) = \mu(c_{\sigma(2)},c_{\sigma(3)},c_{\sigma(4)}) - \mu(c_{\sigma(3)},c_{\sigma(4)}) = \mu(c_2,c_1,c_3) - \mu(c_1,c_3) = 0.88 - 0.70 = 0.18;$
$w_{\sigma(3)} = \mu(A_{\sigma(3)}) - \mu(A_{\sigma(4)}) = \mu(c_{\sigma(3)},c_{\sigma(4)}) - \mu(c_{\sigma(4)}) = \mu(c_1,c_3) - \mu(c_3) = 0.70 - 0.37 = 0.33;$
$w_{\sigma(4)} = \mu(A_{\sigma(4)}) - \mu(A_{\sigma(5)}) = \mu(c_{\sigma(4)}) - 0 = \mu(c_3) - 0 = 0.37;$

So $w_l = w_{\sigma(4)} = 0.37$ and $\sum_1^n w_{ij} = 2.7027$ can be obtained.

Step 3. Calculate the dominance degree. According to Eq. (14), for the convenience of analysis and computation, $\theta = 2.25, \lambda = 1$ and $\widetilde{d}(\widetilde{a}_{ij}, \widetilde{a}_{kj})$ can be calculated by using Eqs. (8)-(9). The evaluation values $\widetilde{a}_{ij}$ and $\widetilde{a}_{kj}$ can be compared by utilizing Definition 10. The dominance degree matrices concerning the criteria $c_1, c_2, c_3$ and $c_4$, could respectively be obtained as follows.

$\Phi_1(a_i, a_k)$

$$= \begin{pmatrix} 0 & -0.2234 & -0.2643 & -0.1000 & -0.2119 \\ 0.1659 & 0 & 0.0732 & 0.1484 & 0.0742 \\ 0.1962 & -0.1000 & 0 & 0.1817 & 0.1285 \\ 0.0742 & -0.1998 & -0.2447 & 0 & -0.1730 \\ 0.1573 & -0.1000 & -0.1730 & 0.1962 & 0 \end{pmatrix}$$

$\Phi_2(a_i, a_k)$

$$= \begin{pmatrix} 0 & 0.1025 & 0.1550 & 0.1162 & -0.2139 \\ -0.2532 & 0 & 0.1397 & 0.0949 & -0.2342 \\ -0.3826 & -0.3449 & 0 & -0.2869 & -0.3826 \\ -0.2869 & 0.2342 & 0.1162 & 0 & -0.2869 \\ 0.0866 & 0.0949 & 0.1550 & 0.1162 & 0 \end{pmatrix}$$

$\Phi_3(a_i, a_k)$

$$= \begin{pmatrix} 0 & 0.1571 & 0 & -0.2213 & -0.2311 \\ -0.1887 & 0 & 0.1112 & -0.2750 & -0.2109 \\ 0 & -0.1335 & 0 & -0.3466 & -0.2583 \\ 0.1842 & 0.2290 & 0.2290 & 0 & 0.1360 \\ 0.1924 & 0.1756 & 0.2151 & -0.1634 & 0 \end{pmatrix}$$

$\Phi_4(a_i, a_k)$

$$= \begin{pmatrix} 0 & 0.0775 & -0.2869 & 0.0949 & 0.1379 \\ -0.2869 & 0 & -0.3514 & 0.0448 & 0.0873 \\ 0.0775 & 0.0949 & 0 & 0.1095 & 0.1484 \\ -0.3514 & -0.1658 & -0.5495 & 0 & 0.0448 \\ -0.5106 & -0.3101 & -0.5495 & -0.1658 & 0 \end{pmatrix}$$

Step 4. Calculate the overall dominance degree.
According to Eq.(15), the overall dominance degree matrix could be shown as follows:

$\delta(a_i, a_k)$

$$= \begin{pmatrix} 0 & 0.1137 & -0.3962 & -0.1102 & -0.5190 \\ -0.5629 & 0 & -0.0263 & 0.0130 & -0.2872 \\ -0.1089 & -0.4835 & 0 & -0.3423 & -0.3641 \\ -0.3798 & -0.3709 & -0.4490 & 0 & -0.2791 \\ -0.0743 & -0.1396 & -0.3525 & -0.0168 & 0 \end{pmatrix}$$

Step 5. Calculate the global value.

