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Solving Two Emden–Fowler Type Equations of Third Order by the Variational Iteration Method

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Abstract: In this paper, we establish two kinds of Emden–Fowler type equations of third order. We investigate the linear and the nonlinear third-order equations, with specified initial conditions, by using the systematic variational iteration method. We corroborate this study by investigating several Emden–Fowler type examples with initial value conditions.

Keywords: Emden–Fowler equation; variational iteration method; Lagrange multipliers.

1 Introduction

Many scientific applications in the literature of mathematical physics and fluid mechanics can be distinctively described by the Emden–Fowler equation

$$y'' + \frac{k}{x}y' + f(x)g(y) = 0, y(0) = y_0, y'(0) = 0, \quad (1)$$

where $f(x)$ and $g(y)$ are some given functions of x and y, respectively, and k is called the shape factor. The Emden–Fowler equation (1) describes a variety of phenomena in fluid mechanics, relativistic mechanics, pattern formation, population evolution and in chemically reacting systems.

For $f(x) = 1$ and $g(y) = y^m$, Eq. (1) becomes the standard Lane–Emden equation of the first order and index m, given by

$$y'' + \frac{k}{x}y' + y^m = 0, y(0) = y_0, y'(0) = 0, \quad (2)$$

The Lane–Emden equation (2) models the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules [1–12] and subject to the classical laws of thermodynamics. Moreover, the Lane–Emden equation of first order is a useful equation in astrophysics for computing the structure of interiors of polytropic stars. On the other side, for $f(x) = 1$ and $g(y) = e^y$, Eq. (1) becomes the standard Lane–Emden equation of the second order that models the non-dimensional density distribution $y(x)$ in an isothermal gas sphere [9–22] Moreover, the Lane–Emden equation (2) describes the temperature variation of a spherical gas cloud under the mutual attraction of its molecules and subject to the laws of thermodynamics. In addition, the Lane-Emden equation of the first kind appears also in other context such as in the case of radiatively cooling, self-gravitating gas clouds, in the mean-field treatment of a phase transition in critical adsorption and in the modeling of clusters of galaxies [17–19].

The Lane-Emden equation was first studied by astrophysicists Jonathan Homer Lane and Robert Emden, where they considered the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics. The well-known Lane-Emden equation has been used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres, the theory of thermionic currents, and in the modeling of clusters of galaxies. The Emden–Fowler equation was studied by Fowler [2] to describe a variety of phenomena in fluid mechanics and relativistic mechanics among others. The singular behavior that occurs at $x = 0$ is the main difficulty of Equations (1) and (2).

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We note that (1) was derived by using the equation

$$x^{-k}\frac{d}{dx}\left(x^{k}\frac{d}{dx}\right)y+f(x)g(y)=0, y(0)=y_0, y'(0)=0, \tag{3}$$

where k is called the shape factor.

The Lane–Emden equation and the Emden–Fowler equation were subjected to a considerable size of investigation, both numerically and analytically. A variety of useful methods were used to obtain exact and approximate solutions as well. Examples of the methods that were applied are the Adomian decomposition method [6, 7, 18], the variational iteration method [8, 10, 14, 23], the homotopy perturbation method [22], the rational Legendre pseudospectral approach [17], and other methods as well [19, 20].

We aim in this work to establish two kinds of Emden–Fowler type equations of third-order. Our approach depends mainly on using different orders of the differential operators involved in the Emden–Fowler sense given in (3). Our next goal of this work is to apply the variational iteration method to handle the developed Emden–Fowler type equations of third-order. Several numerical examples, with specified initial conditions, of each model will be examined to handle the singular point that exists in each model.

2 Constructing Emden–Fowler type equations of third-order

To derive the Emden-Fowler type equations of third-order, we use the sense of (3) and set

$$x^{-k}\frac{d^m}{dx^m}\left(x^{k}\frac{d^n}{dx^n}\right)y+f(x)g(y)=0. \tag{4}$$

To determine third-order equations, it is clear that we should select

$$m+n=3, m,n\geq 1 \tag{5}$$

that leads to the following two choices

$$m=2, n=1, \tag{6}$$

and

$$m=1, n=2. \tag{7}$$

Substituting $m=2, n=1$ in (4) gives

$$x^{-k}\frac{d^2}{dx^2}\left(x^{k}\frac{d}{dx}\right)y+f(x)g(y)=0. \tag{8}$$

This in turn gives the first Emden–Fowler type equation of third order in the form

$$\frac{d^3y}{dx^3}+\frac{2k}{x}\frac{d^2y}{dx^2}+\frac{k(k-1)}{x^2}\frac{dy}{dx}+f(x)g(y)=0, y(0)=y_0, y'(0)=y''(0)=0, \tag{9}$$

or equivalently

$$y'''+\frac{2k}{x}y''+\frac{k(k-1)}{x^2}y'+f(x)g(y)=0, y(0)=y_0, y'(0)=y''(0)=0. \tag{10}$$

Notice that the singular point $x=0$ appears twice as x and x^2 with shape factors $2k$ and $k(k-1)$, respectively. Moreover, the third term vanishes for $k=1$ and the shape factor in this case reduces to 2.

For $f(x)=1$, Eqs. (10) becomes the Lane–Emden type equation of the third-order given by

$$y'''+\frac{2k}{x}y''+\frac{k(k-1)}{x^2}y'+g(y)=0, y(0)=y_0, y'(0)=y''(0)=0. \tag{11}$$

In the other case, we substitute $m=1, n=2$ in (4) to obtain

$$x^{-k}\frac{d}{dx}\left(x^{k}\frac{d^2}{dx^2}\right)y+f(x)g(y)=0. \tag{12}$$

This in turn gives the second kind of Emden–Fowler type equation of third order of the form

$$\frac{d^3y}{dx^3}+\frac{k}{x}\frac{d^2y}{dx^2}+f(x)g(y)=0, y(0)=y_0, y'(0)=y''(0)=0, \tag{13}$$

or equivalently

$$y'''+\frac{k}{x}y''+f(x)g(y)=0, y(0)=y_0, y'(0)=y''(0)=0. \tag{14}$$

Unlike the first kind, the singular point in the second case $x=0$ appears once with shape factor k. Moreover, the first-order term $y'(x)$ term vanishes in the second kind.

For $f(x)=1$, Eqs. (14) becomes the Lane–Emden type equations of the third–order given by

$$y'''+\frac{k}{x}y''+g(y)=0, y(0)=y_0, y'(0)=y''(0)=0. \tag{15}$$

3 Analysis of the method

In this section, we will present the analysis for the use of the variational iteration method (VIM) for a reliable treatment of the two models of the Emden–Fowler type equations of third-order. The main focus will be, as will be seen later, on the derivation of a variety of Lagrange multipliers for each models and investigate several examples as well.

3.1 The first model: the VIM and the Lagrange multipliers

In this section we will present the essential steps for using the variational iteration method and the determination of the Lagrange multipliers for various values of k. Consider the differential equation

$$Ly+Ny=h(t), \tag{16}$$

where L and N are linear and nonlinear operators respectively, and $h(t)$ is the source term.

To use the VIM, a correction functional for equation (16) should be used in the form

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda\,(Ly_n(\xi) + N\tilde{y}_n(\xi))\,d\xi, \tag{17}$$

where λ is a general Lagrange's multiplier, which can be identified optimally via the variational theory, and $\tilde{y}_n$ is a restricted variation, which means $\delta\tilde{y}_n = 0$.

For Eq. (10) the correction functional reads

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(x,t)\left(y_n'''(t) + \frac{2k}{t}y_n''(t) + \frac{k(k-1)}{t^2}y_n'(t) + f(t)\tilde{g}(y_n(t))\right)dt, \tag{18}$$

where $\delta(\tilde{g}(y_n(t))) = 0$.

To determine the optimal value of $\lambda(x,t)$, we take the variation for both sides with respect to $y_n(x)$ to obtain

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta\int_0^x \lambda(x,t)\left(y_n'''(t) + \frac{2k}{t}y_n''(t) + \frac{k(k-1)}{t^2}y_n'(t) + f(t)\tilde{g}(y_n(t))\right)dt, \tag{19}$$

or equivalently

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta\int_0^x \lambda(x,t)\left(y_n'''(t) + \frac{2k}{t}y_n''(t) + \frac{k(k-1)}{t^2}y_n'(t)\right)dt, \tag{20}$$

where we used $\delta(\tilde{g}(y_n(t))) = 0$.

For illustrative purposes, we evaluate the integral at the right side in steps. We first integrate

$$I_1 = \int_0^x \lambda(x,t)y_n'''(t)\,dt. \tag{21}$$

Integrating I_1 by parts three times gives

$$I_1 = (\lambda(x,t)y_n'' - \lambda'(x,t)y_n' + \lambda''(x,t)y_n)\,|_{t=x} - \int_0^x \lambda'''(x,t)y_n(t)\,dt. \tag{22}$$