The global value of the alternative $a_i$ is calculated by using Eq.(16), giving:
$\xi(a_1) = 0.6331; \xi(a_2) = 0.6872; \xi(a_3) = 0.2009; \xi(a_4) = 0; \xi(a_5) = 1.$

Step 6. Rank the alternatives.

Based on Step 5, $\xi(a_4) < \xi(a_3) < \xi(a_1) < \xi(a_2) < \xi(a_5)$ could be obtained, therefore the ranking $a_4 \prec a_3 \prec a_1 \prec a_2 \prec a_5$ is obtained. Thus, the best alternative is $a_5$.

## 5.2 A comparison analysis and discussion

In this section, in order to validate the feasibility of the proposed multiset hesitant fuzzy MCGDM approach based on TODIM and the Choquet integral, a comparative study was conducted with other methods. These methods can be divided into two categories: one is that the criteria are independent of each other, as shown in Xu [20,21], Wei [22], Zhang et al. [24], Chen et al. [27], Xu [28,29], Farhadinia [30], Zhang and Wei [31], Zhang and Xu [32], and Wang et al. [36]; and the other is that the criteria are considered as correlated with one another, as shown in Xia et al. [23] and Yu et al. [45].

The analysis was based on the same illustrative example.

Case 1. The hesitant fuzzy methods with criteria are assumed to be independent of one another. Suppose the weight vector of criteria is known, which can be determined in Step 2, then the compared results can be obtained as shown in Table 1.

Table 1. Comparison of different methods in the case that the criteria are independent

| Methods | Ranking of alternatives |
|---|---|
| Xu [20,21] | $a_3 \prec a_1 \prec a_2 \prec a_5 \prec a_4$ |
| Wei [22] | $a_3 \prec a_1 \prec a_4 \prec a_5 \prec a_2$ |
| Zhang [24] | $a_3 \prec a_4 \prec a_1 \prec a_2 \prec a_5$ |
| Chen et al. [27] | $a_3 \prec a_4 \prec a_2 \prec a_1 \prec a_5$ |
| Xu [28,29] | $a_3 \prec a_1 \prec a_4 \prec a_2 \prec a_5$ |
| Farhadinia [30] | $a_3 \prec a_4 \prec a_2 \prec a_1 \prec a_5$ |
| Zhang and Wei [31] | $a_3 \prec a_4 \prec a_2 \prec a_1 \prec a_5$ |
| Zhang and Xu [32] | $a_4 \prec a_3 \prec a_2 \prec a_1 \prec a_5$ |
| Wang et al. [36] | $a_4 \prec a_3 \prec a_2 \prec a_1 \prec a_5$ |
| Proposed method | $a_4 \prec a_3 \prec a_1 \prec a_2 \prec a_5$ |

From Table 2, it can be seen that the result of the proposed approach is different to the methods of Xu [20, 21] and [22], and the reason is because those methods use an aggregation operator to deal with the hesitant fuzzy formation. It should be noted that it is easy to use an operator when using these methods. However, different aggregation operators also lead to different rankings. Furthermore, it is difficult for decision makers to choose which kind of explicit operators are utilized. Compared with the proposed approach, Zhang [24], Chen et al. [27], Xu [28,29] and Farhadinia [30], Zhang and Wei [31], Zhang and Xu [32] and Wang et al. [36], the ranking results are the same and the best alternative is always . However, those methods have certain shortcomings when using aggregation operators and distance measures. Especially, those distance measures should satisfy the condition that all HFEs must be arranged in ascending order and be of equal length. If the two HFEs being compared have different lengths, then the value of the shorter one should be increased until both are equal. Moreover, the proposed method simultaneously considers the bounded rationality of decision makers and the correlated criteria.

Case 2. The hesitant fuzzy method with the criteria considered to be correlated with one another.