We next integrate the second term of the integral in (20), therefore we set

$$I_2 = \int_0^x \lambda(x,t)\frac{2k}{t}y_n''(t)\,dt. \tag{23}$$

Integrating I_2 by parts twice gives

$$\begin{aligned} I_2 = &\left(\lambda(x,t)\frac{2k}{t}y_n' - \frac{2kt\lambda'(x,t) - 2k\lambda(x,t)}{t^2}y_n(t)\right)|_{t=x} \\ &+ \int_0^x \frac{2kt^2\lambda''(x,t) - 4kt\lambda'(x,t) + 4k\lambda(x,t)}{t^3}y_n(t)\,dt. \end{aligned} \tag{24}$$

We finally integrate the third term of the integral in (20), therefore we set

$$I_3 = \int_0^x \lambda(x,t)\frac{k(k-1)}{t^2}y_n'(t)\,dt. \tag{25}$$

Integrating I_3 by parts once gives

$$I_3 = \left(\lambda(x,t)\frac{k(k-1)}{t^2}y_n\right)|_{t=x} - \int_0^x \frac{k(k-1)t\lambda'(x,t) - 2k(k-1)\lambda(x,t)}{t^3}y_n(t)\,dt. \tag{26}$$

In view of (21)–(26), Eq. (20) becomes

$$\begin{aligned} &\delta y_{n+1}(x) = \delta y_n(x) \\ &+\delta\left(\lambda(x,t)y_n'' + \left[\lambda(x,t)\frac{2k}{t} - \lambda'(x,t)\right]y_n' + \left[\lambda''(x,t) - \frac{2k}{t}\lambda'(x,t) + \frac{2k+k(k-1)}{t^2})y_n\right]\right)|_{t=x} \\ &+\int_0^x\left(-\lambda'''(x,t) + \frac{2k}{t}\lambda''(x,t) - \frac{4k+k(k-1)}{t^2}\lambda'(x,t) + \frac{4k+2k(k-1)}{t^3}\lambda(x,t)\right)\delta y_n. \end{aligned} \tag{27}$$

For arbitrary δy_n, we obtain the stationary condition

$$-\lambda'''(x,t) + \frac{2k}{t}\lambda''(x,t) - \frac{4k+k(k-1)}{t^2}\lambda'(x,t) + \frac{4k+2k(k-1)}{t^3}\lambda(x,t) = 0 \tag{28}$$

and the boundary conditions

$$\begin{aligned} \lambda(x,t)|_{t=x} &= 0, \\ \frac{2k}{t}\lambda(x,t) - \lambda'(x,t)|_{t=x} &= 0, \\ \left(1 + \lambda''(x,t) - \frac{2k}{t}\lambda'(x,t) + \frac{2k+k(k-1)}{t^2}\lambda(x,t)\right)|_{t=x} &= 0. \end{aligned} \tag{29}$$

To determine $\lambda(x,t)$, three cases will be examined:
(i) For the general case where $k \neq 1,2$: solving (28)–(29) gives

$$\lambda(x,t) = -\frac{t^2}{(k-1)(k-2)} - \frac{x^2}{k-1}\left(\frac{t}{x}\right)^{k+1} + \frac{x^2}{k-2}\left(\frac{t}{x}\right)^k. \tag{30}$$

(ii) For $k = 1$: solving (28)–(29) gives

$$\lambda(x,t) = -xt + t^2\left(1 - \ln\frac{t}{x}\right). \tag{31}$$

(iii) For $k = 2$: solving (28)–(29) gives

$$\lambda(x,t) = -\frac{t^3}{x} + t^2\left(1 + \ln\frac{t}{x}\right). \tag{32}$$

The successive approximations y_{n+1}, for $n \geq 0$, of the solution $y(x)$ will be readily obtained upon using any selective function $y_0(x)$. Consequently, the solution

$$y(x) = \lim_{n\to\infty} y_n(x). \tag{33}$$

In other words, the correction functional (18) will give several approximations, and therefore the exact solution is obtained in the limit of the resulting successive approximations.

3.2 Numerical examples for the first kind of the Emden–Fowler type equations

$$y''' + \frac{2k}{x}y'' + \frac{k(k-1)}{x^2}y' + f(x)g(y) = 0, y(0) = \alpha, y'(0) = y''(0) = 0 \tag{34}$$

We will study this equation for a variety of values for the shape factor k and for the given functions $f(x)g(y)$. For comparison reasons we will solve the same examples solved in [1] by using the Adomian decomposition method.

Example 1.

We first consider the Emden–Fowler type equation

$$y''' + \frac{2}{x}y'' - \frac{9}{8}(x^6+8)y^{-5} = 0, y(0)=1, y'(0)=y''(0)=0, \tag{35}$$

obtained by substituting $k=1$ in (34) and by setting $f(x)g(y) = -\frac{9}{8}(x^6+8)y^{-5}$.