The methods proposed by Xia et al. [23] and Yu et al. [45] based on the correlated criteria are used here in the same illustrative example. In order to better validate the method, the fuzzy measure with the values mentioned previously is used. Subsequently the compared results can be obtained as shown in Table 2.

Table 2. Comparison of different methods in the case that the criteria are correlated

| Methods | Ranking of alternatives |
|---|---|
| Xia et al. [29] | $a_3 \prec a_1 \prec a_2 \prec a_4 \prec a_5$ |
| Yu et al. [51] | $a_3 \prec a_1 \prec a_4 \prec a_5 \prec a_2$ |
| Proposed method | $a_4 \prec a_3 \prec a_1 \prec a_2 \prec a_5$ |

From Table 2, it can be seen that the methods proposed by Xia et al.[23] and Yu et al.[45] and the method proposed in this paper have different rankings. They all have considered the interactive phenomena which might occur among criteria, but the method proposed by Xia et al.[23] and Yu et al. [45] involves an aggregation operator. Different aggregation operators might produce different results as discussed earlier, while the proposed approach could overcome those deficiencies and ensure the reasonableness and effectiveness of the decision-making results.

From the comparison analyses presented above, the proposed method for MCDM problems with MHFSs has the following advantages.

Firstly, MHFSs can express the evaluation information more flexibly. They can take into account the repetitive values in HFSs and retain the completeness of original data or the inherent thoughts of decision-makers, which is the prerequisite of guaranteeing accuracy of final outcomes.

Secondly, the proposed comparison method and the distance of MHFEs can overcome these shortcomings in the existing methods and measures that they should satisfy the condition that all elements in HFEs must be arranged in ascending order and be of equal length as we discussed earlier.

Finally, the proposed approach could consider the bounded rationality of decision makers and interaction might affect the criteria in the aggregation process at the same time. This can avoid losing and distorting the preference information provided, which makes the final results better correspond with real life decision-making problems.

## 6 Conclusion

HFSs are considered useful in handling decision-making problems under uncertain situations where decision-makers hesitate when choosing between several values before expressing their preferences about weights and data. MHFSs can deal effectively with the case where some values are repeated more than once in an HFS. In this paper, the operations and a comparison method of MHFSs were discussed. Then a novel approach based on TODIM and the Choquet integral was developed in order to deal with MCGDM problems where the data are MHFSs. Finally, an illustrative example was given to verify the proposed approach. The primary characteristic of the proposed approach is that MHFSs could overcome the shortcomings in traditional HFSs where if two or more decision-makers set the same value, it is only counted once. The new comparison method can also avoid the defects in the existing score functions of HFEs. Furthermore, the proposed approach with MHFSs can better cope with multiset hesitant fuzzy MCGDM problems where the criteria are interdependent or interactive and the decision makers have a bounded rationality. Further research will investigate how to obtain the optimal values of criteria by a specified model within a multiset hesitant fuzzy environment.

## Acknowledgement

This work was supported by the National Natural Science Foundation, P.R. China (Nos. 71271218, 71221061 and 71431006) and the Science Foundation for Doctors of Hubei University of Automotive Technology (BK201405).
The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

## References

[1] L.A. Zadeh, Fuzzy sets, Information and Control **8**, 338-356 (1965).
[2] R. Bellman, L.A. Zadeh, Decision making in a fuzzy environment, Management Science **17**, 141-164 (1970).
[3] R.R. Yager, Multiple objective decision-making using fuzzy sets, International Journal of Man-Machine Studies **9**, 375-382 (1997).
[4] L.A. Zadeh, Fuzzy logic and approximate reasoning, Synthese **30**, 407-428 (1975).
[5] W. Pedrycz, Fuzzy sets in pattern recognition: methodology and methods, Pattern Recognition **23**, 121-146 (1990).
[6] K.T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems **20**, 87-96 (1986).
[7] K.T. Atanassov, Intuitionistic fuzzy sets, Springer, Heidelberg, 1999.
[8] K. T. Atanassov, G. Gargov, Interval valued intuitionistic fuzzy sets, Fuzzy Sets and Systems **31**, 343-349 (1989).
[9] Y.T. Chen, A outcome-oriented approach to multicriteria decision analysis with intuitionistic fuzzy optimistic/pessimistic operators, Expert Systems with Applications **37**, 7762-7774 (2010).
[10] J.Q. Wang, R.R. Nie, H.Y. Zhang, X.H. Chen, Intuitionistic fuzzy multi-criteria decision-making method based on evidential reasoning, Applied Soft Computing **13**, 1823-1831 (2013).
[11] J.Q. Wang, H.Y. Zhang, Multi-criteria decision-making approach based on Atanassov's intuitionistic fuzzy sets with incomplete certain information on weights, IEEE Transactions on Fuzzy Systems **21**, 510-515 (2013).