From (31), the Lagrange multiplier for $k=1$ is given by

$$\lambda(x,t) = -xt + t^2\left(1-\ln\frac{t}{x}\right). \tag{36}$$

The correction functional for (35) becomes

$$y_{n+1}(x) = y_n(x) + \int_0^x \left(-xt+t^2\left(1-\ln\frac{t}{x}\right)\right)\left(y_n'''(t) + \frac{2}{t}y_n''(t) - \frac{9}{8}(t^6+8)y_n^{-5}\right)dt, \tag{37}$$

where $n \geq 0$. By selecting the zeroth approximation $y_0 = 1$, we obtain the following calculated solution approximations

$$\begin{aligned}
y_0(x) &= 1,\\
y_1(x) &= 1+\frac{1}{2}x^3+\frac{1}{576}x^9,\\
y_2(x) &= 1+\frac{1}{2}x^3-\frac{1}{8}x^6+\frac{31}{576}x^9-\frac{2705}{101376}x^{12}+\cdots,\\
y_3(x) &= 1+\frac{1}{2}x^3-\frac{1}{8}x^6+\frac{1}{16}x^9-\frac{3935}{101376}x^{12}+\cdots,\\
y_4(x) &= 1+\frac{1}{2}x^3-\frac{1}{8}x^6+\frac{1}{16}x^9-\frac{5}{128}x^{12}+\cdots,\\
&\cdots.
\end{aligned}$$

To facilitate the computational work, we used few terms of the exponential $y_n(t), n \geq 1$. This in turn gives the series solution

$$y(x) = 1+\frac{1}{2}x^3-\frac{1}{8}x^6+\frac{1}{16}x^9-\frac{5}{128}x^{12}+\cdots, \tag{38}$$

that converges to the exact solution

$$y(x) = \sqrt{1+x^3}. \tag{39}$$

Example 2.

We next consider the linear Emden–Fowler type equation

$$y''' + \frac{4}{x}y'' + \frac{2}{x^2}y' - 9(4+10x^3+3x^6)y = 0, y(0)=1, y'(0)=y''(0)=0, \tag{40}$$

obtained by substituting $k=2$ in (34) and by setting $f(x)g(y) = -9(4+10x^3+3x^6)y$.

From (32), the Lagrange multiplier for $k=2$ is given by

$$\lambda(x,t) = -\frac{t^3}{x} + t^2\left(1+\ln\frac{t}{x}\right). \tag{41}$$

The correction functional for (40) becomes

$$\begin{aligned}
y_{n+1}(x) &= y_n(x)\\
&+ \int_0^x \left(-\frac{t^3}{x}+t^2\left(1+\ln\frac{t}{x}\right)\right)\left(y_n'''(t)+\frac{4}{t}y_n''(t)+\frac{2}{t^2}y_n'(t)-9(4+10t^3+3t^6)y_n(t)\right)dt,
\end{aligned} \tag{42}$$

for $n \geq 0$.

By selecting the zeroth approximation $y_0 = 1$, we obtain the following calculated solution approximations

$$\begin{aligned}
y_0(x) &= 1,\\
y_1(x) &= 1+x^3+\frac{5}{14}x^6+\frac{1}{30}x^9,\\
y_2(x) &= 1+x^3+\frac{1}{2}x^6+\frac{101}{630}x^9+\frac{44}{1365}x^{12}+\cdots,\\
y_3(x) &= 1+x^3+\frac{1}{2}x^6+\frac{1}{6}x^9+\frac{1361}{32760}x^{12}+\cdots,\\
y_4(x) &= 1+x^3+\frac{1}{2}x^6+\frac{1}{3!}x^9+\frac{1}{4!}x^{12}+\cdots,\\
&\cdots.
\end{aligned}$$

This in turn gives the series solution

$$y(x) = 1+x^3+\frac{1}{2!}x^6+\frac{1}{3!}x^9+\frac{1}{4!}x^{12}+\cdots, \tag{43}$$

that converges to the exact solution

$$y(x) = e^{x^3}. \tag{44}$$

Example 3.

We now consider the nonlinear Emden–Fowler type equation

$$y''' + \frac{6}{x}y'' + \frac{6}{x^2}y' - 6(10+2x^3+x^6)e^{-3y} = 0, y(0)=0, y'(0)=y''(0)=0, \tag{45}$$

obtained by substituting $k=3$ in (34) and by setting $f(x)g(y) = -6(10+2x^3+x^6)e^{-3y}$.

From (30), the Lagrange multiplier for $k=3$ is given by

$$\lambda(x,t) = \frac{t^2}{2} - \frac{1}{2}x^2\left(\frac{t}{x}\right)^4 + x^2\left(\frac{t}{x}\right)^3. \tag{46}$$

The correction functional for (45) becomes

$$\begin{aligned}
y_{n+1}(x) &= y_n(x)\\
&+ \int_0^x \left(\frac{t^2}{2}-\frac{1}{2}x^2\left(\frac{t}{x}\right)^4+x^2\left(\frac{t}{x}\right)^3\right)\left(y_n'''(t)+\frac{6}{t}y_n''(t)+\frac{6}{t^2}y_n'(t)-6(10+2t^3+t^6)e^{-3y_n(t)}\right)dt,
\end{aligned} \tag{47}$$

for $n \geq 0$. By selecting the zeroth approximation $y_0 = 0$, we obtain the following calculated solution approximations

$$\begin{aligned}
y_0(x) &= 0,\\
y_1(x) &= x^3+\frac{1}{28}x^6+\frac{1}{165}x^9,\\
y_2(x) &= x^3+\frac{1}{28}x^6+\frac{1}{165}x^9-\frac{199}{2002}x^{12}+\cdots,\\
y_3(x) &= x^3+\frac{1}{28}x^6+\frac{1}{165}x^9-\frac{3391}{14014}x^{12}+\cdots,\\
y_4(x) &= x^3+\frac{1}{28}x^6+\frac{1}{165}x^9-\frac{1}{4}x^{12}+\cdots,\\
&\cdots.
\end{aligned}$$

This in turn gives the series solution

$$y(x) = x^3-\frac{1}{2}x^6+\frac{1}{3}x^9-\frac{1}{4}x^{12}+\cdots, \tag{48}$$

that converges to the exact solution

$$y(x) = \ln(1 + x^3). \tag{49}$$

Example 4.

We conclude this section by considering the Lane–Emden type equation

$$y''' + \frac{8}{x}y'' + \frac{12}{x^2}y' + y^m = 0, y(0) = 1, y'(0) = y''(0) = 0, \tag{50}$$

obtained by substituting $k = 4$ in (34) and by setting $g(y) = y^m$, $f(x) = 1$.

From (30), the Lagrange multiplier for $k = 4$ is given by

$$\lambda(x,t) = -\frac{1}{6}t^2 - \frac{1}{3}x^2\left(\frac{t}{x}\right)^5 + \frac{1}{2}x^2\left(\frac{t}{x}\right)^4. \tag{51}$$

The correction functional for (50) becomes

$$\begin{aligned} &y_{n+1}(x) = y_n(x) \\ &+ \int_0^x \left(-\tfrac{1}{6}t^2 - \tfrac{1}{3}x^2\left(\tfrac{t}{x}\right)^5 + \tfrac{1}{2}x^2\left(\tfrac{t}{x}\right)^4\right)\left(y_n'''(t) + \tfrac{8}{t}y_n''(t) + \tfrac{12}{t^2}y_n^m(t)\right)dt, \end{aligned} \tag{52}$$

for $n \geq 0$. By selecting the zeroth approximation $y_0 = 1$, we obtain the following calculated solution approximations

$$\begin{aligned} y_0(x) &= 1, \\ y_1(x) &= 1 - \frac{1}{90}x^3, \\ y_2(x) &= 1 - \frac{1}{90}x^3 \frac{m}{2592}x^6, \\ y_3(x) &= 1 - \frac{1}{90}x^3 \frac{m}{3880}x^6 - \frac{m(17m-12)}{230947200}x^9, \\ y_4(x) &= 1 - \frac{1}{90}x^3 \frac{m}{3880}x^6 - \frac{m(17m-12)}{230947200}x^9 + \frac{m(679m^2 - 1182m + 528)}{2909934720000}x^{12}, \\ &\cdots. \end{aligned}$$

This in turn gives the series solution

$$y(x) = 1 - \tfrac{1}{90}x^3 + \tfrac{m}{38880}x^6 - \tfrac{m(17m-12)}{230947200}x^9 + \tfrac{m(679m^2-1182m+528)}{2909934720000}x^{12} + \cdots. \tag{53}$$

Substituting $m = 0$ gives the exact solution as

$$y(x) = 1 - \frac{1}{90}x^3. \tag{54}$$

3.3 The second model: the VIM and the Lagrange multipliers

In this section we will follow the analysis presented for the first kind. For Eq. (14) the correction functional reads

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(x,t)\left(y_n'''(t) + \tfrac{k}{t}y_n''(t) + f(t)\tilde{g}(y_n(t))\right)dt, \tag{55}$$

where $\delta(\tilde{g}(y_n(t))) = 0$.