[12] J.Q. Wang, Z.Q. Han, H.Y. Zhang, Multi-criteria group decision-making method based on intuitionistic interval fuzzy information, Group Decision and Negotiation **23**, 715-733 (2014).
[13] P. Zhi, Z. Li, A novel approach to multi-attribute decision making based on intuitionistc fuzzy sets, Expert Systems with Applications **39**, 2560-2566 (2012).
[14] J.Q. Wang, R.R. Nie, H.Y. Zhang, X.H. Chen, New operators on triangular intuitionistic fuzzy numbers and their applications in system fault analysis, Information Sciences **251**, 79-95 (2013).
[15] J. Ye, Multicriteria decision-making method using the correlation coefficient under single-value neutrosophic environment, International Journal of General Systems **42**, 386-394 (2013).
[16] J.J. Peng, J.Q. Wang, J. Wang, H.Y. Zhang, X.H. Chen, Simplified neutrosophic sets and their applications in multi-criteria group decision-making problems, International Journal of Systems Science, In Press, 2015.
[17] J.J. Peng, J.Q. Wang, H.Y. Zhang, X.H. Chen, An outranking approach for multi-criteria decision-making problems with simplified neutrosophic sets, Applied Soft Computing (2015), http://dx.doi.10.1016/j.asoc.2014.08.070.
[18] V. Torra, Hesitant fuzzy sets, International Journal of Intelligent Systems **25**, 529-539 (2010).
[19] V. Torra, Y. Narukawa, On hesitant fuzzy sets and decision, The 18th IEEE International Conference on Fuzzy Systems, Jeju Island, Korea, 2009, pp.1378-1382.
[20] M.M. Xia, Z.S. Xu, Hesitant fuzzy information aggregation in decision making, International Journal of Approximate Reasoning **52**, 395-407 (2011).
[21] B. Zhu, Z.S. Xu, M.M. Xia, Hesitant fuzzy geometric Bonferoni means, Information Sciences **205**, 72-85 (2012).
[22] G.W. Wei, Hesitant fuzzy prioritized operators and their application to multiple attribute decision making, Knowledge-Based Systems **31**, 176-182 (2012).
[23] M.M. Xia, Z.S. Xu, N. Chen, Some Hesitant fuzzy aggregation operators with their application in group decision making, Group Decision and Negotiation **22**, 259-279 (2013).
[24] Z.M. Zhang, C. Wang, D.Z. Tian, K. Li, Induced generalized hesitant fuzzy operators and their application to multiple attribute group decision making, Computers and Industrial Engineering **67**, 116-138 (2014).
[25] W. Zhou, An Accurate method for determining hesitant fuzzy aggregation operator weights and its application to project investment, International Journal of Intelligent Systems **00**, 1-19 (2014).
[26] Z.M. Zhang, Hesitant fuzzy power aggregation operators and their application to multiple attribute group decision making, Information Sciences **234**, 150-181 (2013).
[27] N. Chen, Z.S. Xu, M.M. Xia, Correlation coefficients of hesitant fuzzy sets and their applications to clustering analysis, Applied Mathematical Modelling **37**, 2197-2211 (2013).
[28] Z.S. Xu, M.M. Xia, Distance and similarity measures for hesitant fuzzy sets, Information Sciences **181**, 2128-2138 (2011).
[29] Z.S. Xu, M.M. Xia, On distance and correlation measures of hesitant fuzzy information, International Journal of Intelligent Systems **26**, 410-425 (2011).