To determine the optimal value of $\lambda(x,t)$, we take the variation for both sides with respect to $y_n(x)$ to obtain

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta\int_0^x \lambda(x,t)\left(y_n'''(t) + \tfrac{k}{t}y_n''(t) + f(t)\tilde{g}(y_n(t))\right)dt, \tag{56}$$

or equivalently

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta\int_0^x \lambda(x,t)\left(y_n'''(t) + \frac{k}{t}y_n''(t)\right)dt, \tag{57}$$

where we used $\delta(\tilde{g}(y_n(t))) = 0$.

Proceeding as before, we evaluate the integral at the right side in steps. We first integrate

$$I_1 = \int_0^x \lambda(x,t)y_n'''(t)\,dt. \tag{58}$$

Integrating I_1 by parts three times gives

$$I_1 = \left(\lambda(x,t)y_n'' - \lambda'(x,t)y_n' + \lambda''(x,t)y_n\right)|_{t=x} - \int_0^x \lambda'''(x,t)y_n(t)\,dt. \tag{59}$$

We next integrate the second term of the integral in (57), therefore we set

$$I_2 = \int_0^x \lambda(x,t)\frac{k}{t}y_n''(t)\,dt. \tag{60}$$

Integrating I_2 by parts twice gives

$$\begin{aligned} I_2 = &\left(\lambda(x,t)\tfrac{k}{t}y_n' - \tfrac{kt\lambda'(x,t) - k\lambda(x,t)}{t^2}y_n(t)\right)|_{t=x} \\ &+ \int_0^x \tfrac{kt^2\lambda''(x,t) - 2kt\lambda'(x,t) + 2k\lambda(x,t)}{t^3}y_n(t)\,dt. \end{aligned} \tag{61}$$

In view of (58)–(61), Eq. (57) becomes

$$\begin{aligned} &\delta y_{n+1}(x) = \delta y_n(x) \\ &+\delta\left(\lambda(x,t)y_n'' + \left[\lambda(x,t)\tfrac{k}{t^2} = \lambda'(x,t)\right]y_n' + \left[\lambda''(x,t) - \tfrac{k}{t}\lambda'(x,t) + \tfrac{k}{t^2})y_n\right]\right)|_{t=x} \\ &+ \int_0^x \left(-\lambda'''(x,t) + \tfrac{k}{t}\lambda''(x,t) - \tfrac{2k}{t^2}\lambda'(x,t) + \tfrac{2k}{t^3}\lambda(x,t)\right)\delta y_n. \end{aligned} \tag{62}$$

For arbitrary δy_n, we obtain the stationary condition

$$-\lambda'''(x,t) + \frac{k}{t}\lambda''(x,t) - \frac{2k}{t^2}\lambda'(x,t) + \frac{2k}{t^3}\lambda(x,t) = 0 \tag{63}$$

and the boundary conditions

$$\begin{aligned} \lambda(x,t)|_{t=x} &= 0, \\ \tfrac{k}{t^2}\lambda(x,t) + \lambda'(x,t)|_{t=x} &= 0, \\ \left(1 + \lambda''(x,t) - \tfrac{k}{t}\lambda'(x,t) + \tfrac{k}{t^2}\lambda(x,t)\right)|_{t=x} &= 0. \end{aligned} \tag{64}$$

To determine $\lambda(x,t)$, three cases will be examined:
(i) For the general case where $k \neq 1, 2$: solving (63)–(64) gives

$$\lambda(x,t) = \frac{t^2}{k-2} - \frac{xt}{k-1} - \frac{x^2}{(k-1)(k-2)}\left(\frac{t}{x}\right)^k. \tag{65}$$

(ii) For $k = 1$: solving (63)–(64) gives

$$\lambda(x,t) = -t^2 + xt\left(1 + \ln\frac{t}{x}\right) \tag{66}$$

(iii) For $k = 2$: solving (63)–(64) gives

$$\lambda(x,t) = -xt + t^2\left(1 - \ln\frac{t}{x}\right). \tag{67}$$

The successive approximations y_{n+1}, for $n \geq 0$, of the solution $y(x)$ will be readily obtained upon using any selective function $y_0(x)$. Consequently, the solution

$$y(x) = \lim_{n\to\infty} y_n(x). \tag{68}$$

In other words, the correction functional (56) will give several approximations, and therefore the exact solution is obtained in the limit of the resulting successive approximations.

3.4 Numerical examples for the second kind of the Emden–Fowler type equations

In this section, we study several numerical examples for the second case of the Emden–Fowler type equations of the third-order in the form

$$y''' + \frac{k}{x}y'' + f(x)g(y) = 0, y(0) = \alpha, y'(0) = y''(0) = 0 \tag{69}$$

We will study this equation for a variety of values for the shape factor k and for the given functions $f(x)g(y)$.

Example 1.

We first consider the nonlinear Emden–Fowler type equation

$$y''' + \tfrac{1}{x}y'' + 4x(9+22x^4+x^8)e^{-3y} = 0, y(0) = 0, y'(0) = y''(0) = 0, \tag{70}$$

obtained by substituting $k = 1$ in (69) and by setting $f(x)g(y) = 4x(9+22x^4+x^8)e^{-3y}$.

From (66), the Lagrange multiplier for $k = 1$ is given by

$$\lambda(x,t) = -t^2 + xt\left(1+\ln\frac{t}{x}\right). \tag{71}$$

The correction functional for (70) becomes

$$\begin{aligned} &y_{n+1}(x) = y_n(x) \\ &+ \int_0^x \left(-t^2 + xt\left(1+\ln\tfrac{t}{x}\right)\right)\left(y_n'''(t) + \tfrac{1}{t}y_n''(t) + 4t(9+22t^4+t^8)e^{-3y_n(t)}\right) dt, \end{aligned} \tag{72}$$

for $n \geq 0$. By selecting the zeroth approximation $y_0 = 0$, we obtain the following calculated solution approximations

$$\begin{aligned} y_0(x) &= 0, \\ y_1(x) &= -x^4 - \frac{11}{49}x^8 - \frac{1}{363}x^{12}, \\ y_2(x) &= -x^4 - \frac{1}{2}x^8 - \frac{11129}{35574}x^{12} - \frac{4721}{24200}x^{16} + \cdots, \\ y_3(x) &= -x^4 - \frac{1}{2}x^8 - \frac{1}{3}x^{12} - \frac{295721}{1185800}x^{16} + \cdots, \\ y_4(x) &= -x^4 - \frac{1}{2}x^8 - \frac{1}{3}x^{12} - \frac{1}{4}x^{16} + \cdots, \\ &\cdots. \end{aligned}$$

To facilitate the computational work, we used few terms of the exponential $e^{-3y_n(t)}, n \geq 1$. This in turn gives the series solution

$$y(x) = -x^4 - \frac{1}{2}x^8 - \frac{1}{3}x^{12} - \frac{1}{4}x^{16} - \frac{1}{5}x^{20} + \cdots, \tag{73}$$

that converges to the exact solution

$$y(x) = \ln(1-x^4). \tag{74}$$

Example 2.

We next consider the nonlinear Emden–Fowler type equation

$$y''' + \tfrac{2}{x}y'' + \tfrac{2}{x^2}y' - \tfrac{25}{8}x^2(16+52x^55+7x^{10})y^7 = 0, y(0) = 1, y'(0) = y''(0) = 0, \tag{75}$$

obtained by substituting $k = 2$ in (69) and by setting $f(x)g(y) = -\frac{25}{8}x^2(16+52x5+7x^{10})y^7$.

From (67), the Lagrange multiplier for $k = 2$ is given by

$$\lambda(x,t) = -xt + t^2\left(1-\ln\frac{t}{x}\right). \tag{76}$$

The correction functional for (75) becomes

$$\begin{aligned} &y_{n+1}(x) = y_n(x) \\ &+ \int_0^x \left(-xt + t^2\left(1-\ln\tfrac{t}{x}\right)\right)\left(y_n'''(t) + \tfrac{2}{t}y_n''(t) - \tfrac{25}{8}t^2(16+52t^5+7t^{10})y_n^7(t)\right) dt, \end{aligned} \tag{77}$$

for $n \geq 0$. By selecting the zeroth approximation $y_0 = 1$, we obtain the following calculated solution approximations

$$\begin{aligned} y_0(x) &= 1, \\ y_1(x) &= 1 + \frac{1}{2}x^5 + \frac{13}{72}x^{10} + \frac{1}{144}x^{15}, \\ y_2(x) &= 1 + \frac{1}{2}x^5 + \frac{3}{8}x^{10} + \frac{377}{1296}x^{15} + \cdots, \\ y_3(x) &= 1 + \frac{1}{2}x^5 + \frac{3}{8}x^{10} + \frac{5}{16}x^{15} + \cdots, \\ &\cdots. \end{aligned}$$

This in turn gives the series solution

$$y(x) = 1 + \frac{1}{2}x^5 + \frac{3}{8}x^{10} + \frac{5}{16}x^{15} + \frac{35}{128}x^{20} + \cdots, \tag{78}$$

that converges to the exact solution

$$y(x) = \frac{1}{\sqrt{1-x^5}}. \tag{79}$$

Example 3.