[30] B. Farhadinia, Distance and similarity measures for higher order hesitant fuzzy sets, Knowledge-Based Systems **55**, 43-48 (2014).
[31] N. Zhang, G.W. Wei, Extension of VIKOR method for decision making problem based on hesitant fuzzy set, Applied Mathematical Modeling **37**, 4938-4947 (2013).
[32] X.L. Zhang, Z.S. Xu, The TODIM analysis approach based on novel measured functions under hesitant fuzzy environment, Knowledge-Based Systems **61**, 48-58 (2014).
[33] G. Qian, H. Wang, X.Q. Feng, Generalized hesitant fuzzy sets and their application in decision support system, Knowledge-Based Systems **37**, 357-365 (2013).
[34] B. Zhu, Z.S. Xu, M.M. Xia, Dual Hesitant Fuzzy Sets, Journal of Applied Mathematics, 2012, Article ID 879629, 13 pages.
[35] B. Zhu, Z.S. Xu, Some results for dual hesitant fuzzy sets, Journal of Intelligent and Fuzzy Systems **26**, 1657-1668 (2014).
[36] J.Q. Wang, D.D. Wang, H.Y. Zhang, X.H. Chen, Multi-criteria outranking approach with hesitant fuzzy sets, OR Spectrum **36**, 1001-1019 (2014).
[37] N. Chen, Z.S. Xu, M.M. Xia, Interval-valued hesitant preference relations and their applications to group decision making, Knowledge-Based Systems **37**, 528-540 (2013).
[38] G.W. Wei, X.F. Zhao, Induced hesitant interval-valued fuzzy Einstein aggregation operators and their application to multiple attribute decision making, Journal of Intelligent and Fuzzy Systems **24**, 789-803 (2013).
[39] B. Farhadinia, Correlation for dual hesitant fuzzy sets and dual interval-valued hesitant fuzzy sets, International Journal of Intelligent Systems **29**, 184-205 (2014).
[40] J.J. Peng, J.Q. Wang, J. Wang, X.H. Chen, Multi-criteria decision-making approach with hesitant interval-valued intuitionistic fuzzy set, The Scientific World Journal, 2014, Article ID 868515, 22 pages.
[41] R.M. Rodriguez, L. Martinez, V. Torra, Z.S. Xu, F. Herrera, Hesitant fuzzy sets: state of the art and future directions, International Journal of Intelligent Systems **29**, 495-524 (2014).
[42] G. Choquet, Theory of capacities, Annales de linstitut Fourier **5**, 131-295 (1953).
[43] R.R. Yager, Induced aggregation operators, Fuzzy Sets and Systems **137**,59-69 (2003).
[44] P. Meyer, M. Roubens, On the use of the Choquet integral with fuzzy numbers in multiple criteria decision support, Fuzzy Sets and Systems **157**, 927-938 (2006).
[45] D.J. Yu, Y.Y. Wu, W. Zhou, Multi-criteria decision making based on Choquet integral under hesitant fuzzy environment, Journal of Computational Information Systems **7(12)**, 4506-4513 (2011).
[46] C.Q. Tan, X.H. Chen, Intuitionistic fuzzy Choquet integral operator for multi-criteria decision making, Expert Systems with Applications **37**,149-157 (2010).
[47] C.Q. Tan, A multi-criteria interval-valued intuitionistic fuzzy group decision making with Choquet integral-based TOPSIS, Expert Systems with Applications **38**, 3023C3033 (2011).
[48] H. Bustince, J. Fernandez, J. Sanz, M. Galar, R. Mesiar, A. Kolesarova, Multicriteria decision making by means of interval-valued Choquet integrals, Advances in Intelligent and Soft Computing **107**, 269-278 (2012).