We now consider the nonlinear Emden–Fowler type equation

$$y''' + \tfrac{3}{x}y'' + \tfrac{6}{x^2}y' + 2(4+x^3)y^{-8} = 0, y(0) = 1, y'(0) = y''(0) = 0, \tag{80}$$

obtained by substituting $k = 3$ in (69) and by setting $f(x)g(y) = 2(4+x^3)y^{-8}$.

From (65), the Lagrange multiplier for $k = 3$ is given by

$$\lambda(x,t) = t^2 - \frac{1}{2}xt - \frac{1}{2}x^2(\frac{t}{x})^3. \tag{81}$$

The correction functional for (80) becomes

$$y_{n+1}(x) = y_n(x) + \int_0^x \left(t^2 - \tfrac{1}{2}xt - \tfrac{1}{2}x^2(\tfrac{t}{x})^3\right)\left(y_n'''(t) + \tfrac{3}{t}y_n''(t) + 2(4+t^3)y_n^{-8}\right) dt, \tag{82}$$

for $n \geq 0$. By selecting the zeroth approximation $y_0 = 1$, we obtain the following calculated solution approximations

$$\begin{aligned}
y_0(x) &= 1,\\
y_1(x) &= 1 - \frac{1}{3}x^3 - \frac{1}{105}x^6,\\
y_2(x) &= 1 - \frac{1}{3}x^3 - \frac{1}{9}x^6 - \frac{83}{1575}x^9 + \cdots,\\
y_3(x) &= 1 - \frac{1}{3}x^3 - \frac{1}{9}x^6 - \frac{5}{81}x^9 + \cdots,\\
&\cdots.
\end{aligned}$$

This in turn gives the series solution

$$y(x) = 1 - \frac{1}{3}x^3 - \frac{1}{9}x^6 - \frac{5}{81}x^9 - \frac{10}{243}x^{12} + \cdots, \tag{83}$$

that converges to the exact solution

$$y(x) = (1-x^3)^{\frac{1}{3}}. \tag{84}$$

Example 4.

We conclude this section by considering the linear Emden–Fowler type equation

$$y''' + \frac{4}{x}y'' - (10+10x^3+x^6)y = 0, y(0) = 1, y'(0) = y''(0) = 0, \tag{85}$$

obtained by substituting $k = 4$ in (69) and by setting $f(x)g(y) = -(10+10x^3+x^6)y$.

From (65), the Lagrange multiplier for $k = 4$ is given by

$$\lambda(x,t) = \frac{1}{2}t^2 - \frac{1}{3}xt - \frac{1}{6}x^2(\frac{t}{x})^4. \tag{86}$$

The correction functional for (85) becomes

$$y_{n+1}(x) = y_n(x) + \int_0^x \left(\tfrac{1}{2}t^2 - \tfrac{1}{3}xt - \tfrac{1}{6}x^2(\tfrac{t}{x})^4\right)\left(y_n'''(t) + \tfrac{4}{t}y_n''(t) - (10+10t^3+t^6)y_n(t)\right) dt, \tag{87}$$

for $n \geq 0$. By selecting the zeroth approximation $y_0 = 1$, we obtain the following calculated solution approximations

$$\begin{aligned}
y_0(x) &= 1,\\
y_1(x) &= 1 + \frac{1}{3}x^3 + \frac{1}{24}x^6 + \frac{1}{792}x^9,\\
y_2(x) &= 1 + \frac{1}{3}x^3 + \frac{1}{18}x^6 + \frac{19}{3168}x^9 + \cdots,\\
y_3(x) &= 1 + \frac{1}{3}x^3 + \frac{1}{18}x^6 + \frac{1}{162}x^9,\\
&\cdots.
\end{aligned}$$

This in turn gives the series solution

$$y(x) = 1 + \frac{1}{3}x^3 + \frac{1}{18}x^6 + \frac{1}{162}x^9 + \frac{1}{1944}x^{12} + \cdots, \tag{88}$$

that gives the exact solution

$$y(x) = e^{\frac{x^3}{3}}. \tag{89}$$

4 Conclusion

In this work, we have presented a framework to establish two kinds of Emden–Fowler type equations of third-order. Unlike the standard Emden–Fowler equations where the shape factor is unique, we showed that there are more than one shape factor for equations of order greater than or equal to 3 for one case and only one shape factor for another case. Similarly, the singular point appears once in the standard form, whereas in the established cases, the singular point $x = 0$ may appear twice. We used the variational iteration method for treating linear and nonlinear problems to illustrate our analysis. A variety of Lagrange multipliers was derived where the shape factor s play a major role in its determination. The calculated results from the recursion scheme are effective for all shape factor values k greater than or equal to 1. The obtained results validate the reliability and rapid convergence of the VIM.

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