[49] G.W. Wei, X.F. Zhao, R. Lin, H.J. Wang, Generalized triangular fuzzy correlated averaging operator and their application to multiple attribute decision making, Applied Mathematical Modelling **36**, 2975-2982 (2012).
[50] J.Q. Wang, D.D. Wang, H.Y. Zhang, X.H. Chen, Multi-criteria group dcision making method based on interval 2-tuple linguistic and Choquet integral aggregation operators, Soft Computing, doi:10.1007/s00500-014-1259-z, 2014.
[51] D. Kahneman, A. Tversky, Prospect theory: an analysis of decision under risk, Econometrica **47**, 263-291 (1979).
[52] A. Tversky, D. Kahneman, Advances in prospect theory: cumulative representation of uncertainty, Journal of Risk and Uncertainty **5**, 297-323 (1992).
[53] B. Nicholas, H. Ming, S. Tano, Prospect theory and asset prices, The Quarterly Journal of Economics **116**, 1-53 (2001).
[54] X.D. Li, Behavioral finance theory: research and prospect, Financial Research **1**, 175-190 (2005).
[55] D. Sanjit, N. Ali, Why do people pay taxes? Prospect theory versus expected utility theory, Journal of Economic Behavior & Organization **64**, 171-192 (2007).
[56] J.-Q. Wang, K.-J. Li, H.-Y. Zhang, Interval-valued intuitionistic fuzzy multi-criteria decision-making approach based on prospect score function, Knowledge-Based Systems **27**, 119-125 (2012).
[57] L.F.A. M. Gomes, M.M.P.P. Lima, TODIM: basics and application to multicriteria ranking of projects with environmental impacts, Foundations of Computing and Decision Science **16**, 113-127 (1991).
[58] L.F.A.M. Gomes, M.M.P.P. Lima, From modeling individual preferences to multicriteria ranking of discrete alternatives: a look at prospect theory and the additive difference model, Foundations of Computing and Decision Sciences **17**, 171-184 (1992).
[59] L.F.A.M. Gomes, L.A.D. Rangel, An application of the TODIM method to multicriteria rental evalution of residential properties, European Journal of Operational Research **193**, 204-211 (2009).
[60] L.F.A. M. Gomes, L.A.D. Rangel, F.J.C. Maranhao, Multicriteria analysis of natural gas destination in Brazil: an application of the TODIM method, Mathematical and Computer Modelling **50**, 92-100 (2009).
[61] R.A. Krohling, T.T.M. Souza, Combining propect theory and fuzzy numbers to multi-criteria decision making, Expert Systems with Applications **39**, 11487-11493 (2012).
[62] Z.P. Fan, X. Zhang, F.D. Chen, Y. Liu, Extended TODIM method for hybrid multiple attribute decision making problems, Knowledge-Based Systems **42**, 40-48 (2013).
[63] Z. Wang, G.J. Klir, Fuzzy measure theory, Plenum press, New York, 1992.
[64] B. Farhadinia, A novel method of ranking hesitant fuzzy values for multiple attribute decision-making problems, International Journal of Intelligent Systems **28**, 752-767 (2013).
[65] B. Farhadinia, Information measures for hesitant fuzzy sets and interval-valued hesitant fuzzy sets, Information Sciences **240**, 129-144 (2013).
[66] D. Schmeidler, Subjective probability and expected utility without additivity, Econometrica **57**, 517-587 (1989).


**Juan-juan Peng** received her M.Sc. degree in Computational Mathematics from Wuhan University of Technology, China, in 2007. She is currently a Ph.D. student in School of Business, Central South University; she is also a Lecturer in School of Economics and Management, Hubei University of Automotive Technology, China. Her current research focuses on decision-making theory and application, risk management and control, and information management.

**Jian-qiang Wang** received the Ph.D. degree in Management Science and Engineering from Central South University, Changsha, China, in 2005. He is currently a Professor in School of Business, Central South University. His current research interests include decision-making theory and application, risk management and control, and information management.

**Huan Zhou** graduated from Central South University and obtained the master degree in management of science in 2007. At present, she is studying her doctor degree of business administration at Central South University. Her main research fields are multi-criteria decision making, risk management and control, comprehensive evaluation and uncertainty.

**Xiao-hong Chen** received her Ph.D. in Management from Tokyo University of Technology, Japan, in 1999. She is currently a professor with the School of Business, Central South University, Changsha, China. Her current research interests are in the field of decision theory & method, decision support system, resource-saving and environment-friendly society. She has published academic papers in many journals, including Decision Support System, Expert Systems with Applications, etc.